Constrained rigid body attitude stabilization: an anti-windup approach

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***Abstract*— This paper considers the rigid body attitude stabilization problem with actuator saturation. This problem has been considered in the literature before, but most approaches achieve stability at the expense of strict constraints on the control gains, thereby limiting small-signal performance. The approach proposed here removes the constraints on the gains of the nominal controller and instead deals with the control constraints through an anti-windup mechanism, effectively allowing for more freedom in small signal behavior whilst preserving stability in the large. Although the rigid body stabilization problem is inherently nonlinear, the spirit of the results developed here resemble constrained-control results for linear systems containing imaginary axis poles, despite some subtle differences which need careful consideration.**

***Index Terms*— Constrained control; aerospace; Lyapunov methods**

I. INTRODUCTION

THE attitude control problem for rigid bodies has been studied for many years and is the subject of numerous papers [1]–[3]. It is of vital importance in the aerospace/space industries, especially in application to satellites, landers and UAV's. The attitude control problem has a nonlinear, yet tractable, structure: thus many papers ([1], [2], [4] and references therein) have proposed controllers, and accompanying nonlinear stability analyses, which demonstrate how (almost) global stability, or similar, can be achieved for such systems.

Like all controllers, however, those responsible for attitude control have to contend with actuator saturation. This constrained version of the attitude control problem has thus been studied widely and various approaches have been proposed - [5], [6] and references therein. For instance, [7] considers the attitude control problem using similar ideas to those proposed in [8] for linear integrator chains; almost global stability is proved using PD controllers. In [9], constrained stabilization using reaction wheels and adaptive control is considered. In [5], [10], the controller is partitioned and almost global stability is proved; the controller again has a PD structure. In the aforementioned papers a "one-shot" controller is developed and is responsible for both nominal (unsaturated) and constrained (saturated) performance. Consequently, at least part of the controller is restricted to be "small" to ensure satisfactory large-signal ("almost global") behavior. This is consistent with constrained control of linear systems where low-gain control ([11]) can be exploited to design linear controllers that enforce global, or semi-global, stability requirements. In [12] the constrained attitude control problem is considered by linearizing the attitude dynamics and applying linear constrained control ideas.

The main issue with most of the above approaches is that the price paid for (almost) global-stability is some sort of restriction on the controller. For example, in [5] and [7] the proportional part of the PD controller is restricted to ensure global stability. To overcome limitations such as these, the approach here follows the anti-windup paradigm inherited from the constrained linear system literature (e.g. [13]–[15]). In the anti-windup approach, the so-called *nominal controller* is responsible for controlling the system when saturation is absent, with the so-called *anti-windup* (AW) compensator being activated when saturation occurs, before relinquishing full control once more to the nominal controller after saturation has ceased. The appeal of this approach is that there are **no constraints** on the nominal controller and that the AW compensator is solely responsible for preserving stability and performance when saturation occurs. Moreover, the design of the two control elements is done independently.

Therefore, this paper proposes an AW technique for the constrained rigid body stabilization problem, where the nonlinearities in the plant dynamics are accounted for directly. The scheme provides *global* (but not finite-gain) $\mathcal{L}_2$ performance, which effectively ensures that, under natural conditions, the AW compensator will guarantee a return to ideal, unsaturated behavior - as is typical in the AW approach [13], [15]. The novelty of the scheme is that it can cope with the *non-globally-Lipschitz* and "un-matched" nonlinearities which the technique of [16] requires. Also, the guarantees provided here are *global*; the approaches of [17] and [18] could be applied, but would naturally result in local results, which is less than one expects, given the (almost) global results of [5], [7] and others.

II. PROBLEM FORMULATION

A. Notation

The saturation and deadzone nonlinearities play a central role in the paper and are defined as

$$\mathrm{sat}_{\bar{u}}(u) := \left[\ \mathrm{sat}_{\bar{u}_1}(u_1) \dots \mathrm{sat}_{\bar{u}_m}(u_m)\ \right]' \tag{1}$$

where, with some abuse of notation, $\mathrm{sat}_{\bar{u}_i}(u_i) = \mathrm{sign}(u_i)\min\{|u_i|, \bar{u}_i\}$ and $\bar{u}_i > 0$. The deadzone is the complement of the saturation and is given by

$$\mathrm{Dz}_{\bar{u}}(u) = u - \mathrm{sat}_{\bar{u}}(u) \tag{2}$$

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The cross product of two vectors $a, b \in \mathbb{R}^3$ is expressed in matrix form as

$$a \times b = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{=:[a]^\times} b$$

where $[a]^\times$ is a skew-symmetric matrix. The space $\mathcal{L}_2$ is, similar to [19], used to formally define desirable properties of the AW compensator. A vector $x(t)$ is said to belong to the space $\mathcal{L}_2$ if its $\mathcal{L}_2$ norm is finite i.e.

$$\|x\|_2 := \int_0^\infty \|x(t)\|^2 \, dt < \infty$$

For convenience, $\mathbf{1}_m$ denotes a m-dimensional vector with unity elements. The remaining notation is standard.

