Guaranteed Interval State Estimation for Linear Parameter Varying Systems with Unknown Inputs

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Abstract— **This paper investigates the construction of interval state observer for Linear Parameter Varying (LPV) systems affected by Unknown Input (UI) and bounded perturbations in both state and measurement equations. The parameter may depend on measurement or other exogenous measurable variables and is bounded in a compact set. The use of Sobolev spaces frame as well as High Order Sliding Mode (HOSM) differentiator enables us to properly access to the successive output derivatives, and thus to obtain an easily attainable rank condition for the decoupling of the UI from the state estimation error. In particular, the proposed rank condition relaxes the classical and widely used one. An interval state observer is provided and its boundedness is proved. Finally, an example illustrates the theoretical contribution.**

Index Terms— **Interval Observer, Unknown Input, High Order Sliding Mode, Linear Parameter Varying Systems**

I. Introduction

The problem of state estimation has been of great interest in the field of automatic and control systems for the last decades, and a lot of challenges is still present. Among the famous observers, we can cite the Luenberg, Kalman or H_∞ observers [1][2]. When the studied system is perturbed by some noise or any unknown perturbation, these observers cannot guarantee an exact estimation (even if it can be very good). For some critical application, such an inexact point-wise estimation is not acceptable, and a guarantee of the true state inside given bounds (an interval) is needed.

Several approaches of interval estimation have been developed in that direction, and among them, the algebraic approach [3], as well as the monotone (or cooperative) system approach [4][5] are of main importance. In that former domain, in which the present paper takes place, a lot of results have already been established. In discrete-time case, we can cite among others the work done in [6] for Linear Time Invariant (LTI) systems or in [7] for Linear Time Varying (LTV) systems. In continuous-time case, in which the present paper belongs, several observers have been developed for linear and some classes of nonlinear systems [8][5].

Beyond LTI and LTV systems, another class of systems is of great importance: the Linear Parameter Varying (LPV) systems whose matrices depend on bounded parameters (which may depend on internal or external variables). This kind of systems is of great importance are they are closer to non linear systems than the other classical kind of linear systems (LTI, LTV), whereas most of linear mathematical techniques can be applied on them. In the field of cooperative interval theory, only few works have been done on LPV systems: [4][9][10][11][12]. Both first ones are concerned with the state observation of an continuous-time LPV system without considering any UI. The third one is concerned with the control of such a system. [11] and [13] have dealt with fault diagnosis in LPV systems, and does not consider any perturbation nor unknown input in the measurement equation.

Finally, [12] consider the construction of an interval observer for discrete-time LPV systems with unknown inputs, and with known parameter.

The present paper deals with continuous-time LPV systems in the presence of an Unknown Inputs. Let consider the continuous LPV system described by:

$$\begin{cases} \dot{x}(t) &= A_\rho x(t) + D_\rho d(t) + F_\rho w(t) \\ y(t) &= C_\rho x(t) + E_\rho d(t) + v(t) \\ \psi(t) &= C_\rho x(t) + E_\rho d(t) \end{cases}, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state of the system, $d \in \mathbb{R}^{n_d}$ denotes the Unknown Input (UI) vector, $y \in \mathbb{R}^{n_y}$ is the output, and $\psi \in \mathbb{R}^{n_y}$ is the free-noise output. $w \in \mathbb{R}^{n_y}$ and $v \in \mathbb{R}^{n_y}$ are respectively the state and measurement perturbations. A_ρ, C_ρ, F_ρ, D_ρ and E_ρ are parameter varying matrices with appropriate dimensions, $\rho \in \Omega$ being the parameter. $\Omega \subset \mathbb{R}^{n_\rho}$ is assumed to be a convex and compact set. In the present paper, the parameter ρ is assumed to be known.

The main goal of this paper is to propose a way of making a guaranteed estimation of the state of system (1) using cooperative interval approach. Only few works have been done in that sense. In LTI case, without any unknown input on the measurement equation (matrix E is null), the problem has been solved in [14] under the specific condition $rank(CD) = rank(D)$. This condition enables the width of the state estimation interval not to be affected by the presence of the UI.

In a continuous-time LPV system, in the case both UI and bounded perturbations are present in both state and measurement equation, and in the case the condition $rank(CD) = rank(D)$ is not satisfied, no observer has been developed to the best of the authors knowledge. The novelty of the present work is to treat those cases in a unified way. More precisely, the present work focuses on the state interval estimations of the continuous LPV system (1). One of the main difficulty

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of dealing with UI in continuous-state systems, is that the decoupling of the UI needs to compute the derivative of successive output derivatives, and derivation of a noisy equation is not easy and raises several issues. In this paper, this derivation is made thanks to the use of High Order Sliding Mode differentiator.

