IEEE 021 American Control Conference, 25-28 May 2021, New Orleans, LA
DOI:10.23919/ACC50511.2021.9483418

Nonlinear Optimal Missile Guidance for Stationary Target Interception with Pendulum Motion Perspective

Namhoon Cho

***Abstract*— This study outlines the set of equations constituting the necessary conditions that should be solved to determine the optimal guidance command for a missile to intercept a stationary target along a desired impact direction at a prespecified final time. Unlike the earlier studies on nonlinear optimal guidance problems, the present study formalises the optimal control problem with both final time and final state fixed. The pure control effort quadratic norm is considered as the performance index to be minimised. A noticeable finding from the study of the necessary conditions is that the flight path angle of the optimal trajectory obeys the simple pendulum dynamics. Full characterisation of the exact optimal solution requires numerically solving a set of four nonlinear algebraic equations with respect to four unknowns.**

I. Introduction

Optimal control theory is well-suited for many practical control system applications involving various constraints. The design of an optimal control law inherently gives attention to the evolution of a given dynamic system in an interval of time, as the system dynamics enters into the formulation as the dynamic constraint. The optimality of a control solution with respect to an objective function can be quantitatively assessed only if the state history for a time interval can be properly evaluated. This in turn leads to the understanding about the nature of optimal control that it entails internal prediction of the system dynamics. In this respect, the problems that require the state at a given final time to exactly reach a specific desired point or to closely approach within a region nearby the goal point can take great advantages of the optimal control theory, in comparison to the other broad categories of control design schemes.

The guidance of aerospace vehicles including missiles is one of many application areas that finds the predictive nature of optimal control theory useful in practice [1], [2]. A missile should hit the target accurately at the final time for a successful mission accomplishment while not demanding too much effort in steering, i.e., heading correction. The requirement of reducing steering effort is the point where the prediction enters into the problem. The amount of necessary heading correction is obtained by the deviation of the current heading angle from the *ideal* collision condition of each instance. Here, the ideal collision course is usually defined as the direction of flight on which the instantaneous *prediction* of the uncontrolled trajectory eventually intercepts the target. Therefore, many modern guidance algorithms have been developed on the basis of optimal control principle which essentially incorporates predictive behaviour.

However, the studies on the optimal missile guidance laws are mostly confined to the linear design and analysis in the domain where the near-collision-course assumptions are valid. The main reason for the limitation comes from the fact that, in general, the analytical solution for the predicted trajectories can be obtained in closed-form only for the linear dynamic systems. The assumptions such as the small deviation in the displacement from a reference line, the small angle approximation for the heading error, or else, are taken to linearise the engagement kinematics. Then, a majority of literature formalise the optimal control design problem into that of minimising a quadratic performance index subject to a linearised engagement kinematics, i.e., the Linear Quadratic Regulator (LQR) formulation [3]–[6]. In this way, a optimal feedback control law satisfying the terminal constraints can be derived easily, since the two-point boundary value problem arose from the first order necessary condition can be solved by using the closed-form equations for the state and costate trajectories.

As the efforts to overcome the limitations observed in the previous studies due to linearisation of system dynamics, the optimal design of new guidance laws or the optimality analysis of existing guidance laws based on nonlinear formulation have been conducted in several existing studies. An approach was to investigate the form and the behaviour of the exact optimal guidance solution by establishing the problem in the nonlinear setting with the exact planar engagement kinematic equations [7]–[10]. Although the final solutions for the problem formulations considered in each of [7]–[10] turn out to be at best the semi-analytical forms that require numerical computations to fully determine the optimal guidance command, the attempts made for the quantitative analyses of the nonlinear optimal guidance problems support having a more complete understandings about the exact optimal guidance trajectories. In [7], the semi-analytical form of the optimal guidance law for an interceptor pursuing a manoeuvring evader was derived considering full nonlinear engagement kinematics in the free final time formulation and without the impact angle constraint. The performance index to be minimised was a linear combination of the flight time and the control effort. The optimal guidance laws in the free final time nonlinear setting was studied in [9] with the pure control effort as the objective function for both cases with and without the impact angle constraint. The optimal solutions were compared with the proportional navigation guidance law. Most recently, [10] showed that the free final time minimum control nonlinear optimal guidance problem