B. Unit Quarternion

The plant is modeled using unit quaternions, a good summary of which can be found in [20]. The *Hamilton* convention is followed with $q \in \mathbb{H}$ being defined as

$$q = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \cos\beta/2 \\ \vec{e}\sin\beta/2 \end{bmatrix} \in \mathbb{H} \tag{3}$$

where $\vec{e}$ is the Euler axis and β is the rotation angle around the Euler axis. q_0 and $\vec{q}$ are the scalar and vector parts of the quaternion, respectively. The unit quaternion does not evolve on a standard linear vector space, but rather on the set

$$\mathbb{H} := \left\{ q : \ q_0^2 + \vec{q}\,'\vec{q} = 1, \quad q_0 \in \mathbb{R}, \vec{q} \in \mathbb{R}^3 \right\} \tag{4}$$

The conjugate of a unit quaternion $q \in \mathbb{H}$ is given as

$$q^* = \begin{bmatrix} q_0 \\ -\vec{q} \end{bmatrix} \tag{5}$$

and the quaternion product between two unit quaternions $q_a, q_b \in \mathbb{H}$ is given by

$$q_a \otimes q_b = \begin{bmatrix} q_{a0}q_{b0} - \vec{q}_a\,'\vec{q}_b \\ q_{a0}\vec{q}_b + q_{b0}\vec{q}_a + \vec{q}_a \times \vec{q}_b \end{bmatrix} \tag{6}$$

The quaternion error between $q_a, q_b \in \mathbb{H}$ is defined as

$$q_e := q_a^* \otimes q_b$$

The rotation matrix, $R(q)$, describing the attitude of a rigid body, can be expressed via the following identity.

$$R(q) = I_3 + 2q_0[\ \vec{q}\ ]^\times + 2([\ \vec{q}\ ]^\times)^2 \tag{7}$$

The rotation matrix is orthogonal and hence $R(q)^{-1} = R(q)'$. Note that $\dot{R}(q) = R(q)[\vec{\omega}]^\times$ where $\vec{\omega} \in \mathbb{R}^3$ is the angular velocity of the rigid body, expressed in body axis coordinates.

C. Plant Dynamics

The dynamics of a rigid body can be expressed *via* the kinematics using quaternions, and the angular velocity dynamics, expressed in the body axis. The kinematics are given by

$$\begin{bmatrix} \dot{q}_0 \\ \dot{\vec{q}} \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} -\vec{q}\,' \\ q_0 I_3 + [\ \vec{q}\ ]^\times \end{bmatrix}}_{T(q)} \vec{\omega} \tag{8}$$

This can be combined with the angular velocity dynamics to obtain the plant equations as

$$\dot{q} = T(q)\vec{\omega} \tag{9}$$

$$J\dot{\vec{\omega}} = -\vec{\omega} \times J\vec{\omega} + u + u_d \tag{10}$$

where $J > 0$ is the positive definite inertia matrix; $u \in \mathbb{R}^3$ is a vector of control input torques; and $u_d \in \mathbb{R}^3$ is a vector of disturbance torques.

D. Controller Dynamics and Closed-loop

Keeping with much of the attitude control literature, the nominal controller is assumed to be of PD type. This is not necessary but will simplify presentation. Therefore, noting the plant (9)-(10) contains no saturation, the controller is

$$u = K_1\vec{q} + K_2\vec{\omega} \tag{11}$$

where $K_1, K_2 \in \mathbb{R}^{3\times 3}$ are matrices which are designed such that the interconnection of (9), (10) and (11) is (almost) globally asymptotically stable. These controller gains can be constructed according to [1], [7] and yield the closed loop:

$$\begin{cases} \dot{q} &= T(q)\vec{\omega} \\ J\dot{\vec{\omega}} &= -\vec{\omega} \times J\vec{\omega} + K_1\vec{q} + K_2\vec{\omega} + u_d \end{cases} \tag{12}$$

This system is referred to as the *nominal closed-loop system*. It is a standing assumption throughout the paper that this system has favorable stability and performance properties and represents, in some sense, *ideal* behavior. The following formal assumption is made.

Assumption 1: K_1, K_2 and $u_d(t)$ are such that $\vec{\omega}(t)$ in (12) is bounded for all $t \geq 0$ and such that $\lim_{t\to\infty} \vec{\omega}(t) = 0$.

Assumption 1 is stronger than that used in linear AW (see [19]) due to the additional constraint on $\vec{\omega}$. This is because, as demonstrated later, the AW compensator is driven not only by the deadzone function (as in standard linear AW) but also by the states of the nominal system. A wider class of disturbances may be included if a more sophisticated controller is used.

III. ANTI-WINDUP STRUCTURE

A. Saturated Closed-loop System

A more realistic model of the control input torques is

$$u = \mathrm{sat}_{\bar{u}}(u_s) \tag{13}$$

The control signal u_s is generated by supplementing the nominal control signal (11) with outputs from an AW compensator in the following manner:

$$u_s = K_1\vec{q}_e + K_2\vec{\omega}_e + \Lambda_1\vec{q}_a + \Lambda_2\vec{\omega}_a \tag{14}$$

where

$$\vec{\omega}_e = \vec{\omega} - R(q_e)\vec{\omega}_a \tag{15}$$

$$q_e = q_a^* \otimes q \tag{16}$$

with $q_a = [q_{a0}, \vec{q}_a]' \in \mathbb{H}$ and $\vec{\omega}_a \in \mathbb{R}^3$ being states of the AW compensator, and $\Lambda_1, \Lambda_2 \in \mathbb{R}^{3\times 3}$ gains which represent the design freedom in the AW compensator. Inspired by [16] (see also [19], [21]), the AW compensator has the form

$$\begin{cases} \dot{q}_a &= T(q_a)\vec{\omega}_a \\ JR(q_e)\dot{\vec{\omega}}_a &= F(\vec{\omega}, \vec{\omega}_a) - \mathrm{Dz}_{\bar{u}}(u_s) + \Lambda_1\vec{q}_a + \Lambda_2\vec{\omega}_a \end{cases} \tag{17}$$

