Navigation with Probabilistic Safety Constraints: Convex Formulation

Joseph Moyalan, Yongxin Chen, and Umesh Vaidya

Abstract— We consider the problem of navigation with safety constraints. The safety constraints are probabilistic, where a given set is assigned a degree of safety, a number between zero and one, with zero being safe and one being unsafe. The deterministic unsafe set will arise as a particular case of the proposed probabilistic description of safety. We provide a convex formulation to the navigation problem with probabilistic safety constraints. The convex formulation is made possible by lifting the navigation problem in the dual space of density using linear transfer operator theory methods involving Perron-Frobenius and Koopman operators. The convex formulation leads to an infinite-dimensional feasibility problem for probabilistic safety verification and control design. The finite-dimensional approximation of the optimization problem relies on the data-driven approximation of the linear transfer operator.

I. INTRODUCTION

The obstacle avoidance problem is one of the necessities for robots and unmanned vehicles operating in a structured or unstructured environment. The classical approach to obstacles modeling and avoidance is deterministic in nature [1]–[3]. However, increasingly there is a shift in focus for obstacle modeling and avoidance to a probabilistic or stochastic setting. This shift in focus is mainly driven by navigation applications in unstructured or off-road terrain where environmental uncertainty is a norm.

The authors of [4] investigate the probabilistic collision checking between uncertain configurations for two objects, which is referred to as collision chance constraints. They provide a probability of collision that accounts for both robot and obstacle uncertainty. In [5], an investigation of risk metrics, i.e., mapping from a random variable corresponding to costs to a real number, for different probabilistic applications is done. The authors of [5] provide various axioms that help in the selection of risk metrics which is utilized in the cost function. However, finding such risk metrics for everyday applications is, in general, not trivial. In [6], the authors tackle the problem of autonomous traverse in an unknown environment by efficient risk and constraintaware kinodynamic motion planning using sequential quadratic programming-based (SQP) model predictive control (MPC). They propose a tail risk assessment using the Conditional Value-at-Risk (CVaR). However, the problem is formulated in a nonlinear setting with highly non-convex constraints on the robot's motion. In [7], a new probabilistic navigation function (PNF) has been proposed to work in a stochastic environment. The proposed PNF provides an analytic proof for convergence and considers both the geometries and the location probability functions without inflating the ambient space dimension. But so far, all the formulations involving navigation with probabilistic descriptions of the obstacle set are formulated as non-convex problems. This non-convex nature of the problem is one of the main challenges in developing systematic analysis and synthesis methods.

In this paper, we provide a probabilistic description of safety using a probability measure. The probability measure is used to assign a varying degree of safety for any given set based on the value of the probability measure on the set. Larger the value of probability measure on a set more unsafe the set. Furthermore, the deterministic description of an unsafe set will arise as a particular case of the proposed probabilistic description. We consider the problem of navigation with probabilistic safety constraints. The main contribution is in providing convex formulation to the verification and control synthesis for the navigation problem. The convex formulation is made possible by lifting the navigation problem in the space of density using methods from linear operator theory involving Perron-Frobenius and Koopman operators [8]. This convex formulation builds on our past work on providing convex formulation to navigation, stabilization, and optimal control problems [2], [9]–[11]. The convex formulation leads

J. Moyalan and U. Vaidya are with the Department of Mechanical Enginerring, Clemson University, Clemson SC, USA, jmoyala@clemson.edu, uvaidya@clemson.edu

Y. Chen is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA, USA, yongchen@gatech.edu

to an infinite-dimensional convex feasibility problem for the probabilistic safety verification and synthesis problems. The infinite-dimensional feasibility problem is approximated using a data-driven method developed for the finite-dimensional approximation of the linear transfer operator.

The rest of the paper is structured as follows. In Section II, we provide a brief introduction to our framework's necessary preliminaries. Section III consists of a probabilistic description of unsafe sets, and the main theoretical results are described in section IV. In Section V, we develop the computational framework based on the linear operator framework. This is followed by simulation examples in Section VI and a conclusion in Section VII.