Namhoon Cho is with the Centre for Autonomous and Cyber-Physical Systems, School of Aerospace, Transport and Manufacturing, Cranfield University, Cranfield, MK43 0AL, Bedfordshire, United Kingdom. e-mail: nhcho91@gmail.com

addressed in [9] is ill-posed. Also, [10] presented that the problem with the objective function similar to that of [7] eventually reduces to finding zeros of single real-valued function since the necessary conditions are shown to be parametrised with respect to a scalar.

In addition, the optimality of proportional navigation guidance for stationary target interception was studied without any linearisation in [11] considering a weighted control effort as the performance index. The result presented in [11] is basically an LQR solution, because the original nonlinear engagement kinematics is found to be transformable into a linear one by taking the range as the independent variable and the term with close relation to the zero-effort-miss as the state variable.

In summary, the nonlinear optimal guidance problems were studied in an effort to close the approximation gap in the linearisation-based result from the true optimal solution for better accuracy of the resultant optimal control solution.

This study presents a preliminary attempt made in the formulation and the derivation of necessary conditions for the optimal guidance of a missile intercepting a stationary target interception subject to the nonlinear engagement kinematics, in an effort to extend analytical understandings on the exact optimal solution. Unlike the earlier studies on the nonlinear optimal guidance problems of which formulations considered free final time without impact angle constraint, the present study derives the necessary conditions for the optimal guidance law considering both the final time and also the final flight path angle as fixed values.

The remainder of the paper is organised as follows: Section II formally describes the missile guidance problem with the constraints under consideration. In Sec. III, the Lagrange multiplier method is applied to derive the necessary conditions of the optimal control problem for the three-dimensional case with vector formulation. The necessary conditions are more detailed for the planar case in Sec. IV by showing that the flight path angle and thus each component of the velocity vector behaves like a pendulum. The optimal guidance law and the state solutions are derived in terms of the elliptic integrals which require further numerical procedures to be fully determined. Section V summarises the concluding remarks.

II. Problem Formulation

Let us consider a missile with a constant speed V, which starts to fly from a given initial point with a given initial velocity. There is a stationary target located at a distinct point in the space, and it has a weakness along a certain direction. It is usually desired to let the missile to hit the target along that particular direction of weakness, and making the time of the collision controllable will lead to additional tactical benefits.

Let p_i, p_f be the initial position of the missile and that of the target, respectively, and let v_i, v_f be the initial and the desired final velocity of the missile, respectively. Let T be the desired time of the collision. The aim is to find a "good" candidate for the function $a(t)$ representing the acceleration of the missile at the given time instance t, with the following constraints:

1) *Initial/final velocity constraint*:

$$\int_0^T a(t)\,dt = v_f - v_i,$$

2) *Initial/final position constraint*:

$$\int_0^T \left(\int_0^t a(s)\,ds + v_i \right) dt = p_f - p_i,$$

3) *Constant speed constraint*:

$$\left\langle a(t), \int_0^t a(s)\,ds + v_i \right\rangle = 0$$

for all $t \in [0, T]$.

There are many possibilities in precisely defining the meaning of a "good" control function. As in the usual minimum effort missile guidance problems, this study considers having a small $\mathcal{L}_2$-norm as a suitable way of interpretation. Now, the problem can be formalised as a constrained optimization problem with the above constraints and with the objective given by $J = \int_0^T \|a(t)\|^2\,dt$.