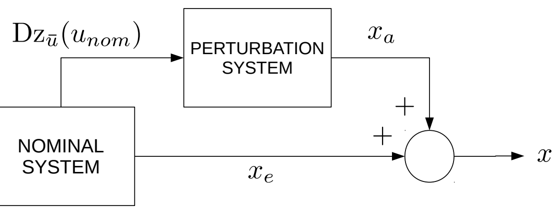

Fig. 1. Partitioning of system with saturation and anti-windup.

Section IV will show how the function $F(\cdot,\cdot):\mathbb{R}^3\times\mathbb{R}^3\mapsto\mathbb{R}^3$ and the gains Λ_1 and Λ_2 can be chosen to bestow certain stability and performance properties on the closed-loop.

B. Anti-windup objectives

The objectives of linear AW design were formalized in [19] (and [13], [22]) and, in essence, require that (i) in the absence of saturation, nominal control continues unimpeded; and (ii) after saturation has ceased nominal behavior will resume. These notions are illustrated in Fig. 1, where the real state (x) is the sum/difference of the states of a system representing the ideal (unsaturated) system (x_e) and those of a system representing a perturbation due to saturation and AW (x_a). If saturation does not take place in steady state, one expects the perturbation system states (x_a) to converge to zero, meaning the actual states (x) then converge to the ideal states (x_e).

The above approach is adopted here and achieved using a coordinate transformation. First, $F(\vec{\omega},\vec{\omega}_a)$ is chosen as

$$F(\vec{\omega},\vec{\omega}_a)=-\vec{\omega}\times J\vec{\omega}+\vec{\omega}_e\times J\vec{\omega}_e-JR(q_e)[\vec{\omega}_e]^{\times}\vec{\omega}_a \quad (18)$$

Then, pre-multiplying the derivative of (15) by J and using (10), (13) and (17) yields:

$$\begin{aligned}J\dot{\vec{\omega}}_e&=-\vec{\omega}\times J\vec{\omega}+\mathrm{sat}_{\bar{u}}(u_s)-F(\vec{\omega},\vec{\omega}_a)+\mathrm{Dz}_{\bar{u}}(u_s)\\&\quad-\Lambda_1\vec{q}_a-\Lambda_2\vec{\omega}_a-J\dot{R}(q_e)\vec{\omega}_a+u_d\\&=-\vec{\omega}_e\times J\vec{\omega}_e+u_s-\Lambda_1\vec{q}_a-\Lambda_2\vec{\omega}_a+u_d\\&=-\vec{\omega}_e\times J\vec{\omega}_e+K_1\vec{q}_e+K_2\vec{\omega}_e+u_d\end{aligned} \quad (19)$$

Differentiating equation (16) yields:

$$\dot{q}_e=\frac{d}{dt}(\vec{q}_a^{\,*})\otimes\vec{q}+\vec{q}_a^{\,*}\otimes\dot{\vec{q}} \quad (20)$$

Using the expressions for $\dot{\vec{q}}_a$ and $\dot{\vec{q}}$ from equations (17) and (9), and the error definitions from (15) and (16), after lengthy algebra (see Section 6 of [20]) it can be proved that

$$\dot{q}_e=T(q_e)\vec{\omega}_e \quad (21)$$

In the $(q_e,\vec{\omega}_e,q_a,\vec{\omega}_a)$ coordinates, the system dynamics are

$$\begin{cases}\dot{q}_e&=\;T(q_e)\vec{\omega}_e\\J\dot{\vec{\omega}}_e&=\;-\vec{\omega}_e\times J\vec{\omega}_e+K_1\vec{q}_e+K_2\vec{\omega}_e+u_d\end{cases} \quad (22)$$

$$\begin{cases}\dot{q}_a&=\;T(q_a)\vec{\omega}_a\\JR(q_e)\dot{\vec{\omega}}_a&=\;F(\vec{\omega},\vec{\omega}_a)-\mathrm{Dz}_{\bar{u}}(u_s)+\Lambda_1\vec{q}_a+\Lambda_2\vec{\omega}_a\end{cases} \quad (23)$$

Equations (22)-(23) represent the dynamics of the *real, physical* system, but they have an alluring form: (22) has precisely the same form as (12) and, by the standing assumption and Assumption 1, behaves in a desirable manner and is such that $\lim_{t\to\infty}\vec{\omega}_e(t)=0$. The physical states of interest are q and $\vec{\omega}$ and, by equations (15)-(16), convergence to ideal behavior is guaranteed if they converge to the ideal states q_e and $\vec{\omega}_e$.

IV. MAIN RESULTS

A. Two useful Lemmas

Proofs of these Lemmas are provided in the Appendix.