The paper is organized as follow. Some preliminaries are introduced in section II. The main results are given and proved in section III. Finally, an example illustrating the theoretical contribution is given in section IV.

II. Preliminaries

A. Notations

For any matrix $A \in \mathcal{M}_{n,p}(\mathbb{R})$, $tr(A)$ denotes its trace, A^T its transpose, and $A^\dagger$ its pseudo-inverse. If $A = A^T \in \mathcal{M}_{n,n}(\mathbb{R})$ is a symmetric matrix, then $A > 0$ (resp. $A < 0$, $A \geq 0$, and $A \leq 0$) means that A is a positive definite (resp. negative definite, nonnegative, nonpositive) matrix.

The operators $\leq$, $\geq$, $<$ and $>$ are understood component wise for vectors and matrices, and the notation $x \in [\underline{x};\overline{x}]$ means that $\underline{x} \leq x \leq \overline{x}$ (where x is any vector or matrix). If $A \in \mathcal{M}_{n,m}(\mathbb{R})$, let set $A^+ = max\{0, A\}$, $A^- = A - A^+$ (such that $A = A^+ + A^-$ with $A^+ \geq 0$ and $A^- \leq 0$). $|A| = A^+ - A^-$ is the element-wise absolute value of A. Note that for any matrices A and B with compatible dimensions, $|AB| \leq |A|.|B|$ and $|A + B| \leq |A| + |B|$.

Finally, for $n \geq 1$ and $r \geq 1$, let set $U_{n,r} = \begin{bmatrix} I_n & 0_{n,r\times n} \end{bmatrix}$.

B. Definitions and useful properties

Let recall some definitions and useful properties.

Lemma 1 (Relation Order [4]): Let be a vector $x \in [\underline{x};\overline{x}]$ and a matrix $A \in [\underline{A};\overline{A}]$. Then:

$$\underline{A}^+\underline{x}^+ + \overline{A}^+\underline{x}^- + \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \leq \overline{A}^+\overline{x}^+ + \underline{A}^+\overline{x}^- + \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \tag{2}$$

Definition 1: Let $A = (a_{i,j}) \in \mathcal{M}_{n,n}(\mathbb{R})$ be a square matrix. A is said to be Metzler if $a_{i,j} \geq 0$ for $i \neq j$.

Lemma 2 (Cooperative property [15]): Given a non-autonomous system described by $\dot{x}(t) = Ax(t) + B(t,x)$, where $x \in \mathbb{R}^n_+$, A is Metzler and $B : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a map such that $B(t,x) \geq 0$ for all $t \geq 0$ and $x \geq 0$. Then, $x(t) \geq 0$, $\forall t \geq 0$, provided that $x(0) \geq 0$.

Lemma 3 (Asymptotic convergence): Given a system described by $\dot{x}(t) = A(t)x(t) + B(t)$ where $A(t)$ is Metzlez for all $t \geq 0$, $A(t) \leq \overline{A}, \forall t \geq 0$, $0 \leq B(t) \leq \overline{B}$, $\forall t \geq 0$, and $\overline{A}$ is Hurwitz. Then:

$$0 \leq x(t) \leq x_m(t), \qquad \forall t \geq 0, \tag{3}$$

where $x_m(t)$ is solution of $\dot{x}_m(t) = \overline{A}x_m(t) + \overline{B}$, and $x_m(0) = x(0) \geq 0$. Besides:

$$\lim_{t\to\infty} x(t) \leq -\overline{A}^{-1}\overline{B}, \tag{4}$$

providing that $x(0) \geq 0$.

Proof: Let consider the unknown vector $x_e = x_m - x$. It comes that $\dot{x}_e = \overline{A}x_e + A_e(t)x + B_e(t)$, where $A_e(t) = \overline{A} - A(t) \geq 0$ and $B_e(t) = \overline{B} - B(t) \geq 0$. Using lemma 2 on equation $\dot{x}(t) = A(t)x(t) + B(t)$, it comes that $x_m(t) \geq 0$ for all $t \geq 0$, and thus $A_e(t)x + B_e(t) \geq 0$, which leads to $x_e(t) \geq 0, \forall t \geq 0$, again thanks to lemma 2 (recall that $\overline{A}$ is Metzler). Thus, equation (3) holds. Besides, because $\overline{A}$ is Hurwitz, it comes: $lim_{t\to\infty} x_m(t) = -\overline{A}^{-1}\overline{B}$ (note that $\overline{A}^{-1}$ exists as $\overline{A}$ is Hurwitz, and thus 0 is not an eigenvalue of $\overline{A}$). Then, thanks to equation (3), the second result of the lemma is proved. ■

C. About the parameter varying matrices

Throughout all this paper, the following assumption on the parameter ρ holds.