II. PRELIMINARIES AND NOTATIONS

Notations: \mathbb{R}^n denotes the *n* dimensional Euclidean space and $\mathbb{R}^n_{\geq 0}$ is the positive orthant. Given $\mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{Y} \subseteq \mathbb{R}^{m}$, let $\mathcal{L}_1(\mathbf{X}, \mathbf{Y}), \mathcal{L}_{\infty}(\mathbf{X}, \mathbf{Y})$, and $\mathcal{C}^k(\mathbf{X}, \mathbf{Y})$ denote the space of all real valued integrable functions, essentially bounded functions, and space of k times continuously differentiable functions mapping from X to Y respectively. If the space Y is not specified then it is understood that the underlying space is \mathbb{R} . $\mathcal{B}(\mathbf{X})$ denotes the Borel σ -algebra on X and $\mathcal{M}(\mathbf{X})$ is the vector space of real-valued measure on $\mathcal{B}(\mathbf{X})$. $\mathbf{s}_t(\mathbf{x})$ denotes the solution of dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ starting from initial condition x. X_0 , X_r are assumed to be the initial set and final terminal set respectively. With no loss of generality we will assume that $\mathbf{X}_r = \{0\}$. Let \mathcal{N}_{δ} be the neighborhood of \mathbf{X}_r for some fixed $\delta > 0$ i.e., $\mathbf{X}_r \subset \mathcal{N}_{\delta}$. Let $\mathbf{X}_1 := \mathbf{X} \setminus \mathcal{N}_{\delta}$. Let $h_0(\mathbf{x})$ be the probability density function corresponding to the probability measure μ_0 with support on set \mathbf{X}_0 .

A. Perron-Frobenius and Koopman Operator

Consider a dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n.$$
 (1)

where the vector field is assumed to be $\mathbf{F}(\mathbf{x}) \in \mathcal{C}^1(\mathbf{X}, \mathbb{R}^n)$. There are two different ways of lifting the finite dimensional nonlinear dynamics from state space to infinite dimension space of functions namely using Koopman and Perron-Frobenius operators. The definitions of these operators along with the infinitesimal generators of these operators are defined as follows [8].

Definition 1 (Koopman Operator): $\mathbb{U}_t : \mathcal{L}_{\infty}(\mathbf{X}) \to \mathcal{L}_{\infty}(\mathbf{X})$ for dynamical system (1) is defined as

$$[\mathbb{U}_t \varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x})). \tag{2}$$

The infinitesimal generator for the Koopman operator is given by

$$\lim_{t \to 0} \frac{(\mathbb{U}_t - I)\varphi}{t} = \mathbf{F}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) =: \mathcal{K}_{\mathbf{F}}\varphi, \ t \ge 0.$$
(3)
Definition 2 (Perron-Frobenius Operator):

 $\mathbb{P}_t : \mathcal{L}_1(\mathbf{X}) \to \mathcal{L}_1(\mathbf{X})$ for dynamical system (1) is defined as

$$\left[\mathbb{P}_{t}\psi\right](\mathbf{x}) = \psi(\mathbf{s}_{-t}(\mathbf{x})) \left|\frac{\partial \mathbf{s}_{-t}(\mathbf{x})}{\partial \mathbf{x}}\right|,\tag{4}$$

where $|\cdot|$ stands for the determinant. The infinitesimal generator for the P-F operator is given by

$$\lim_{t \to 0} \frac{(\mathbb{P}_t - I)\psi}{t} = -\nabla \cdot (\mathbf{F}(\mathbf{x})\psi(\mathbf{x}))$$
$$=: \mathcal{P}_{\mathbf{F}}\psi, \ t > 0.$$
(5)

These two operators are dual to each other where the duality is expressed as follows.

$$\int_{\mathbf{X}} [\mathbb{U}_t \varphi](\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{X}} [\mathbb{P}_t \psi](\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}.$$

Property 1: These two operators enjoy positivity and Markov properties which are used in the finite dimension approximation of these operators.

1) Positivity: The P-F and Koopman operators are positive operators i.e., for any $0 \leq \varphi(\mathbf{x}) \in \mathcal{L}_{\infty}(\mathbf{X})$ and $0 \leq \psi(\mathbf{x}) \in \mathcal{L}_{1}(\mathbf{X})$, we have

$$[\mathbb{P}_t \psi](\mathbf{x}) \ge 0, \quad [\mathbb{U}_t \varphi](\mathbf{x}) \ge 0, \quad \forall t \ge 0.$$
 (6)

 Markov Property: The P-F operator satisfies Markov property i.e.,

$$\int_{\mathbf{X}} [\mathbb{P}_t \psi](\mathbf{x}) d\mathbf{x} = \int_{\mathbf{X}} \psi(\mathbf{x}) d\mathbf{x}.$$
 (7)

Assumption 1: We assume that $\mathbf{x} = 0$ is locally stable equilibrium point for the system (1) with local domain of attraction denoted by \mathcal{N}_{δ} for a fixed $\delta > 0$. We let $\mathbf{X}_1 := \mathbf{X} \setminus \mathcal{N}_{\delta}$

Definition 3 (Almost everywhere (a.e.) uniform stability): The equilibrium point is said to be a.e. uniform stable w.r.t. measure $\mu_0 \in \mathcal{M}(\mathbf{X})$ if for any given ϵ , there exists a time $T(\epsilon)$ such that

$$\int_{T(\epsilon)}^{\infty} \mu_0(B_t) dt < \epsilon, \tag{8}$$

where $B_t := \{ \mathbf{x} \in \mathbf{X}_1 : \mathbf{s}_t(\mathbf{x}) \in B \}$ for every set $B \in \mathcal{B}(\mathbf{X}_1)$.