Lemma 1: Consider the signals $u_1(t),u_2(t)\in\mathbb{R}^m$ and assume that $\mathrm{Dz}_{\bar{u}-\bar{v}}(u_1)\in\mathcal{L}_2$ where $\bar{u}$ and $\bar{v}$ are vectors with positive elements and are such that $\bar{u}_i-\bar{v}_i>0$ for all $i\in\{1,\dots,m\}$. Then

$$\mathrm{Dz}_{\bar{u}}(u_1+u_2)\in\mathcal{L}_2$$

if $|u_{2,i}(t)|<\bar{v}_i$ for all $i\in\{1,\dots,m\}$ and all $t\geq 0$.

Lemma 2: Consider two scalars $w_i,u_i\in\mathbb{R}$ and a positive scalar $\bar{u}_i$. Suppose $|w_i|<\bar{u}_i$. Then, with $\epsilon=\bar{u}_i-|w_i|>0$, the following inequality holds:

$$u_i[w_i-\mathrm{sat}_{\bar{u}_i}(w_i-u_i)]\geq\min\{\epsilon|u_i|,|u_i|^2\}\quad\forall u_i\in\mathbb{R} \quad (24)$$

B. Anti-windup Design

Theorem 1: Consider the closed-loop saturated system given by (9), (10), (13), (14), (17) and (18). Let Assumption 1 be satisfied and let the AW gains be chosen as

$$\Lambda_1=-\lambda_1R(q_e)\qquad\Lambda_2=-\tilde{\Lambda}_2R(q_e)$$

where $\tilde{\Lambda}_2>0$ is a diagonal matrix and $\lambda_1>0$ is a positive scalar. Let $u_{nom}=K_1\vec{q}_e+K_2\vec{\omega}_e$. Then the following properties hold:

1) If the initial AW states satisfy
$$\{q_{a0}(0),\vec{q}_a(0),\tilde{\omega}_a(0)\}=\{1,0_3,0_3\}$$
and $\mathrm{Dz}_{\bar{u}}(u_{nom})=0\quad\forall t\geq 0$ then $\vec{q}_a=0$ and $\vec{\omega}_a=0$ for all $t\geq 0$.
2) If $\mathrm{Dz}_{\bar{u}-\lambda_1\mathbf{1}_3}(u_{nom})\in\mathcal{L}_2$ then $\lim_{t\to\infty}\vec{q}_a(t)=0$ and $\lim_{t\to\infty}\vec{\omega}_a(t)=0$.

In the above theorem, u_{nom} represents the control signal associated with the system (22), which has the same form as the ideal unconstrained system (9)-(11); thus u_{nom} represents the nominal (ideal) control signal. Therefore, Item 1 of Theorem 1 ensures that if the ideal control signal does not saturate, then the AW compensator is never activated in the physical system. Item 2 then effectively ensures that, if the ideal control signal u_{nom} eventually falls to levels strictly within the saturation bounds ($|u_{nom,i}|\leq\bar{u}_i-\lambda_1$), then the AW states decay to zero. By equations (15)-(16), this implies that, as $t\to\infty$

$$\vec{\omega}\to\vec{\omega}_e\quad\text{and}\quad\vec{q}\to\vec{q}_e$$

and hence nominal behaviour will resume asymptotically in the physical system (see Figure 1 and also Section IV-D).

C. Proof of Theorem 1

The proof of Theorem 1 is given in several parts below.

1) Analysis of anti-windup dynamics: The system (9), (10), (13), (14), (17) and (18) can be re-written as equations (22)-(23), and, since system (22) represents the nominal system, the proof of convergence to nominal behavior purely involves an analysis of the AW dynamics in equation (23). It is more convenient to analyze the AW dynamics using the $\tilde{\omega}_a:=R(q_e)\vec{\omega}_a$ coordinates. Since $R(q_e)$ is an orthogonal matrix, the mapping $\vec{\omega}_a \mapsto \tilde{\omega}_a$ is bijective and hence convergence of $\tilde{\omega}_a$ implies convergence of $\vec{\omega}_a$. Thus, $\dot{\tilde{\omega}}_a$ is given by,

$$\dot{\tilde{\omega}}_a = R(q_e)\dot{\vec{\omega}}_a + R(q_e)[\vec{\omega}_e]^\times \vec{\omega}_a \tag{25}$$

Using equations (17) and (18), it follows that

$$\begin{aligned} J\dot{\tilde{\omega}}_a &= F(\vec{\omega}, \vec{\omega}_a) - \mathrm{Dz}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 \vec{\omega}_a + JR(q_e)[\vec{\omega}_e]^\times \vec{\omega}_a \\ &= -\vec{\omega} \times J\vec{\omega} + \vec{\omega}_e \times J\vec{\omega}_e - \mathrm{Dz}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a \end{aligned} \tag{26}$$

It is convenient to define

$$\tilde{F}(\vec{\omega}_e, \tilde{\omega}_a) := -\vec{\omega} \times J\vec{\omega} + \vec{\omega}_e \times J\vec{\omega}_e \tag{27}$$

$$= -\vec{\omega}_e \times J\tilde{\omega}_a - \tilde{\omega}_a \times J(\vec{\omega}_e + \tilde{\omega}_a) \tag{28}$$

The dynamics (23) can then equivalently be written as

$$\begin{cases} \dot{q}_a = T(q_e)R'(q_e)\tilde{\omega}_a \\ J\dot{\tilde{\omega}}_a = \tilde{F}(\vec{\omega}_e, \tilde{\omega}_a) - \mathrm{Dz}_{\bar{u}}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a \end{cases} \tag{29}$$

and the control signal (14) can be equivalently written as

$$u_s = K_1 \vec{q}_e + K_2 \vec{\omega}_e + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a \tag{30}$$

Subsequent analysis will use equations (29) and (30).