III. Derivation of Necessary Condition: Three-Dimensional Vector Formulation

The problem under consideration can be solved by using the Lagrange multiplier method. However, there are infinitely many constraints, since for each $t \in [0, T]$ there is a corresponding constant speed constraint. Therefore, it is natural to consider a form of Lagrange multiplier method in a Banach space. Precise and rigorous mathematical setting may require some intricacies. This study avoids a very serious approach to the mathematical completeness while putting focus on the formulaic structure of the solution. However, to apply a Banach space version of Lagrange multiplier method, it will be helpful to sketch some ideas about the space where the "multiplier" belongs. Since $a(t)$ is the second derivative of the path, it is natural to consider it as an element in the Hilbert space $H^{-2}([0, T])$. Then the anti-derivative $v(t) := \int_0^t a(s)\,ds + v_i$ belongs to $H^{-1}([0, T])$ and the function $t \mapsto \langle a(t), v(t)\rangle$ belongs to $W^{-1,1}([0, T])$. Thus, the corresponding multiplier is thought to be Lipschitz over $[0, T]$. Keeping this in mind, the Lagrangian can be defined as follows:

$$\begin{aligned}
\mathcal{L}(a) := &\int_0^T \|a(t)\|^2\,dt \\
&+ \int_0^T \lambda_s(t) \left\langle a(t), \int_0^t a(s)\,ds + v_i \right\rangle dt \\
&+ \left\langle \lambda_v, \int_0^T a(t)\,dt + v_i - v_f \right\rangle \\
&+ \left\langle \lambda_p, \int_0^T \left(\int_0^t a(s)\,ds + v_i \right) dt + p_i - p_f \right\rangle
\end{aligned} \tag{1}$$

where $\lambda_s(t)$ is Lipschitz over $[0,T]$. Define for each $t \in [0,T]$,

$$I(a;t) := \|a(t)\|^2 + \lambda_s(t)\left\langle a(t), \int_0^t a(s)\,ds + v_i \right\rangle + \langle \lambda_v, a(t)\rangle + \left\langle \lambda_p, \int_0^t a(s)\,ds \right\rangle \tag{2}$$

then

$$\mathcal{L}(a) = \int_0^T I(a;t)\,dt - \langle \lambda_v, v_f - v_i\rangle - \langle \lambda_p, p_f - p_i - v_i T\rangle .$$

By the Banach space version of the Lagrange multiplier method, every directional derivative of $\mathcal{L}(a)$ at the optimal point is zero. To compute the derivative, assume that $a(t)$ is a critical point, $h \in \mathbb{R}$, and φ is a smooth function on $[0,T]$, then

$$\begin{aligned}
&\left.\frac{\partial I(a+h\varphi;t)}{\partial h}\right|_{h=0} \\
=&2\langle a(t), \varphi(t)\rangle + \lambda_s(t)\left\langle \varphi(t), \int_0^t a(s)\,ds + v_i\right\rangle \\
&+ \lambda_s(t)\left\langle a(t), \int_0^t \varphi(s)\,ds\right\rangle \\
&+ \langle \lambda_v, \varphi(t)\rangle + \left\langle \lambda_p, \int_0^t \varphi(s)\,ds\right\rangle \\
=&\left\langle \varphi(t), 2a(t) + \lambda_s(t)\left(\int_0^t a(s)\,ds + v_i\right) + \lambda_v\right\rangle \\
&+ \left\langle \int_0^t \varphi(s)\,ds, \lambda_s(t)a(t) + \lambda_p\right\rangle,
\end{aligned}$$

so consequently,

$$\begin{aligned}
&\left.\frac{\partial \mathcal{L}(a+h\varphi)}{\partial h}\right|_{h=0} \\
&= \int_0^T \left\langle \varphi(t), 2a(t) + \lambda_s(t)\left(\int_0^t a(s)\,ds + v_i\right) + \lambda_v\right\rangle dt \\
&+ \int_0^T \int_0^t \langle \varphi(s), \lambda_s(t)a(t) + \lambda_p\rangle\, ds\,dt = 0.
\end{aligned} \tag{3}$$

Applying Fubini's theorem, the second term can be rewritten as

$$\begin{aligned}
&\int_0^T \int_s^T \langle \varphi(s), \lambda_s(t)a(t) + \lambda_p\rangle\, dt\,ds \\
&= \int_0^T \int_t^T \langle \varphi(t), \lambda_s(s)a(s) + \lambda_p\rangle\, ds\,dt \\
&= \int_0^T \left\langle \varphi(t), \int_t^T (\lambda_s(s)a(s) + \lambda_p)\,ds\right\rangle dt,
\end{aligned}$$

thus

$$\int_0^T \left\langle \varphi(t), 2a(t) + \lambda_s(t)\left(\int_0^t a(s)\,ds + v_i\right) + \lambda_v + \int_t^T (\lambda_s(s)a(s) + \lambda_p)\,ds\right\rangle dt = 0. \tag{4}$$