The following theorem is from [12, Theorem 13] providing necessary and sufficient condition for a.e. uniform stability.

Theorem 1: The equilibrium point $\mathbf{x} = 0$ for system (1) satisfying Assumption 1 is a.e. uniformly stable w.r.t. measure μ_0 if and only if there exists a function $\rho(\mathbf{x}) \in C^1(\mathbf{X} \setminus \{0\}, \mathbb{R}_{\geq 0}) \cap \mathcal{L}_1(\mathbf{X}_1)$ and satisfies

$$\nabla \cdot (\mathbf{F}\rho) = h_0. \tag{9}$$

where $h_0 \in \mathcal{L}_1(\mathbf{X}, \mathbb{R}_{\geq 0})$ is assumed to be the density function corresponding to the measure μ_0 .

B. Data-Driven Approximation: Naturally Structured Dynamic Mode Decomposition

Naturally structured dynamic mode decomposition (NSDMD) is a modification of Extended Dynamic Mode Decomposition (EDMD) algorithm [13], one of the popular algorithms for Koopman approximation from data. The modifications are introduced to incorporate the natural properties of these operators namely positivity and Markov. For the continuous-time dynamical system (1), consider snapshots of data set obtained as time-series data from single or multiple trajectories

$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M], \quad \mathcal{Y} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_M],$$
(10)

where $\mathbf{x}_i \in \mathbf{X}$ and $\mathbf{y}_i \in \mathbf{X}$. The pair of data sets are assumed to be two consecutive snapshots i.e., $\mathbf{y}_i = \mathbf{s}_{\Delta t}(\mathbf{x}_i)$, where $\mathbf{s}_{\Delta t}$ is solution of (1). Let $\mathbf{\Psi} = [\psi_1, \dots, \psi_N]^{\top}$ be the choice of basis functions. The popular Extended Dynamic Mode Decomposition (EDMD) algorithm provides the finite-dimensional approximation of the Koopman operator as the solution of the following least square problem.

$$\min_{\mathbf{K}} \| \mathbf{G}\mathbf{K} - \mathbf{A} \|_F, \tag{11}$$

where,

$$\mathbf{G} = \frac{1}{M} \sum_{m=1}^{M} \boldsymbol{\Psi}(\mathbf{x}_m) \boldsymbol{\Psi}(\mathbf{x}_m)^{\top}, \qquad (12)$$

$$\mathbf{A} = \frac{1}{M} \sum_{m=1}^{M} \boldsymbol{\Psi}(\mathbf{x}_m) \boldsymbol{\Psi}(\mathbf{y}_m)^{\top}, \qquad (13)$$

with $\mathbf{K}, \mathbf{G}, \mathbf{A} \in \mathbb{R}^{N \times N}$, $\|\cdot\|_F$ stands for Frobenius norm. The above least square problem admits an analytical solution

$$\mathbf{K}_{EDMD} = \mathbf{G}^{\dagger} \mathbf{A}.$$
 (14)

Convergence results for EDMD algorithms in the limit as the number of data points and basis functions go to infinity are provided in [14], [15]. In this paper, we work with Gaussian Radial Basis Function (RBF) for the finite-dimensional approximation of the linear operators. Under the assumption that the basis functions are positive, like the Gaussian RBF, the NSDMD algorithm propose following convex optimization problem for the approximation of the Koopman operator that preserves positivity and Markov property in Property 1.

$$\min_{\hat{\mathbf{P}}} \| \hat{\mathbf{G}} \hat{\mathbf{P}} - \hat{\mathbf{A}} \|_F$$
(15)
s.t. $[\hat{\mathbf{P}}]_{ij} \ge 0, \quad \hat{\mathbf{P}} \mathbb{1} = \mathbb{1},$

where,

$$\hat{\mathbf{G}} = \mathbf{G} \mathbf{\Lambda}^{-1}, \ \hat{\mathbf{A}} = \mathbf{A} \mathbf{\Lambda}^{-1}, \ \& \ \mathbf{\Lambda} = \int_{\mathbf{X}} \mathbf{\Psi} \mathbf{\Psi}^{\top} d\mathbf{x}, (16)$$