2) Proof of Item 1: The dynamics (29) are driven by the control signal u_s and some states of the ideal system $\vec{\omega}_e$. However, because the initial AW compensator state is

$$\{q_{a0}(0), \vec{q}_a(0), \tilde{\omega}_a(0)\} = \{1, 0_3, 0_3\}$$

which is an equilibrium point of the system (29), then $u_s(0) = u_{nom}(0)$. Therefore if $\mathrm{Dz}_{\bar{u}}(u_{nom}) = 0$ for all $t \geq 0$, then the states will not diverge from this equilibrium point.

3) Proof of Item 2 - Lyapunov function: Consider the following Lyapunov function, as used, for example, in [7]:

$$V = \frac{1}{2}\tilde{\omega}_a' J\tilde{\omega}_a + 2(1 - q_{a0})\eta \tag{31}$$

where $\eta > 0$ is a scalar to be determined. Note that $V > 0$ for all $(q_a, \tilde{\omega}_a) \in \mathbb{H} \times \mathbb{R}^3$. The time derivative is

$$\begin{aligned} \dot{V} &= \tilde{\omega}_a' \left[\tilde{F}(\vec{\omega}_e, \tilde{\omega}_a) - \mathrm{Dz}_{\bar{u}}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a\right] \\ &\quad + \eta \vec{q}_a{}' R'(q_e)\tilde{\omega}_a \\ &= \tilde{\omega}_a' \left[-\vec{\omega}_e \times J\tilde{\omega}_a - \tilde{\omega}_a \times J(\vec{\omega}_e + \tilde{\omega}_a)\right] \\ &\quad + \tilde{\omega}_a' \left[-\mathrm{Dz}_{\bar{u}}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a\right] + \eta \vec{q}_a{}' R'(q_e)\tilde{\omega}_a \\ &= -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a \\ &\quad + \tilde{\omega}_a' \left[-\mathrm{Dz}_{\bar{u}}(u_s) + \Lambda_1 \vec{q}_a + \Lambda_2 R'(q_e)\tilde{\omega}_a\right] + \eta \tilde{\omega}_a' R(q_e)\vec{q}_a \end{aligned} \tag{32}$$

Choosing $\Lambda_1 = -\lambda_1 R(q_e)$, and $\eta = \lambda_1$ gives

$$\begin{aligned} \dot{V} &= -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - \tilde{\omega}_a' (\mathrm{Dz}_{\bar{u}}(u_s) - \Lambda_2 R'(q_e)\tilde{\omega}_a) \\ &= -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - \tilde{\omega}_a' (u_s - \mathrm{sat}_{\bar{u}}(u_s) - \Lambda_2 R'(q_e)\tilde{\omega}_a) \end{aligned}$$

Recalling $u_{nom} = K_1 \vec{q}_e + K_2 \vec{\omega}_e$ and using (14), then gives

$$\dot{V} = -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - \tilde{\omega}_a' (u_{nom} - \lambda_1 R(q_e)\vec{q}_a - \mathrm{sat}_{\bar{u}}(u_s)) \tag{33}$$

Defining $\tilde{u} := u_{nom} - \lambda_1 R(q_e)\vec{q}_a$ this becomes

$$\dot{V} = -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - \tilde{\omega}_a' (\tilde{u} - \mathrm{sat}_{\bar{u}}(\tilde{u} + \Lambda_2 R'(q_e)\tilde{\omega}_a)) \tag{34}$$

Choosing $\Lambda_2 = -\tilde{\Lambda}_2 R(q_e)$, as stipulated, then gives

$$\begin{aligned} \dot{V} &= -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - \tilde{\omega}_a' (\tilde{u} - \mathrm{sat}_{\bar{u}}(\tilde{u} - \tilde{\Lambda}_2 \tilde{\omega}_a)) \\ &= -\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a - u_a' \tilde{\Lambda}_2^{-1} (\tilde{u} - \mathrm{sat}_{\bar{u}}(\tilde{u} - u_a)) \end{aligned} \tag{35}$$

where $u_a := \tilde{\Lambda}_2 \tilde{\omega}_a$. Noting that, for $\rho_1 = \lambda_{\max}(J)$,

$$-\tilde{\omega}_a' [\vec{\omega}_e]^\times J\tilde{\omega}_a \leq \rho_1 \|\vec{\omega}_e\| \, \|\tilde{\omega}_a\|^2 = \sum_{i=1}^{3} \rho_1 \|\vec{\omega}_e\| \, |\tilde{\omega}_{a,i}|^2$$

$\dot{V}$ can be bounded by,

$$\dot{V} \leq \sum_{i=1}^{3} \underbrace{\rho_1 \|\vec{\omega}_e\| |\tilde{\omega}_{a,i}|^2 - \tilde{\lambda}_{2,i}^{-1} u_{a,i} (\tilde{u}_i - \mathrm{sat}_{\bar{u}_i}(\tilde{u}_i - u_{a,i}))}_{W_i} \tag{36}$$

where $u_{a,i} = \lambda_{2,i} \tilde{\omega}_{a,i}$ and $\tilde{\Lambda}_2 = \mathrm{diag}(\tilde{\lambda}_{2,1}, \tilde{\lambda}_{2,2}, \tilde{\lambda}_{2,3})$. Therefore, $\dot{V}$ will be negative semi-definite if

$$W_i \leq -\alpha_i(|\tilde{\omega}_{a,i}|) \quad \forall \;\; \tilde{\omega}_{a,i} \neq 0 \quad \forall i \in \{1, 2, 3\} \tag{37}$$

where $\alpha_i(\cdot)$ are positive definite functions determined below.