Assumption 1: All LPV matrices depend **continuously** on the parameter $\rho \in \Omega$.

Remark 1: Ω being a compact, Assumption 1 implies that for each matrix M_ρ, the subset $\{M_\rho, \rho \in \Omega\}$ is also a compact (this is due to Heine theorem which states that the image of a compact set by a continuous function is a compact set), and thus is bounded. In particular, the sum, the difference or any product of such matrices stay bounded. The inverse of such an LPV matrix stays bounded if for all possible values of the parameter, the inverse exists.

D. Sobolev Spaces and HOSM differentiator

The main difficulty in continuous observation of systems affected by UI is the derivation of the outputs. Indeed, the derivative is needed in order to decouple the state estimation error from the UI (so that the estimation is guaranteed regardless of the UI variations). If the output is noise-free, there is no difficulty to do that, and any differentiator can be used. However, as it is the case in this paper, the presence of noise in the output is a big difficulty while making the derivation of it. This justify the following assumption.

Assumption 2: The noises w (resp. v) are assumed to be $s-1$ (resp. s) times differentiable, and all their respective derivatives are bounded: $|w| \leq \overline{w}$, $|\dot{w}| \leq \overline{\dot{w}}$, ..., $|w^{(s-1)}| \leq \overline{w^{(s-1)}}$, $|v| \leq \overline{v}$, $|\dot{v}| \leq \overline{\dot{v}}$, ..., $|v^{(s)}| \leq \overline{v^{(s)}}$, where $\overline{w}$, $\overline{\dot{w}}$, ..., $\overline{w^{(s-1)}}$, $\overline{v}$, $\overline{\dot{v}}$ and $\overline{v^{(s)}}$ are positive constants.

In a more formal way, and as in [16], the previous assumption can be stated under the frame of Sobolev spaces. A Sobolev space is a normed vector space, whose elements are functions such that their norm and the norm of their derivatives up to a given order are part of $\mathcal{L}_p$ for a given p. More formally, the following definition holds.

Definition 2: Let be s and p two integers (possibly ∞). The Sobolev space $\mathcal{W}^{s,p}_n$ is defined by:

$$\mathcal{W}^{s,p}_n = \{z : [0,\infty] \to \mathbb{R}^n \quad | \quad \frac{\partial^i z}{\partial t^i} \in \mathcal{L}^n_p([0,\infty]), \forall i = 0,...,s\}. \tag{5}$$

The associated norm is the following:

$$||z||^n_{s,p} = [\textstyle\sum_{i=0}^{s}(||\frac{\partial^i z(t)}{\partial t^i}||_{\mathcal{L}^n_p})^p]^{1/p} = (\sum_{i=0}^{s}\int_0^{+\infty}||\frac{\partial^i z(t)}{\partial t^i}||^p \mathrm{d}t)^{1/p} \tag{6}$$

Now, assumption 2 can be stated as:

Assumption 3: There exist an integer s such that $w \in \mathcal{W}_{n_w}^{s-1,\infty}$, $v \in \mathcal{W}_{n_y}^{s,\infty}$ and $\rho \in \mathcal{W}_{n_\rho}^{s,\infty}$.

Remark 2: In order to complete remark 1 it is worth noting that all matrices depending continuously of ρ or one of its successive derivatives till the order s are bounded. And it is also the case of the sum, the difference, the product or the inverse (provided that it exists) of such matrices.

Besides the previous assumption, the output and the Unknown Input (UI) vector are also assumed to be s times differentiable (without bounded assumption here).

Assumption 4: The output y and the UI vector d in (1) are both s times differentiable.

In a practical point of view, even if the noises are bounded, the derivation remains a problem, as classical method of derivation (as Euler differentiator) can introduce additive perturbations. Thus, and in order to tackle this problem, the calculation of the successive derivatives of y in the presence of the noise v is done thanks to the following High Order Sliding Mode differentiator detailled [17]:

$$\begin{cases} \dot{q}_0 = \nu_0, \qquad \nu_0 = -\lambda_0 |q_0 - y(t)|^{\frac{s}{s+1}} sign(q_0 - y(t)) + q_1 \\ \dot{q}_i = \nu_i, \qquad i = 1, ..., s-1 \\ \nu_i = -\lambda_i |q_i - \nu_{i-1}|^{\frac{s-i}{s-i+1}} sign(q_i - \nu_{i-1}) + q_{i+1} \\ \dot{q}_s = -\lambda_s sign(q_s - \nu_{s-1}) \end{cases} \tag{7}$$

where λ, $k = 0, ..., s$ are positive constants to be tuned (a procedure is proposed in [17]).