with G and A are as defined in 12 and 1 is a vector of all ones. All the matrices in Eq. (16) are pre-computed from the data. In fact, since the basis functions are assumed to be Gaussian RBF, the constant Λ matrix can be computed explicitly as

$$\Lambda_{i,j} = (\frac{\pi\sigma^2}{2})^{n/2} \exp^{\frac{-\|\mathbf{c}_i - \mathbf{c}_j\|^2}{2\sigma^2}}, i, j = 1, 2, \dots, N,$$

where $\mathbf{c}_i, \mathbf{c}_j$ are the centers of the ψ_i and ψ_j Gaussian RBFs respectively. The constraints in (15) ensure that finite-dimensional approximation preserves the positivity property and Markov property respectively. The approximation for the P-F operator and its generator are obtained as the solution of the optimization problem (15) as

$$\mathbb{P}_{\Delta t} \approx \hat{\mathbf{P}}^{\top} =: \mathbf{P}, \qquad \mathcal{P}_{\mathbf{F}} \approx \frac{\hat{\mathbf{P}}^{\top} - \mathbf{I}}{\Delta t} =: \mathbf{M}.$$
 (17)

III. PROBABILISTIC DESCRIPTION OF UNSAFE SET

In this paper, we consider a probabilistic description of the unsafe set defined as follows. Let $p(\mathbf{x})$ is the probability density function describing the unsafe set and $\mu_p \in \mathcal{M}(\mathbf{X})$ is the associated probability measure i.e., $d\mu_p(\mathbf{x}) = p(\mathbf{x})d\mathbf{x}$. Let $A \in \mathcal{B}(\mathbf{X})$, then the probability that the set A is unsafe is defined using $p(\mathbf{x})$ as

$$\operatorname{Prob}(A \text{ is Unsafe}) = \int_{A} p(\mathbf{x}) d\mathbf{x} =: \mu_p(A). \quad (18)$$

The above definition include the deterministic description of unsafe set as the special case. In particular, if \mathbf{X}_u is an unsafe set then it can described using the following definition of the uniform probability density function

$$p(\mathbf{x}) = \frac{1}{m(\mathbf{X}_u)} \mathbb{1}_{\mathbf{X}_u}(\mathbf{x})$$
(19)

where $m(\cdot)$ is the Lebesgue measure and $\mathbb{1}_{\mathbf{X}_u}$ is the indicator function of the set \mathbf{X}_u . So by allowing $p(\mathbf{x})$ to be nonuniform probability density function we can consider cases where different regions of the state space \mathbf{X} have varying degree of safety. Note that when $p(\mathbf{x})$ is supported on the entire set \mathbf{X} , then the entire region of the state space \mathbf{X} is potentially hazardous; however, the degree of hazard is determined by the probability density function $p(\mathbf{x})$. In particular, if

$$\mu_p(A_1) < \mu_p(A_2)$$

where $A_i \in \mathcal{B}(\mathbf{X})$, then the region A_2 is more hazardous than region A_1 . Strictly speaking, in the above described probabilistic setting, it may not be appropriate to use the terminology of a safe or unsafe set as it has a connotation of carrying binary information. Consider the case when $p(\mathbf{x})$ is supported on the entire space \mathbf{X} , then potentially the entire space \mathbf{X} is hazardous. However, with nonuniform probability density $p(\mathbf{x})$, some regions of \mathbf{X} are more favorable to navigate and hence potentially less hazardous than others. The objective then is to navigate through regions of \mathbf{X} where μ_p is small. With some abuse of terminology, we will continue to refer to the sets in \mathbf{X} as safe and unsafe where the degree of safety is characterized by $p(\mathbf{x})$. We defined the probability of collision as follows.

Definition 4: The probability of collision with the unsafe set under the system dynamics (1) with the initial condition distributed w.r.t. probability measure μ_0 is given by

$$\int_0^\infty \int_{\mathbf{X}} p(\mathbf{s}_t(\mathbf{x})) d\mu_0(\mathbf{x}) dt$$
 (20)

The objective is to minimize the above collision probability while navigating the system dynamics from some initial set \mathbf{X}_0 , supported on measure μ_0 to some final terminal set \mathbf{X}_r asymptotically.

The reason for using the formula (20) in the definition of collision probability stems from the following Lemma. The following Lemma essentially points to the fact that the formula (20) is a measure of occupancy in the unsafe region, and this connection is evident in the case of a deterministic unsafe set.