Since smooth functions form a dense subspace, it follows that

$$2a(t) + \lambda_s(t)\left(\int_0^t a(s)\,ds + v_i\right) + \int_t^T (\lambda_s(s)a(s) + \lambda_p)\,ds = 0. \tag{5}$$

Differentiating the both sides with t, we get

$$2v''(t) + \lambda_s'(t)v(t) + \lambda_s(t)v'(t) - (\lambda_s(t)v'(t) + \lambda_p) = 0, \tag{6}$$

which leads to

$$2v''(t) + \lambda_s'(t)v(t) = \lambda_p. \tag{7}$$

Since $a(t)$ is orthogonal to $v(t)$, we get

$$\langle \lambda_p, a(t)\rangle = 2\langle v''(t), a(t)\rangle = \frac{d\|a(t)\|^2}{dt}, \tag{8}$$

Integrating both sides gives

$$\|a(t)\|^2 - \|a(0)\|^2 = \langle \lambda_p, v(t) - v_i\rangle . \tag{9}$$

Now, since $\langle a(t), v(t)\rangle = 0$, we know that

$$\frac{d\langle a(t), v(t)\rangle}{dt} = \langle v''(t), v(t)\rangle + \|a(t)\|^2 = 0, \tag{10}$$

so it follows that

$$\begin{aligned}
\langle v''(t), v(t)\rangle &= -\|a(t)\|^2 \\
&= -\langle \lambda_p, v(t)\rangle + \langle \lambda_p, v_i\rangle - \|a(0)\|^2 .
\end{aligned} \tag{11}$$

From (7), we know

$$2\langle v''(t), v(t)\rangle + \lambda_s'(t)\|v(t)\|^2 = \langle \lambda_p, v(t)\rangle , \tag{12}$$

thus

$$\begin{aligned}
\lambda_s'(t) &= \frac{\langle \lambda_p, v(t)\rangle}{V^2} - \frac{2\langle v''(t), v(t)\rangle}{V^2} \\
&= \frac{3\langle \lambda_p, v(t)\rangle}{V^2} + \frac{2(\|a(0)\|^2 - \langle \lambda_p, v_i\rangle)}{V^2}.
\end{aligned} \tag{13}$$

Therefore, we arrive at the following second-order nonlinear differential equation:

$$2v''(t) + \left(\frac{3\langle \lambda_p, v(t)\rangle}{V^2} + \frac{2(\|a(0)\|^2 - \langle \lambda_p, v_i\rangle)}{V^2}\right)v(t) = \lambda_p. \tag{14}$$

IV. Two-Dimensional Case

Now, assume that the missile path lies only on a 2-dimensional plane. Figure 1 shows the problem geometry. The variables depicted in Fig. 1 will be explained below.

Because the missile flies with a constant speed, its velocity can be written as

$$v(t) = V(\cos\theta(t), \sin\theta(t)). \tag{15}$$

where θ denotes the flight path angle as defined in Fig. 1. The acceleration can be represented as

$$a(t) = V\theta'(t)(-\sin\theta(t), \cos\theta(t)), \tag{16}$$

so applying it to (8) gives

$$\begin{aligned} \frac{d}{dt}\left(V^2(\theta'(t))^2\right) &= 2V^2\theta'(t)\theta''(t) \\ &= \langle \lambda_p, a(t) \rangle \\ &= V\theta'(t)(\lambda_{py}\cos\theta(t) - \lambda_{px}\sin\theta(t)). \end{aligned} \tag{17}$$

Write $\lambda_p = \|\lambda_p\|(\cos\alpha, \sin\alpha)$, then

$$\begin{aligned} \theta''(t) &= \frac{\|\lambda_p\|}{2V}(\sin\alpha\cos\theta(t) - \cos\alpha\sin\theta(t)) \\ &= -\frac{\|\lambda_p\|}{2V}\sin(\theta(t)-\alpha). \end{aligned} \tag{18}$$