The values q, $k = 1..s$, solutions of (7), are approximations of the successive derivatives of the free-output ψ. Due to the presence of the noise, the differentiation suffer from some estimation error (error between q and $\psi^{(k)}$). An upper bound of this error is given by the following theorem.

Theorem 4: [17] Let $y = \psi + v : \mathbb{R}_+ \to \mathbb{R}$ be a s times continuously differentiable signal, with $|v| \le \overline{v}$, then there exist $0 \le T < \infty$, and some constants $\mu > 0$, $k = 0, ..., s$ (dependent on λ, $k = 0, ..., s$ only) such that for all $t \ge T$:

$$|q(t) - \psi^{(k)}(t)| \le \mu |v(t)|^{\frac{s-k+1}{s+1}}, \qquad k = 0, ..., s. \tag{8}$$

The previous theorem claims that the errors on the calculation of the successive derivatives of ψ with (7) are bounded. Using the bounds given by that theorem, it comes any $k = 0, ..., s$, and $t \ge T$:

$$q(t) = \psi^{(k)}(t) + \beta(t), \tag{9}$$

where $|\beta(t)| \le \overline{\beta} := \mu \overline{v}^{\frac{s-k+1}{s+1}}$ for any $t \ge T$.

E. Some useful relationships

Before establishing the main results, let state some useful relationships. The first one is the compact form of the expression of the output, as well as its successive derivative in a unique vector: it can be established (the calculation is straightforward using the previous notations) that for all $r \ge 1$ and $s \le r$, $y^{(s)}$ is under the form:

$$y^{(s)} = \mathcal{A}_{r,\rho}^s x + \mathcal{D}_{r,\rho}^s \tilde{d}_s + \mathcal{F}_{r,\rho}^s \tilde{w}_s + \tilde{v}_s, \tag{10}$$

where $\tilde{d}_s = \begin{bmatrix} d & \dot{d} & ... & d^{(s)} \end{bmatrix}^T$, $\tilde{w}_s = \begin{bmatrix} w & \dot{w} & ... & w^{(s-1)} \end{bmatrix}^T$, $\tilde{v}_s = \begin{bmatrix} \beta_0 & \beta_1 & ... & \beta_s \end{bmatrix}^T$, and the matrices $\mathcal{A}_{r,\rho}^s$, $\mathcal{D}_{r,\rho}^s$ and $\mathcal{F}_{r,\rho}^s$ are given recursively by:
$\mathcal{A}_{r,\rho}^{s+1} = \dot{\mathcal{A}}_{r,\rho}^s + \mathcal{A}_{r,\rho}^s A_\rho$, with $\mathcal{A}_{r,\rho}^0 = C_\rho$,
$\mathcal{D}_{r,\rho}^{s+1} = \begin{bmatrix} \mathcal{A}_{r,\rho}^s D_\rho & \mathcal{D}_{r,\rho}^s \end{bmatrix} + \begin{bmatrix} \dot{\mathcal{D}}_{r,\rho}^s & 0_{n_y,n_d} \end{bmatrix}$, with $\mathcal{D}_{r,\rho}^0 = E_\rho$, and
$\mathcal{F}_{r,\rho}^{s+1} = \begin{bmatrix} \mathcal{A}_{r,\rho}^s F_\rho & \mathcal{F}_{r,\rho}^s \end{bmatrix} + \begin{bmatrix} \dot{\mathcal{F}}_{r,\rho}^s & 0_{n_y,n_w} \end{bmatrix}$, with $\mathcal{F}_{r,\rho}^0 = 0_{n_y,n_w}$.

Therefore, it comes that:

$$\mathcal{Y}_r = \mathcal{A}_{r,\rho} x + \mathcal{D}_{r,\rho} \tilde{d}_r + \mathcal{F}_{r,\rho} \tilde{w}_r + \tilde{v}_r, \tag{11}$$

where $\mathcal{Y}_r = \begin{bmatrix} q_0 \\ q_1 \\ ... \\ q_r \end{bmatrix}$, and $\mathcal{A}_{r,\rho} = \begin{bmatrix} \mathcal{A}_{r,\rho}^0 \\ \mathcal{A}_{r,\rho}^1 \\ \mathcal{A}_{r,\rho}^2 \\ ... \\ \mathcal{A}_{r,\rho}^r \end{bmatrix}$, $\mathcal{D}_{r,\rho} = \begin{bmatrix} \mathcal{D}_{r,\rho}^0 & 0_{n_y,rn_d} \\ \mathcal{D}_{r,\rho}^1 & 0_{n_y,(r-1)n_d} \\ \mathcal{D}_{r,\rho}^2 & 0_{n_y,(r-2)n_d} \\ & ... \\ & \mathcal{D}_{r,\rho}^r \end{bmatrix}$

and $\mathcal{F}_{r,\rho} = \begin{bmatrix} \mathcal{F}_{r,\rho}^0 & 0_{n_y,rn_w} \\ \mathcal{F}_{r,\rho}^1 & 0_{n_y,(r-1)n_w} \\ \mathcal{F}_{r,\rho}^2 & 0_{n_y,(r-2)n_w} \\ & ... \\ & \mathcal{F}_{r,\rho}^r \end{bmatrix}$.