Lemma 1: Let
$$p(\mathbf{x}) = \frac{1}{m(\mathbf{X}_u)} \mathbb{1}_{\mathbf{X}_u}(\mathbf{x})$$
. If
$$\int_0^\infty \int_{\mathbf{X}} p(\mathbf{s}_t(\mathbf{x})) d\mu_0(\mathbf{x}) dt = 0$$
(21)

then

$$\int_{\mathbf{X}_1} \mathbb{1}_{\mathbf{X}_u}(\mathbf{s}_t(\mathbf{x})) h_0(\mathbf{x}) d\mathbf{x} = 0, \quad \forall t \ge 0.$$
(22)

i.e., the amount of time system trajectories spend in the unsafe set \mathbf{X}_u starting from the positive measure set of initial condition corresponding to the initial set, \mathbf{X}_0 , with density $h_0(\mathbf{x})$ is equal to zero.

Proof: Proof by contradiction. Assume (22) is not true, i.e., there exists some time t_0 for which

$$\int_{\mathbf{X}_1} \mathbb{1}_{\mathbf{X}_u}(\mathbf{s}_{t_0}(\mathbf{x})) h_0 d\mathbf{x} = \int_{\mathbf{X}_1} [\mathbb{U}_{t_0} \mathbb{1}_{\mathbf{X}_u}](\mathbf{x}) h_0 d\mathbf{x} > 0$$

Then using the continuity property of the Koopman semi-group, we know there exists a Δ such that

$$\int_{t_0}^{t_0+\Delta} \int_{\mathbf{X}_1} [\mathbb{U}_{t_0} \mathbb{1}_{\mathbf{X}_u}]((\mathbf{x})) h_0(\mathbf{x}) d\mathbf{x} dt > 0.$$

We have

$$0 < \int_{t_0}^{t_0+\Delta} \int_{\mathbf{X}_1} [\mathbb{U}_{t_0} \mathbb{1}_{\mathbf{X}_u}](\mathbf{x}) h_0(\mathbf{x}) d\mathbf{x} dt \le \\ \int_0^\infty \int_{\mathbf{X}_1} [\mathbb{U}_{t_0} \mathbb{1}_{\mathbf{X}_u}](\mathbf{x}) h_0(\mathbf{x}) d\mathbf{x} dt \\ = \int_0^\infty \int_{\mathbf{X}} p(\mathbf{s}_t(\mathbf{x})) d\mu_0(\mathbf{x}) dt = 0.$$

So by minimizing the quantity in (20) for a general non-uniform probability density function $p(\mathbf{x})$, we are essentially trying to minimize the occupancy in the unsafe set.

IV. MAIN RESULTS

The main results of this paper are presented in the following two subsections and provide for a convex formulation to the probabilistic safety verification and control synthesis problems.

A. Probabilistic Safety Verification

In this section, we show that the probabilistic safety navigation problem can be verified convexly. In particular, the stochastic safety navigation problem can be written as convex optimization problem. The stochastic safety navigation problem can be stated as follows.

Definition 5 (Stochastic Safety Navigation Problem): The problem consists of navigating almost every (a.e.) (w.r.t. Lebesgue measure) trajectories of the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{23}$$

starting from some initial set, X_0 , to some final set X_r asymptotically, while ensuring that the probability of collision with the obstacle set should be less than or equal to γ i.e.,

$$\int_0^\infty \int_{\mathbf{X}_1} p(\mathbf{s}_t(\mathbf{x})) d\mu_0(\mathbf{x}) dt \le \gamma.$$
 (24)

The main results of this section provides convex formulation to the stochastic safety verification problem as defined in Definition 5. Before that we make following assumption on the system dynamics (23).

Assumption 2: We assume that the final destination set is locally stable with local domain of attraction N_{δ} for a fixed $\delta > 0$.

Theorem 2: Under Assumption 2, the stochastic safety navigation problem (Definition 5) can be verified convexly if there exists a density function $\rho(\mathbf{x}) \in \mathcal{L}_1(\mathbf{X}_1) \cap \mathcal{C}^1(\mathbf{X}_1, \mathbb{R}_{>0})$ and satisfies

$$\nabla \cdot (\mathbf{F}(\mathbf{x})\rho(\mathbf{x})) = h_0(\mathbf{x}), \text{ a.e. } \mathbf{x} \in \mathbf{X}_1$$
 (25)

$$\int_{\mathbf{X}_{+}} p(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} \le \gamma \tag{26}$$