Define $\varphi(t) := \theta(t) - \alpha$ and $\varepsilon := \frac{\|\lambda_p\|}{V}$ so that

$$\varphi''(t) = -\frac{\varepsilon}{2}\sin\varphi(t). \tag{19}$$

The governing equation above is the differential equation for simple pendulum. Interpreting V as the length of the pendulum, we find that the vector λ_p corresponds to the (doubled) gravity vector. Figure 2 illustrates this relation visually as an analogy between the nonlinear optimal guidance and the pendulum motion in the velocity space.

Multiplying $\varphi'(t)$ to the both sides, we get

$$\varphi'(t)\varphi''(t) = \frac{1}{2}\frac{d(\varphi'(t))^2}{dt} = -\frac{\varepsilon}{2}\varphi'(t)\sin\varphi(t),$$

Integrating both sides gives

$$\frac{1}{2}(\varphi'(t))^2 - \frac{1}{2}(\varphi'(0))^2 = \frac{\varepsilon}{2}(\cos\varphi(t) - \cos\varphi(0)). \tag{20}$$

Now, define

$$E := \frac{1}{2}(\varphi'(0))^2 + \frac{\varepsilon}{2}(1-\cos\varphi(0))$$

so that

$$\frac{1}{2}(\varphi'(t))^2 + \frac{\varepsilon}{2}(1-\cos\varphi(t)) = E.$$

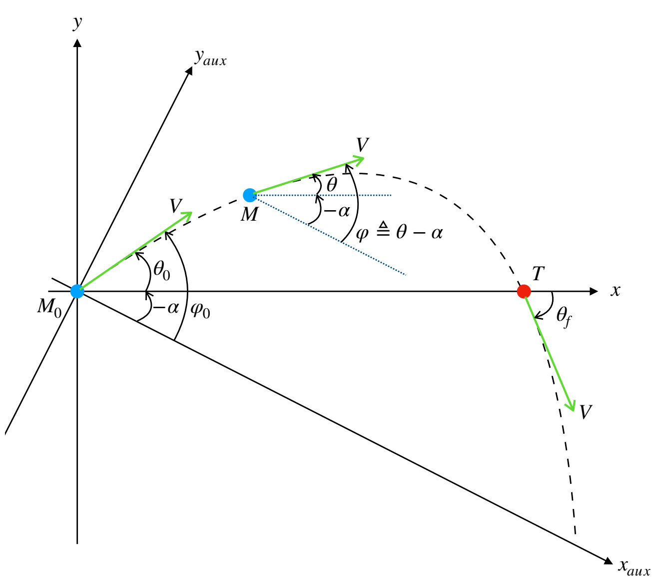

Fig. 1. Guidance Problem Geometry

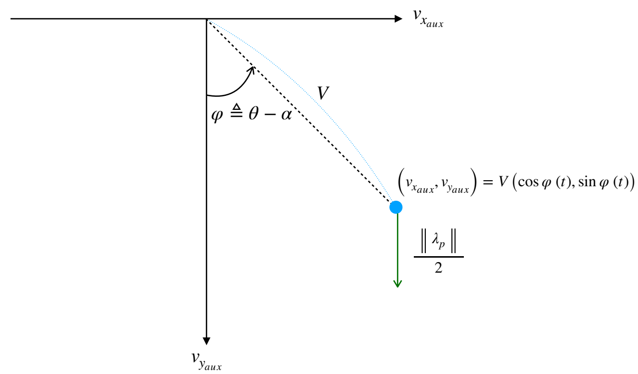

Fig. 2. Pendulum in Velocity Space

Here, E is the total energy of the pendulum, and ε is the potential energy of the pendulum when the pendulum is totally inverted. The solution behaviour for the case $\varepsilon \geq E$, in which the pendulum stops at certain height and then swings back, and the case $\varepsilon \leq E$, in which the pendulum swings over and over are different. For both cases, the solution to the differential equation can be written in terms of the *Jacobi elliptic functions*.