Besides, and according to equation (9), the following inequalities holds for any time $t \ge T$ (with T given by theorem 4):

$$\tilde{w}_r \le \overline{W}_r, \qquad \tilde{v}_r \le \overline{V}_r, \tag{12}$$

where $\overline{W}_r = \begin{bmatrix} \overline{w} & \overline{\dot{w}} & ... & \overline{w^{(r-1)}} \end{bmatrix}^T$ and $\overline{V}_r = \begin{bmatrix} \overline{\beta}_0 & \overline{\beta}_1 & ... & \overline{\beta}_r \end{bmatrix}^T$.

III. Main result

A. Assumptions

In order to establish the main results of the paper, the needed assumptions are introduced here. The first assumption states a rank condition in order to decouple the unknown input from the observer estimation error (in that way the unknown input won't have any impact on the estimation performance).

Assumption 5: There exists an integer r such that:

$$rank(\mathcal{D}_{r,\rho}) = rank(\begin{bmatrix} \mathcal{D}_{r,\rho} \\ D_\rho U_{n_d,r} \end{bmatrix}), \qquad \forall \rho \in \Omega \tag{13}$$

This assumption relaxes the rank assumption $rank(C_\rho D_\rho) = rank(D_\rho)$ of [14]. This former is equivalent to the case in which $E_\rho = 0$ and $r = 1$.

According to assumption 5, for any $\rho \in \Omega$, there exists a gain K_ρ satisfying the following equation:

$$K_\rho \mathcal{D}_{r,\rho} = D_\rho U_{n_d,r}, \tag{14}$$

and the solution is given by:

$$K_\rho = G_\rho + X_\rho H_\rho, \tag{15}$$

where $G_\rho = D_\rho U_{n_d,r} \mathcal{D}_{r,\rho}^+$, $H = I_{(r+1)n_y} - \mathcal{D}_{r,\rho} \mathcal{D}_{r,\rho}^\dagger$, and X_ρ is any arbitrary (gain) matrix (which means that for any matrix X_ρ, K_ρ given by (15) is solution of equation (14)).

The second assumption ensures the stability of the observer, as well as the desired order relation between the lower bound $\underline{x}$, the state x and the upper bound $\overline{x}$: $\underline{x} \le x \le \overline{x}$.

Assumption 6: There exist a constant transformation matrix $\mathcal{T} \in GL_{n_x}(\mathbb{R})$, and for any $\rho \in \Omega$, there exist a gain matrix $X_\rho \in \mathcal{M}_{n_x,(r+1)n_d}(\mathbb{R})$, such that:

1) the matrix N_ρ is Hurwitz,
2) the matrix $\mathcal{T}N_\rho\mathcal{T}^{-1}$ is Metzler,

where $N_\rho = A_\rho - K_\rho\mathcal{A}_{r,\rho} = A_\rho - G_\rho\mathcal{A}_{r,\rho} - X_\rho H_\rho\mathcal{A}_{r,\rho}$.

The Hurwitz property is needed in order to make a stable observer, whereas the Metzler property is needed in order to satisfy the increasing order of the state bounds: $\underline{x} \le x \le \overline{x}$.

Note that the transformation with the invertible matrix $\mathcal{T}$ conserves the eigenvalues of N_ρ, and thus $TN_\rho T^{-1}$ has the same eigenvalues (with multiplicity) as N_ρ and is therefore also Hurwitz. Thus, the design principle consists in constructing X_ρ such that N_ρ is Hurwitz, and then $\mathcal{T}$ such that $\mathcal{T}N_\rho\mathcal{T}^{-1}$ is Metzler for any $\rho \in \Omega$.

The following lemma can be applied in some case in order to prove the existence of a matrix $\mathcal{T}$ satisfying assumption 6.