4) Proof of Item 2 - Lemmas: Recall that $\tilde{u} = u_{nom} - \lambda_1 R(q_e)\vec{q}_a$ and observe that

$$|(\lambda_1 R(q_e)\vec{q}_a)_i| \leq \lambda_1 \|R(q_e)\| \|q_a\| \leq \lambda_1$$

By the statement of Item 2, $\mathrm{Dz}_{\bar{u} - \lambda_1 \mathbf{1}_3}(u_{nom}) \in \mathcal{L}_2$. Thus, by Lemma 1 it follows that

$$\mathrm{Dz}_{\bar{u}}(u_{nom} - \lambda_1 R(q_e)\vec{q}_a) = \mathrm{Dz}_{\bar{u}}(\tilde{u}) \in \mathcal{L}_2$$

Noting absolute continuity of $\tilde{u}$, this implies that there exists a finite time $t_1 > 0$ such that

$$|\tilde{u}_i| < \bar{u}_i \quad \forall t > t_1 \quad \forall i \in \{1, 2, 3\}$$

Therefore, applying Lemma 2 to W_i in equation (36) we have, for $t > t_1$,

$$\begin{aligned} W_i &\leq \rho_1 \|\vec{\omega}_e\| \, |\tilde{\omega}_{a,i}|^2 - \tilde{\lambda}_{2,i}^{-1} \min\left\{\epsilon_i |u_{a,i}|, |u_{a,i}|^2\right\} \\ &= \rho_1 \|\vec{\omega}_e\| \, |\tilde{\omega}_{a,i}|^2 - \min\left\{\epsilon_i |\tilde{\omega}_{a,i}|, \tilde{\lambda}_{2,i} |\tilde{\omega}_{a,i}|^2\right\} \end{aligned} \tag{38}$$

for some $\epsilon_i > 0$. Next, due to the structure of (22), Assumption 1 implies $\vec{\omega}_e$ asymptotically converges to zero, so there is a time $t_2 > t_1$ where $\|\vec{\omega}_e\| \leq \delta(t_2)$ for some arbitrarily small $\delta(t_2)$ (by choosing t_2 large enough). Hence, for all $t > t_2$ we have

$$W_i \leq \rho_1 \delta(t_2) |\tilde{\omega}_{a,i}|^2 - \min\left\{\epsilon_i |\tilde{\omega}_{a,i}|, \tilde{\lambda}_{2,i} |\tilde{\omega}_{a,i}|^2\right\} \tag{39}$$

Now for each i consider two cases:

A) : $\epsilon_i |\tilde{\omega}_{a,i}| < \tilde{\lambda}_{2,i} |\tilde{\omega}_{a,i}|^2$. This obviously implies

$$\epsilon_i < \tilde{\lambda}_{2,i} |\tilde{\omega}_{a,i}| \tag{40}$$

and, by equation (36), that W_i is bounded by

$$W_i \leq -|\tilde{\omega}_{a,i}| (\epsilon_i - \rho_1 \delta(t_2) |\tilde{\omega}_{a,i}|) \tag{41}$$

Thus $W_i < 0 \;\; \forall \tilde{\omega}_{a,i} \neq 0$ if $\epsilon_i > \rho_1 \delta(t_2) |\tilde{\omega}_{a,i}|$. Combining this with inequality (40), gives the inequality

$$\rho_1 \delta(t_2) |\tilde{\omega}_{a,i}| < \lambda_{2,i} |\tilde{\omega}_{a,i}\| \Rightarrow \rho_1 \delta(t_2) < \lambda_{2,i} \tag{42}$$

Note this inequality holds for sufficiently small $\delta(t_2)$.

B) : $\epsilon_i|\tilde{\omega}_{a,i}| > \tilde{\lambda}_{2,i}|\tilde{\omega}_{a,i}|^2$. In this case W_i is bounded by

$$W_i \leq \rho_1 \delta(t_2)|\tilde{\omega}_{a,i}|^2 - \tilde{\lambda}_{2,i}|\tilde{\omega}_{a,i}|^2$$

Thus $W_i < 0 \quad \forall \tilde{\omega}_{a,i} \neq 0$ under the same conditions as given in inequality (42).

Thus t_2 can always be chosen sufficiently large so that $\delta(t_2)$ is sufficiently small to satisfy inequality (42) for each $i = \{1, 2, 3\}$. This implies for all $t_2 > 0$

$$\dot{V} \leq -\sum_{i=1}^{3} \alpha_i(|\tilde{\omega}_{a,i}|) \leq -\alpha(\|\tilde{\omega}_a\|) \tag{43}$$

for some positive definite functions $\alpha_i(.)$ and $\alpha(.)$. Hence, $\dot{V}$ is negative semi-definite and, from La Salle's invariance principle, the state $(q_a, \tilde{\omega}_a)$ converges to the largest invariant set such that $\dot{V} = 0$, i.e. the state-converges to

$$\Omega := \left\{ q_a \in \mathbb{H}, \tilde{\omega}_a \in \mathbb{R}^3 \ : \ \dot{V} = 0 \right\} = \{ q_a \in \mathbb{H}, \tilde{\omega}_a = 0_3 \}$$