A. *Case 1 ($\varepsilon \geq E$)*

Note that

$$2E - \varepsilon(1-\cos\varphi(t)) = (\varphi'(t))^2 \geq 0$$

for all t, so

$$\frac{2E}{\varepsilon} \geq 1 - \cos\varphi(t), \quad \cos\varphi(t) \geq 1 - \frac{2E}{\varepsilon},$$

therefor $\varphi(t)$ should be in the range $[-\varphi_{\max}, \varphi_{\max}]$ where $\varphi_{\max} := \cos^{-1}\left(1 - \frac{2E}{\varepsilon}\right) \leq \pi$ so that $E = \frac{\varepsilon}{2}(1-\cos\varphi_{\max})$. Now, the equation becomes

$$(\varphi'(t))^2 = \varepsilon(\cos\varphi(t) - \cos\varphi_{\max}). \tag{21}$$

The general form of the solutions is

$$\varphi(t) = \pm 2\sin^{-1}\left(\sqrt{\frac{E}{\varepsilon}}\,\mathrm{sn}\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)\right) \tag{22}$$

with the elliptic modulus $k = \sqrt{\frac{E}{\varepsilon}}$, where τ is a constant. Note that the Jacobi elliptic functions are defined with respect to the inversion of the elliptic integral of the first kind

$$u = f(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2\sin^2 t}} \tag{23}$$

through the following relations.

$$\begin{aligned} \mathrm{sn}(u;k) &= \sin\phi \\ \mathrm{cn}(u;k) &= \cos\phi \\ \mathrm{dn}(u;k) &= \sqrt{1-k^2\sin^2\phi} \end{aligned} \tag{24}$$

From the formula $\mathrm{sn}' = \mathrm{cn}\cdot\mathrm{dn}$, we get

$$\varphi'(t) = \pm\frac{2}{\sqrt{1-\frac{E}{\varepsilon}\,\mathrm{sn}^2\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)}}\sqrt{\frac{E}{2}}\times \\ \mathrm{cn}\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)\mathrm{dn}\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right), \tag{25}$$

Using the identities $\mathrm{sn}^2+\mathrm{cn}^2=1$ and $\mathrm{dn}^2+k^2\mathrm{sn}^2=1$, we confirm again that

$$\begin{aligned}(\varphi'(t))^2 &= 2E\,\mathrm{cn}^2\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)\\ &= 2E\left(1-\mathrm{sn}^2\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)\right)\\ &= 2E\left(1-\frac{\varepsilon}{E}\sin^2\frac{\varphi(t)}{2}\right)\\ &= \varepsilon\left(\cos\varphi(t)+\frac{2E}{\varepsilon}-1\right)\\ &= \varepsilon(\cos\varphi(t)-\cos\varphi_{\max}).\end{aligned} \tag{26}$$

Now, there are four constants to determine; α, E, ε, and τ, and four equations we have; $\theta(0)$, $\theta(T)$, $\int_0^T\cos\theta(t)\,dt$, and $\int_0^T\sin\theta(t)\,dt$. Note that the boundary conditions that should be solved for the unknown constants can be summarised as follows:

$$\begin{aligned}\theta(0) &= \mathrm{atan2}(v_{iy}, v_{ix}),\\ \theta(T) &= \mathrm{atan2}(v_{fy}, v_{fx}),\\ \int_0^T\cos\theta(t)\,dt &= \frac{1}{V}(p_{fx}-p_{ix}-v_{ix}T),\\ \int_0^T\sin\theta(t)\,dt &= \frac{1}{V}(p_{fy}-p_{iy}-v_{iy}T),\end{aligned} \tag{27}$$

where $v_i=(v_{ix},v_{iy})$, $v_f=(v_{fx},v_{fy})$, $p_i=(p_{ix},p_{iy})$, and $p_f=(p_{fx},p_{fy})$.