Lemma 5 (Coordinates Transformation property [9]): Let N_ρ be a matrix satisfying the interval constraint: $N_0 - \tilde{N} \le N_\rho \le N_0 + \tilde{N}$, for all $\rho \in \Omega$, with constant matrices $N_0 = N_0^T$ and $\tilde{N} \ge 0$. If there exist a constant $\mu \in \mathbb{R}_+$ and a diagonal matrix Δ such that $\mu \ge n\|\tilde{N}\|_{max}$, and the Metzler matrix $\mu\mathbf{1}_{n,n} - \Delta$ has the same eigenvalues as N_0 (where $\mathbf{1}_{n,n}$ is the matrix whose all elements are equal to 1). Then, there exists an orthogonal matrix $\mathcal{T}$ such that $\mathcal{T}N_\rho\mathcal{T}^{-1}$ is Metzler for any $\rho \in \Omega$.
If lemma 5 cannot be applied, other methods exist in order to construct a matrix $\mathcal{T}$ satisfying 6. For example a sylvester approach can be used [5], or a time varying similarity transformation [18].

By making the non singular transformation $z = \mathcal{T}x$ suggested by assumption 6, system (1) is described in the new coordinates by:

$$\begin{cases} \dot{z} &= \mathcal{T}A_\rho\mathcal{T}^{-1}z + \mathcal{T}D_\rho d + \mathcal{T}F_\rho w \\ y &= C_\rho\mathcal{T}^{-1}z + E_\rho d + v \end{cases}. \tag{16}$$

B. Main results

Let consider the following observer structure:

$$\begin{cases} \dot{\overline{z}} &= \mathcal{T}N_\rho\mathcal{T}^{-1}\overline{z} + \mathcal{T}K_\rho\mathcal{Y}_r + (L_\rho^+ - L_\rho^-)\overline{W}_r + (M_\rho^+ - M_\rho^-)\overline{V}_r \\ \dot{\underline{z}} &= \mathcal{T}N_\rho\mathcal{T}^{-1}\underline{z} + \mathcal{T}K_\rho\mathcal{Y}_r - (L_\rho^+ - L_\rho^-)\overline{W}_r - (M_\rho^+ - M_\rho^-)\overline{V}_r \end{cases}, \tag{17}$$

where $L_\rho = \mathcal{T}(K_\rho\mathcal{F}_{r,\rho} - F_\rho U_{r-1,n_w})$ and $M_\rho = \mathcal{T}K_\rho$.

Theorem 6: Let assume that the parameter $\rho \in \Omega$ is known at any time. Let assumptions 1, 3, 4, 5 and 6 hold. Then, let consider the time instant $T \ge 0$ given by theorem 4 from which equation (8) holds, and let assume that $\underline{z}(T) \le z(T) \le \overline{z}(T)$. Then the following relation holds for any $t \ge T$:

$$\underline{z}(t) \le z(t) \le \overline{z}(t). \tag{18}$$

Besides, there exists $\gamma_z > 0$ such that:

$$\lim_{t\to\infty} e_z(t) \le \gamma_z, \tag{19}$$

where $e_z(t) = \overline{z}(t)\underline{z}(t)$ is the total interval error.
Note that equation (19) gives an asymptotic bound of the width of the interval observer.

Proof: In the following proof, the variables are considered at a time $t \ge T$. In particular, relations (12) hold. Let calculate the dynamics of the upper and lower interval errors. It comes:

$$\begin{aligned}
&\dot{\overline{e}}_z = \dot{\overline{z}} - \dot{z} \\
&= \mathcal{T}N_\rho\mathcal{T}^{-1}\overline{z} + \mathcal{T}K_\rho\mathcal{Y}_r + (L_\rho^+ - L_\rho^-)\overline{W}_r + (M_\rho^+ - M_\rho^-)\overline{V}_r \\
&-\mathcal{T}A_\rho\mathcal{T}^{-1}z - \mathcal{T}D_\rho d - \mathcal{T}F_\rho w \\
&= \mathcal{T}N_\rho\mathcal{T}^{-1}\overline{e}_z + \mathcal{T}(N_\rho + K_\rho\mathcal{A}_{r,\rho} - A_\rho)\mathcal{T}^{-1}z + \mathcal{T}(K_\rho\mathcal{D}_{r,\rho} - D_\rho U_{r,n_d})\tilde{d}_r \\
&+L_\rho\tilde{w}_r + M_\rho\tilde{v}_r + (L_\rho^+ - L_\rho^-)\overline{W}_r + (M_\rho^+ - M_\rho^-)\overline{V}_r
\end{aligned} \tag{20}$$

and similarly:

$$\begin{aligned}
&\dot{\underline{e}}_z = \dot{z} - \dot{\underline{z}} \\
&= \mathcal{T}N_\rho\mathcal{T}^{-1}\underline{e}_z + \mathcal{T}(A_\rho - K_\rho\mathcal{A}_{r,\rho} - N_\rho)\mathcal{T}^{-1}z + \mathcal{T}(D_\rho U_{r,n_d} - K_\rho\mathcal{D}_{r,\rho})\tilde{d}_r \\
&-L_\rho\tilde{w}_r - M_\rho\tilde{v}_r + (L_\rho^+ - L_\rho^-)\overline{W}_r + (M_\rho^+ - M_\rho^-)\overline{V}_r.
\end{aligned} \tag{21}$$