Proof: Following the results of Theorem 1, since $\rho(\mathbf{x})$ satisfies (25) it follows that the $\mathbf{X}_r = \{0\}$ is a.e. uniform stable w.r.t. measure $d\mu_0(\mathbf{x}) = h_0(\mathbf{x})d\mathbf{x}$ i.e., a.e. initial condition from set \mathbf{X}_0 supported on measure μ_0 will be attracted to the final set \mathbf{X}_r . Furthermore, the density function $\rho(\mathbf{x})$ that satisfies (25) can be expressed using the following integral formula

$$\rho(\mathbf{x}) = \int_0^\infty [\mathbb{P}_t h_0](\mathbf{x}) dt \tag{27}$$

We next show that (26) implies (24) i.e., the probability of collision is less than or equal to γ . We have,

$$\int_0^\infty \int_{\mathbf{X}_1} p(\mathbf{s}_t(\mathbf{x})) d\mu_0(\mathbf{x}) dt = \int_0^\infty \int_{\mathbf{X}_1} [\mathbb{U}_t p](\mathbf{x}) d\mu_0(\mathbf{x}) dt$$

Now using duality between the Koopman and P-F operator and the fact that $\frac{d\mu_0}{d\mathbf{x}} = h_0$, we obtain

$$\int_0^\infty \int_{\mathbf{X}_1} [\mathbb{U}_t p](\mathbf{x}) d\mu_0(\mathbf{x}) dt = \int_0^\infty \int_{\mathbf{X}_1} p(\mathbf{x}) [\mathbb{P}_t h_0](\mathbf{x}) d\mathbf{x} dt$$

Exchanging the spatial and temporal integration and using the integral formula for the $\rho(\mathbf{x})$ in (27) we obtain

$$\int_0^\infty \int_{\mathbf{X}_1} p(\mathbf{x}) [\mathbb{P}_t h_0](\mathbf{x}) d\mathbf{x} dt = \int_{\mathbf{X}_1} p(\mathbf{x}) \rho(\mathbf{x}) d\mathbf{x}$$

B. Convex Approach to Control Synthesis with Stochastic Safety Constraints

In this section, we formulate control synthesis for stochastic safety navigation problem as a convex optimization problem. The convex control synthesis is possible under the assumption that control system is affine in input. Consider a control affine dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{28}$$

where $\mathbf{u} \in \mathbb{R}$ is the control input. For the simplicity of presentation we restrict the discussion to the case of single input. It is straight forward to extend this framework to multi-input case. The objective is to design the control input so that almost all trajectories of the system (28) starting from the initial set \mathbf{X}_0 to the final set \mathbf{X}_r minimizes the probability of collision with the obstacle set to less than equal to γ (Eq. (24)) Following assumption is made on the control synthesis problem for safety navigation.

Assumption 3: We assume that the control for stochastic safety navigation is feedback in nature i.e., $u = k(\mathbf{x}) \in C^1(\mathbf{X})$ and that the final destination set \mathbf{X}_r is locally stable with local domain of attraction \mathcal{N}_{δ} for a fixed $\delta > 0$.

Following is the main results of this section.

Theorem 3: Under Assumption 3, the control synthesis problem for the stochastic safety navigation with norm constraints on the control input of the form, $|u| \leq M$, can be formulated as following convex feasibility problem in terms of variable $\rho \in \mathcal{L}_1(\mathbf{X}_1) \cap \mathcal{C}^1(\mathbf{X}_1, \mathbb{R}_{>0})$ and $\bar{\rho} \in \mathcal{C}^1(\mathbf{X}_1, \mathbb{R})$.

$$\nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\rho}) = h_0, \text{ a.e. } \mathbf{x} \in \mathbf{X}_1$$
 (29)

$$\int_{\mathbf{X}_1} p(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} \le \gamma \tag{30}$$

$$|\bar{\rho}(\mathbf{x})| \le M\rho(\mathbf{x}) \tag{31}$$

The control input is given by

$$u = k(\mathbf{x}) = \frac{\bar{\rho}(\mathbf{x})}{\rho(\mathbf{x})}$$
(32)

Proof: The proof of this Theorem follows along similar lines to the proof of Theorem 2 applied to feedback control system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})k(\mathbf{x})$. This gives us

$$\nabla \cdot \left((\mathbf{f} + \mathbf{g}k)\rho \right) = h_0$$

The above equation is bilinear in the two unknowns ρ and k and can be converted to linear equation in terms of new unknown variables and ρ and $\bar{\rho}$ defined as $\bar{\rho}(\mathbf{x}) := k(\mathbf{x})\rho(\mathbf{x})$. After the change of variables from $(\rho, k) \to (\rho, \bar{\rho})$, we obtain (29). The derivation of (30) follows along the same lines of corresponding derivation in Theorem 2. The condition (31) ensures that the formula for the control input in (32) is well defined and that the constraints on the control input i.e., $|u| \leq M$ are satisfied.