Using the dynamics (29), this implies that $\vec{q}_a \to 0$ (and hence $q_{a0} \to \pm 1$). Also, since $\tilde{\omega}_a = R(q_e)\vec{\omega}_a$, then $\tilde{\omega}_a \to 0$ implies convergence of $\vec{\omega}_a$ to zero. □□

D. Convergence to nominal behavior

Theorem 1 effectively shows that if the ideal control signal u_{nom} eventually falls to values within the saturation limits, the AW states $\vec{\omega}_a$ and $\vec{q}_a$ converge to zero, and hence the scalar quaternion q_{a0} converges ± 1. Expanding (16) gives

$$q_e = \begin{bmatrix} q_{e0} \\ \vec{q}_e \end{bmatrix} = \begin{bmatrix} q_0 q_{a0} + \vec{q}\,' \vec{q}_a \\ -q_0 \vec{q}_a + q_{a0}\vec{q} - \vec{q} \times \vec{q}_a \end{bmatrix}$$

and since $q_{a0} \to \pm 1$ and $\vec{q}_a \to 0$, this implies that $q \to \pm q_e$. Also, from (15), it is clear that $\vec{\omega} \to \vec{\omega}_e$ as $t \to \infty$. Therefore the system converges to nominal behaviour. This also implies that the control signal u_s will converge to the nominal control signal and thus, eventually, will leave the saturated regime.

E. Comments

Linearization. Linearizing the attitude dynamics of a rigid body yields double-integrator-type behavior (see e.g. [12]). Thus if one were to apply AW to the linearized system, the best one could achieve would be *global* $\mathcal{L}_2$ performance, *without finite* $\mathcal{L}_2$ *gain*. A similar result has been proved here without resorting to linearization. The main difference is that, here, convergence of the AW state requires

$$\mathrm{Dz}_{\bar{u}-\lambda_1 \mathbf{1}_3}(u_{nom}) \in \mathcal{L}_2 \tag{44}$$

where scalar $\lambda_1 > 0$ is chosen by the designer and must be such that $\bar{u}_i - \lambda_1 > 0$. This condition is *stricter* than in AW for linear systems ([19]) where $\lambda_1 \equiv 0$. As $\lambda_1 \to 0$, (44) approaches the standard condition, but λ_1 cannot be zero since Lyapunov function (31) will not be positive definite.

Tuning. The designer needs to chose four parameters: $\tilde{\Lambda}_2 = \mathrm{diag}(\tilde{\lambda}_{2,1}, \tilde{\lambda}_{2,2}, \tilde{\lambda}_{2,3})$ and $0 < \lambda_1 < \bar{u}_i$. It is less clear *how* to choose these parameters. Simulations have indicated a "sweet spot" is obtained with moderate values of all four gains, but further investigation is required.

V. SIMULATION RESULTS

The effectiveness of the AW compensator design is illustrated on a rigid body with $J = \mathrm{diag}(2,\ 1,\ 0.5)$ kg-m^2. Control signals u_s produce torques about the three principal axes, which are bounded by $\bar{u} = 0.25$. For the simulations, $K_1 = -2I_3$, $K_2 = -I_3$, $\lambda_1 = 0.1$, and $\tilde{\Lambda}_2 = I_3$, initial conditions of all states (both the rigid body and the AW compensator) are zero and a disturbance torque u_d is applied in the form of a square wave with period of 8 seconds, beginning at $t = 1$ second and lasts for 1.5 periods. Results presented are Euler angles $\theta = [\theta_1\ \theta_2\ \theta_3]'$ calculated from the quaternion vector q [23], the angular velocities $\vec{\omega} = [\omega_1\ \omega_2\ \omega_3]'$ and the control signals $u_s = [u_{s,1}\ u_{s,2}\ u_{s,3}]'$.

Fig. 2 illustrates the response to uniaxial disturbance $u_d = [2\ 0\ 0]'$. Significant improvement is achieved with AW, while the control signal without AW "bounces" between the saturation limits for almost 100 seconds.

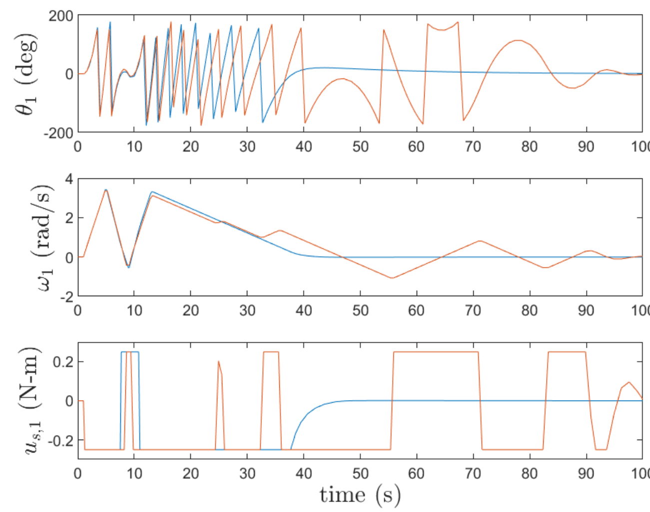

Fig. 2. Response to disturbance $\boldsymbol{u_d} = [\mathbf{2\ 0\ 0}]'$. —— With anti-windup; —— without anti-windup.