Then, from the definition of N_ρ in assumption 6 and from the definition of K_ρ in assumption 5, the following equalities hold:

$$\begin{cases} 0 &= N_\rho + K_\rho\mathcal{A}_{r,\rho} - A_\rho \\ 0 &= K_\rho\mathcal{D}_{r,\rho} - D_\rho U_{r,n_d} \end{cases}. \tag{22}$$

Besides, recalling that $L_\rho = L_\rho^+ + L_\rho^-$ and $M_\rho = M_\rho^+ + M_\rho^-$, it comes:

$$\begin{cases} \dot{\overline{e}}_z &= \mathcal{T}N_\rho\mathcal{T}^{-1}\overline{e}_z + L_\rho^+(\overline{W}_r + \tilde{w}_r) - L_\rho^-(\overline{W}_r - \tilde{w}_r) \\ &+M_\rho^+(\overline{V}_r + \tilde{v}_r) - M_\rho^-(\overline{V}_r - \tilde{v}_r) \\ \dot{\underline{e}}_z &= \mathcal{T}N\mathcal{T}^{-1}\underline{e}_z + L_\rho^+(\overline{W}_r - \tilde{w}_r) - L_\rho^-(\overline{W}_r + \tilde{w}_r) \\ &+M_\rho^+(\overline{V}_r - \tilde{v}_r) - M_\rho^-(\overline{V}_r + \tilde{v}_r) \end{cases}. \tag{23}$$

Let set: $J_\rho = \mathcal{T}N_\rho\mathcal{T}^{-1}$, $\overline{S}_\rho = L_\rho^+(\overline{W}_r + \tilde{w}_r) - L_\rho^-(\overline{W}_r - \tilde{w}_r) + M_\rho^+(\overline{V}_r + \tilde{v}_r) - M_\rho^-(\overline{V}_r - \tilde{v}_r)$ and $\underline{S}_\rho = L_\rho^+(\overline{W}_r - \tilde{w}_r) - L_\rho^-(\overline{W}_r + \tilde{w}_r) + M_\rho^+(\overline{W}_r - \tilde{v}_r) - M_\rho^-(\overline{V}_r + \tilde{v}_r)$. The interval errors become:

$$\begin{cases} \dot{\overline{e}}_z &= J_\rho\overline{e}_z + \overline{S}_\rho \\ \dot{\underline{e}}_z &= J_\rho\underline{e}_z + \underline{S}_\rho \end{cases}. \tag{24}$$

Recalling that $|\tilde{w}_r| \le \overline{W}_r$ and $|\tilde{v}_r| \le \overline{V}_r$, it comes that $\overline{S}_\rho \ge 0$ and $\underline{S}_\rho \ge 0$. Besides, from assumption 6, J_ρ is Metlzler. Therefore, according to lemma 2 with initial time $t = T$, and using the initial order relation $\underline{z}(T) \le z(T) \le \overline{z}(T)$, it comes that for all $t \ge T$, $\overline{e}_z(t) \ge 0$ and $\underline{e}_z(t) \ge 0$, which proves the first statement of the theorem.

In order to prove the second statement, let us define the total non-negative error: $e_z = \overline{e}_z + \underline{e}_z$. It comes:

$$\dot{e}_z(t) = J_\rho e_z(t) + S_\rho, \tag{25}$$

with $S_\rho = \overline{S}_\rho + \underline{S}_\rho = 2|L_\rho^+ - L_\rho^-|\overline{W}_r + 2|M_\rho^+ - M_\rho^-|\overline{V}_r$. J being Metzler and Hurwitz according to assumption 6, and S_ρ being positive, lemma 3 can be applied and leads to :

$$\lim_{t\to\infty} e_z \le \gamma_z, \tag{26}$$

where $\gamma_z = -J^{-1}S$, with $J = sup\{J_\rho, \rho \in \Omega\}$ and $S = sup\{S_\rho, \rho \in \Omega\}$, which proves the second statement of the theorem and concludes the proof. ■

From the observer defined in equations (17) and theorem 6, the solution x of system (1) is given by the following corollary from [19].