The integrations $\int_0^T\cos\theta(t)\,dt$ and $\int_0^T\sin\theta(t)\,dt$ can be written in terms of $\int_0^T\cos\varphi(t)\,dt$ and $\int_0^T\sin\varphi(t)\,dt$, which are then related to elliptic integral of second kind. First, let us compute $\int_0^T\cos\varphi(t)\,dt$. First, $\cos\varphi(t)$ can be represented as

$$\cos\varphi(t) = 1-2\sin^2\frac{\varphi(t)}{2} = 1-\frac{2E}{\varepsilon}\,\mathrm{sn}^2\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right),$$

and therefore

$$\begin{aligned}\int_0^T\cos\varphi(t)\,dt &= T-\frac{2E}{\varepsilon}\int_0^T\mathrm{sn}^2\left(\sqrt{\frac{\varepsilon}{2}}(t+\tau)\right)\,dt\\ &= T-\frac{2E}{\varepsilon}\sqrt{\frac{2}{\varepsilon}}\int_{\sqrt{\frac{\varepsilon}{2}}\tau}^{\sqrt{\frac{\varepsilon}{2}}(T+\tau)}\mathrm{sn}^2u\,du.\end{aligned} \tag{28}$$

Using the identity

$$\int_0^{\sin^{-1}(\mathrm{sn}u)}\sqrt{1-k^2\sin^2\varphi}\,d\varphi = u-k^2\int_0^u\mathrm{sn}^2w\,dw,$$

we can write $\int_0^T\cos\varphi(t)\,dt$ in terms of an incomplete elliptic integral of the second kind:

$$\begin{aligned}&\int_0^T\cos\varphi(t)\,dt\\ &= T-\frac{2E}{\varepsilon}\sqrt{\frac{2}{\varepsilon}}\cdot\frac{\varepsilon}{E}\int_{\sin^{-1}\left(\mathrm{sn}\left(\sqrt{\frac{\varepsilon}{2}}(T+\tau)\right)\right)}^{\sin^{-1}\left(\mathrm{sn}\left(\sqrt{\frac{\varepsilon}{2}}\tau\right)\right)}\sqrt{1-\frac{E}{\varepsilon}\sin^2\varphi}\,d\varphi\\ &= T+2\sqrt{\frac{2}{\varepsilon}}\int_{\sin^{-1}\left(\mathrm{sn}\left(\sqrt{\frac{\varepsilon}{2}}\tau\right)\right)}^{\sin^{-1}\left(\mathrm{sn}\left(\sqrt{\frac{\varepsilon}{2}}(T+\tau)\right)\right)}\sqrt{1-\frac{E}{\varepsilon}\sin^2\varphi}\,d\varphi.\end{aligned} \tag{29}$$

The integration $\int_0^T\sin\varphi(t)\,dt$ is simpler;

$$\sin\varphi(t) = -\frac{2}{\varepsilon}\varphi''(t),$$

therefore

$$\begin{aligned}&\int_0^T\sin\varphi(t)\,dt\\ &= -\frac{2}{\varepsilon}(\varphi'(T)-\varphi'(0))\\ &= \pm\frac{2\sqrt{2E}}{\varepsilon}\left(\mathrm{cn}\left(\sqrt{\frac{\varepsilon}{2}}\tau\right)-\mathrm{cn}\left(\sqrt{\frac{\varepsilon}{2}}(T+\tau)\right)\right).\end{aligned} \tag{30}$$

However, it is intractable to solve these equations to get the constants α, E, ε, and τ analytically.

B. Case 2 ($\varepsilon \le E$)

In this case, the general form of the solutions is

$$\varphi(t) = \pm 2\sin^{-1}\left(\mathrm{sn}\left(\sqrt{\frac{E}{2}}(t+\tau)\right)\right) \tag{31}$$

with the elliptic modulus $k=\sqrt{\frac{\varepsilon}{E}}$, where τ is a constant. Note that

$$\varphi'(t) = \pm\frac{2}{\sqrt{1-\mathrm{sn}^2\left(\sqrt{\frac{E}{2}}(t+\tau)\right)}}\sqrt{\frac{E}{2}}\times \\ \mathrm{cn}\left(\sqrt{\frac{E}{2}}(t+\tau)\right)\mathrm{dn}\left(\sqrt{\frac{E}{2}}(t+\tau)\right), \tag{32}$$