Fig. 3 illustrates the response to biaxial disturbance $u_d = [2\ 2\ 0]'$. The settling time for θ_1 and θ_3 (not shown) is similar with and without AW compensation. However, for this particular test case, θ_2 settles to the origin within 70 seconds without AW compensation, while the same response with AW compensation settles beyond the 100 second period presented. Despite this, the angular velocities $\vec{\omega}$ and the control signals u_s in all three axes settle much faster with AW compensation, e.g., ω_2 approaches zero in roughly half the time with AW compensation than without.

VI. CONCLUSIONS

This paper has considered the constrained attitude stabilization problem for rigid bodies using an AW approach. The AW approach ensures, effectively, global recovery of unconstrained behaviour. It is emphasized that this approach *places no restrictions on the nominal controller gains*: the AW compensator takes care of the saturation present. For brevity, the approach has been developed assuming a nominal PD controller, which is clearly restrictive. However, an extension to more general controllers is envisaged to be straightforward, with few technical complications.

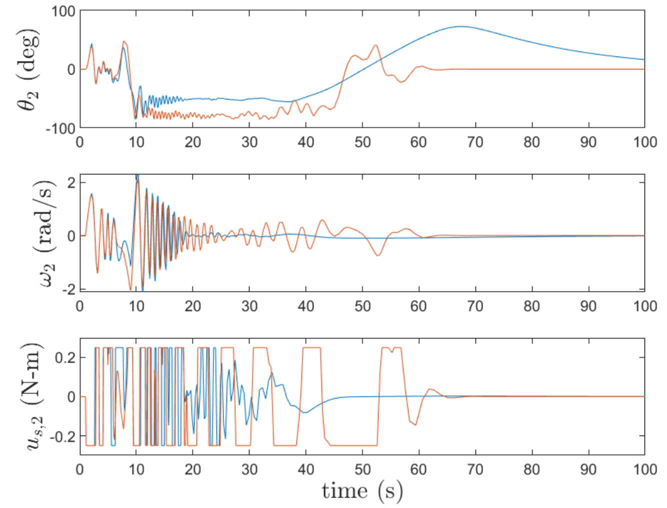

Fig. 3. Response to disturbance $\boldsymbol{u_d} = [\mathbf{2\ 2\ 0}]'$. —— With anti-windup; —— without anti-windup.

A weakness in the current approach is that the non-unique covering of $SO(3)$ by the quaternion representation has not been fully addressed, meaning the AW compensator may produce "stable" but "strange" behavior, due to the unwinding phenomenon [3]. A more careful treatment borrowing ideas from [7], [24] could be used to resolve this problem. This paper has concentrated on global $\mathcal{L}_2$ performance, but clearly would benefit from more specific assurances, such as (local) finite-gain bounds, or (local) decay rate estimates. These topics are the subject of continuing research.

APPENDIX

A. Proof of Lemma 1

$$\begin{aligned}\|\mathrm{Dz}_{\bar{u}}(u_1+u_2)\| &= \inf_{|w_i|<\bar{u}_i} \|u_1+u_2-w\| \\ &= \inf_{|w_i-u_{2i}+u_{2i}|<\bar{u}_i} \|u_1-(w-u_2)\| \\ &\le \inf_{|w_i-u_{2i}|<\bar{u}_i-|u_{2i}|} \|u_1-(w-u_2)\| \\ &\le \inf_{|w_i-u_{2i}|<\bar{u}_i-\psi_i} \|u_1-(w-u_2)\| \\ &= \inf_{|\tilde{w}_i|<\bar{u}_i-\psi_i} \|u_1-\tilde{w}\| = \|\mathrm{Dz}_{\bar{u}-\psi}(u_1)\|\end{aligned}$$

Now, because $x \in \mathcal{L}_2$ iff $\|x\| \in \mathcal{L}_2$ the result follows. □□

B. Proof of Lemma 2

This Lemma is similar to Lemma 1 in [25].

i) $|w_i - u_i| \le \bar{u}_i$. By direct calculation, it follows that

$$u_i[w_i - \mathrm{sat}_{\bar{u}_i}(w_i-u_i)] = |u_i|^2$$

ii) $w_i - u_i > \bar{u}_i$. In this case $\mathrm{sat}_{\bar{u}_i}(w_i-u_i) = \bar{u}_i$, so

$$u_i[w_i - \mathrm{sat}_{\bar{u}_i}(w_i-u_i)] = |u_i||w_i-\bar{u}_i|\mathrm{sign}(w_i-\bar{u}_i)\mathrm{sign}(u_i)$$

Now, since $w_i < \bar{u}_i$, it is clear that

$$\mathrm{sign}(w_i-\bar{u}_i) = -1, \quad w_i-\bar{u}_i > u_i$$

and since $|w_i| < \bar{u}_i$, this implies $\mathrm{sign}(u_i) = -1$. Hence,

$$\begin{aligned}u_i'[w_i - \mathrm{sat}_{\bar{u}_i}(w_i-u_i)] &= |u_i||w_i-\bar{u}_i| \\ &\ge |u_i|\,||w_i|-\bar{u}_i| \\ &= |u_i|(\bar{u}_i-|w_i|) =: |u_i|\epsilon\end{aligned}$$

iii) $w_i - u_i < -\bar{u}_i$. Symmetric to case ii). □□

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