Corollary 7 ([19]): Let consider the following equations:

$$\begin{cases} \overline{x}(t) = \mathcal{S}^+\overline{z}(t) + \mathcal{S}^-\underline{z}(t) \\ \underline{x}(t) = \mathcal{S}^+\underline{z}(t) + \mathcal{S}^-\overline{z}(t) \end{cases}, \tag{27}$$

with $\mathcal{S} = \mathcal{T}^{-1}$. Then, under the conditions of theorem 6, and providing that $\underline{x}(T) \leq x(T) \leq \overline{x}(T)$, there exist a time instant T such that:

1)
$$\overline{x}(t) \leq x(t) \leq \overline{x}(t), \qquad \forall t \geq T, \tag{28}$$

2) the total non-negative error $e_x = \overline{e}_x + \underline{e}_x$ satisfies:

$$lim_{t\to\infty} e_x \leq \gamma_x := |\mathcal{S}|\gamma_z, \tag{29}$$

where $\overline{e}_z = \overline{z} - z$ and $\underline{e}_z = z - \underline{z}$ are the upper and lower bound errors.

IV. Illustrative example

Consider the following LTI system:

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 4 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + v \end{cases}, \tag{30}$$

where $x \in \mathbb{R}^3$, $y \in \mathbb{R}^2$, $d \in \mathbb{R}^2$, $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^2$. The noises are simulated according to $w = 0.5\cos(4t + 0.4)$ and $v = -0.8\sin(1.2t + 0.7)\begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Thus, the noises and all their derivatives are bounded and assumption 3 holds. The unknown input is simulated to be: $d(t) = \begin{bmatrix} 12\cos(t) & -4\sin(t) \end{bmatrix}^T$.

We can see that, in this example $rank(CD) \neq rank(D)$, and thus, the observer given in [14] cannot be applied. However, for $r = 2$, $rank(\mathcal{D}_r) = rank(\begin{bmatrix} \mathcal{D}_r \\ U_r D \end{bmatrix}) = 3$, and thus assumption 5 is satisfied. Then, the pair $(A - G\mathcal{A}_r, H\mathcal{A}_r)$ being observable, X can be chosen such that N is Hurwitz. Finally, the matrix $\mathcal{T}$ is constructed using a Sylvester transformation.

Simulation is launched during 10 seconds with a constant time step equal to 10^{-4}. The estimations of the successive output derivatives are obtained using the HOSM differentiator presented in equations (7) with $s = 2$, $\lambda_0 = 2.0L^{1/3}$, $\lambda_1 = 1.5L^{1/2}$, $\lambda_2 = 1.1L$, according to [20] and [17], where L is an estimation of the upper bound of the norm of the $(s+1)$-th derivative of ψ : $\psi^{(s+1)} < L$. The results are presented on figures 1 (the estimation for x_3 gives similar results and is given due to a lack of space).

In that particular example, the boundedness of the output derivatives estimation errors was not achieved in the first time of the simulation, but a bit later (after about 1 second) due to the form of the signal. In fact, it is consistent with theorem 4 which states that the boundedness is true after a certain time T (here, $T = 1s$). A way of dealing with that is to take for the interval observers the lowest and highest possible values of x (as it is usually taken for the initialization of the observers when no other prior knowledge are available) up to the time T. Besides T is estimated by looking for the instant from which the difference between the measured output and the output calculated by the HOSM is lowest than the associated bound given by theorem 4 (more on this in [20] or [17]).

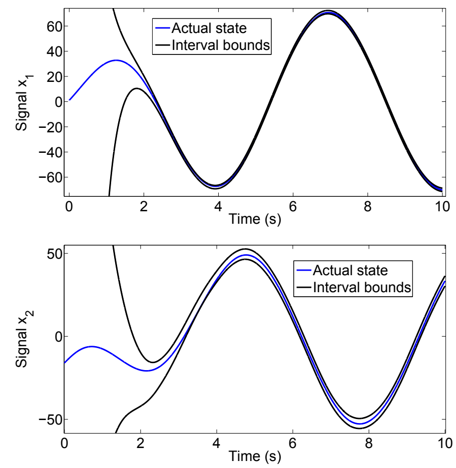

Fig. 1: Example LTI: State Interval Estimation

V. Conclusions

In this paper, state observers for LPV systems with Unknown Inputs are proposed. The use of Sobolev spaces as well as HOSM differentiator enables to guarantee boundedness of output derivatives (with knowledge of the derivative), when the output is affected by perturbations belonging to Sobolev spaces. Based on that, state observers are proposed for Linear Parameter Varying systems with UI, when the parameter (as well as its derivatives) is known. A Theorem is given for the guaranteed estimation, as well as for the (asymptotic) boundededness of the estimation error. Finally, an example illustrates the theoretical contribution of the paper.

As a future work, the study of the case in which the parameter is not known may be treated.

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