V. COMPUTATIONAL FRAMEWORK

For the finite dimensional approximation of the infinite dimensional convex feasibility problems (26) and (31), we need to construct the approximation of the generator corresponding to vector field f and g i.e., $\nabla \cdot (\mathbf{f}\rho)$ and $\nabla \cdot (\mathbf{g}\bar{\rho})$. Following notations from (5) and (17), let the approximations of the two generators corresponding to vector fields \mathbf{f} and \mathbf{g} be denoted by

$$\mathcal{P}_{\mathbf{f}} \approx \mathbf{M}_0, \quad \mathcal{P}_{\mathbf{g}} \approx \mathbf{M}_1$$
 (33)

We use NSDMD algorithm outlined in Section II-B for this approximation with $\Psi(\mathbf{x}) = (\psi_1, \dots, \psi_N)^{\top}$ as the basis functions. Let $h_0(\mathbf{x})$, $p(\mathbf{x})$, $\rho(\mathbf{x})$, and $\bar{\rho}(\mathbf{x})$ be expressed in terms of the basis function

$$h_0(\mathbf{x}) = \mathbf{\Psi}^\top \mathbf{m}, \ \rho(\mathbf{x}) \approx \mathbf{\Psi}^\top \mathbf{v}, \ \bar{\rho}(\mathbf{x}) \approx \mathbf{\Psi}^\top \mathbf{w}$$
$$p(\mathbf{x}) \approx \mathbf{\Psi}^\top \mathbf{u}. \tag{34}$$

Using (34), we can also write

$$\int_{\mathbf{X}_{1}} p(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} \approx \mathbf{u}^{\top} \left[\int_{\mathbf{X}_{1}} \Psi(\mathbf{x})\Psi^{\top}(\mathbf{x})d\mathbf{x} \right] \mathbf{v} \quad (35)$$
$$= \mathbf{u}^{\top} \mathbf{D} \mathbf{v} \quad (36)$$

where $\mathbf{D} := \int_{\mathbf{X}_1} \boldsymbol{\Psi}(\mathbf{x}) \boldsymbol{\Psi}^{\top}(\mathbf{x}) d\mathbf{x}$.

Assumption 4: We assume that all the basis functions are positive and are linearly independent.

Remark 4: In this paper, we use Gaussian RBF to obtain all the simulation results i.e., $\psi_k(\mathbf{x}) =$ $\exp^{-\frac{\|\mathbf{x}-\mathbf{c}_k\|^2}{2\sigma^2}}$ where \mathbf{c}_k is the center of the k^{th} Gaussian RBF.

Using Assumption 4, the finite dimensional approximation of the convex synthesis problem from Theorem 3 can be written as

$$-\mathbf{M}_0\mathbf{v} - \mathbf{M}_1\mathbf{w} = \mathbf{m} \tag{37}$$

$$\mathbf{u}^{\top}\mathbf{D}\mathbf{v} \leq \gamma$$
 (38)

$$\mathbf{v} \ge 0, \quad |\mathbf{w}| \le M \mathbf{v} \tag{39}$$

where the inequalities are assumed to the element wise. The feedback control input is then obtained using the following formula

$$u = k(\mathbf{x}) = \frac{\boldsymbol{\Psi}^{\top}(\mathbf{x})\mathbf{w}}{\boldsymbol{\Psi}^{\top}(\mathbf{x})\mathbf{v}}$$
(40)

VI. SIMULATION RESULTS

All the simulation results in this paper are obtained using Gaussian RBF. The value of γ is taken to be 0.1 for all the examples. The following rules of thumb are abided in selecting centers and σ parameters for the Gaussian RBF. The RBF centers are chosen to be uniformly distributed in the state space at a distance of d. The σ for the Gaussian RBF is chosen such that $d \leq 3\sigma \leq 1.5d$. All the simulation results are

performed using MATLAB on a desktop computer with 16GB RAM. The optimization problem is solved using CVX. We present simulation results involving two examples and in both the examples we design local controller around a small neighbourhood of \mathbf{X}_r to stabilize the equilibrium point. Let us consider this small neighbourhood to be \mathbf{X}_L which is defined as follows:

$$\mathbf{X}_L = \{ \mathbf{x} \in \mathbf{X} : |\mathbf{x} - \mathbf{X}_r| \le \epsilon, \epsilon > 0 \}$$

Here, the value of $\epsilon = 0.002$. We design an optimal linear quadratic regulator (LQR) controller for \mathbf{X}_L from the linearized dynamics of the model to guarantee the convergence in \mathbf{X}_L .