Using the identities $\mathrm{sn}^2+\mathrm{cn}^2=1$ and $\mathrm{dn}^2+k^2\mathrm{sn}^2=1$, we verify that

$$\begin{aligned}(\varphi'(t))^2 &= 2E\,\mathrm{dn}^2\left(\sqrt{\frac{E}{2}}(t+\tau)\right)\\ &= 2E\left(1-\frac{\varepsilon}{E}\mathrm{sn}^2\left(\sqrt{\frac{E}{2}}(t+\tau)\right)\right)\\ &= 2E\left(1-\frac{\varepsilon}{E}\sin^2\frac{\varphi(t)}{2}\right)\\ &= 2E-\varepsilon(1-\cos\varphi(t)).\end{aligned} \tag{33}$$

Again, the algebraic equations should be solved for the unknown constants.

V. Conclusions

The necessary conditions that should be solved to determine the optimal guidance command for a missile to intercept a stationary target at a desired final time with a desired impact angle was developed in this study. An analogy between the velocity components obeying the necessary conditions and the simple pendulum motion was found in the planar engagement case. The equations for the optimal input and the state were written in terms fo elliptic integrals by utilising the pendulum analogue. It was shown that the closed-form guidance law in the nonlinear setting was not available and that the optimal guidance law could be obtained by numerically solving a determined system of nonlinear algebraic equations consisting of four nonlinear equations with four unknown constants. Further works are required to develop an efficient numerical solution strategy by seeking for the possibilities of problem reduction. It is conjectured that the necessary conditions can be parametrised with a fewer number of normalised scalar variables by cleverly exploiting the periodicity and the symmetry of the pendulum solution, facilitating the numerical root-finding to search for a single solution.

References

[1] J. Z. Ben-Asher, *Optimal Control Theory with Aerospace Applications*. Reston, VA: American Institute of Aeronautics and Astronautics, 2010.

[2] J. M. Longuski, J. J. Guzmán, and J. E. Prussing, *Optimal Control with Aerospace Applications*. New York, NY: Springer-Verlag, 2014.

[3] A. Bryson, "Linear Feedback Solutions for Minimum Effort Interceptions, Rendezvous and Soft Landing," *AIAA Journal*, vol. 3, no. 8, pp. 1542–1544, 1965.

[4] R. G. Cottrell, "Optimal Intercept Guidance for Short-Range Tactical Missiles," *AIAA Journal*, vol. 9, no. 7, pp. 1414–1415, 1971.

[5] E. Kreindler, "Optimality of Proportional Navigation," *AIAA Journal*, vol. 11, no. 6, pp. 878–880, 1973.

[6] G. M. Anderson, "Comparison of Optimal Control and Differential Game Intercept Missile Guidance Laws," *Journal of Guidance, Control, and Dynamics*, vol. 4, no. 2, pp. 109–115, 1981.

[7] M. Guelman and J. Shinar, "Optimal Guidance Law in the Plane," *Journal of Guidance, Control, and Dynamics*, vol. 7, no. 4, pp. 471–476, 1984.

[8] ——, "Errata: Optimal Guidance Law in the Plane," *Journal of Guidance, Control, and Dynamics*, vol. 8, no. 1, p. 160, 1985.

[9] P. Lu and F. R. Chavez, "Nonlinear Optimal Guidance," in *AIAA Guidance, Navigation, and Control Conference*, Keystone, CO, August 2006.

[10] Z. Chen and T. Shima, "Nonlinear Optimal Guidance for Intercepting a Stationary Target," *Journal of Guidance, Control, and Dynamics*, vol. 42, no. 11, pp. 2418–2431, 2019.

[11] I.-S. Jeon and J.-I. Lee, "Optimality of Proportional Navigation Based on Nonlinear Formulation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 46, no. 4, pp. 2051–2055, 2010.

2021-07-28

Nonlinear optimal missile guidance for stationary target interception with pendulum motion perspective

Cho, Namhoon

IEEE