Example 1. We consider the problem of navigating a boat on a river [16]. The dynamics of the model is given by:

$$\dot{x_1} = 1 + 0.125 \cos(0.5x_1) - 0.125 \sin(0.5x_2); \quad \dot{x_2} = u;$$

We define the sets to be:

- $\mathbf{X} \triangleq \{\mathbf{x} \in \mathbb{R}^2 : 0 \le x_1 \le 10, 0 \le x_2 \le 10\}$
- $\mathbf{X}_0 \triangleq \{\mathbf{x} \in \mathbf{X} : 0 \le x_1 \le 1, 5 \le x_2 \le 7\}$ $\mathbf{X}_u \triangleq \{\mathbf{x} \in \mathbf{X} : (2\sin(x_1) + 6 x_2)(x_2 2\sin(x_1) 6 x_2)(x_2 2\sin(x_1)) \le 1\}$ $(4) \le 0 \ or \ x_1 \le 0$
- $\mathbf{X}_r \triangleq [9.7, 3.8]^T$.

We have used 2500 radial basis functions as lifting functions and $\Delta t = 0.01$. Fig. 1 displays the open and closed loop trajectories navigating the river dynamics. We can see from this plot that the controller obtained using the finite-dimensional approximation of the feasibility problem (37)-(39) is able to avoid unsafe set successfully. Fig. 2 displays the color plots of the density function $\rho(\mathbf{x})$ which clearly displays the differences in the density values between safe and unsafe sets. In particular, the value of the density function on the unsafe set is equal to zero.

Example 2. Consider the following controlled dynam-

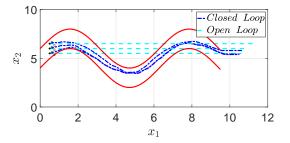


Fig. 1: Open and Closed loop trajectories navigating the river model prescribed in Example 1.

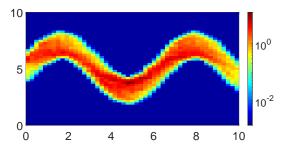


Fig. 2: Density function obtained from convex optimization in Example 1.

ics of nonlinear system:

$$\dot{x_1} = -0.125 + 0.125\cos(0.5x_1) - 0.125\sin(0.5x_2)$$

 $\dot{x_2} = u$

For this example, we choose

- $\mathbf{X} \triangleq \{\mathbf{x} \in \mathbb{R}^2 : -8 \le x_1 \le 8, -8 \le x_2 \le 8\}$
- $\mathbf{X}_0 \triangleq \{\mathbf{x} \in \mathbf{X} : 5.3 \le x_1 \le 6.7, -0.7 \le x_2 \le 0.7\}$
- $\mathbf{X}_{u1} \triangleq \{\mathbf{x} \in \mathbf{X} : 2 \le x_1 \le 0.1, 0.1 \le x_2 \le 0\}$ $\mathbf{X}_{u1} \triangleq \{\mathbf{x} \in \mathbf{X} : 2 \le x_1 \le 4, -3 \le x_2 \le 0\}$ $\mathbf{X}_{u2} \triangleq \{\mathbf{x} \in \mathbf{X} : 2 \le x_1 \le 4, 0 \le x_2 \le 3\}$ $\mathbf{X}_{u3} \triangleq \{\mathbf{x} \in \mathbf{X} : -1 \le x_1 \le 7, -3 \le x_2 \le 3\}$

•
$$\mathbf{X}_r \triangleq [0,0]^T$$

 X_{u3} represents the outer deterministic obstacle set where the value of $p(\mathbf{x}) = 1$ everywhere. It confines our feasible region to a rectangular block. X_{u1} and X_{u2} represents inner obstacle sets with probability measure $\mu_p(\mathbf{X}_{u1})$ and $\mu_p(\mathbf{X}_{u2})$ respectively. They divides our rectangular feasible set into two smaller regions as shown in Fig. 3. We design the problem such that trajectories starting from \mathbf{X}_0 finds it's way to \mathbf{X}_r by passing through either X_{u1} or X_{u2} depending upon their probability measure. $\Delta t = 0.01$. The probability measure of X_{u1} and X_{u2} are varied to show their effect on the trajectories as can be seen in Fig. 4-8. The values of density function $\rho(\mathbf{x})$ obtained from the convex optimization problem varies with respect to the probability measure of unsafe sets which can be observed in Fig. 5-9. It is observed that values of $\rho(\mathbf{x})$ in the obstacle sets \mathbf{X}_{u1} and \mathbf{X}_{u2} increases when there is a decrease in the probability density function of the obstacle set and vice-versa. This is to be expected from the fact that $\rho(\mathbf{x})$ is an occupancy measure and obstacle set with high probability density function will have fewer trajectories entering into them.

VII. CONCLUSIONS

We consider the problem of navigation with probabilistic safety constraints. We provide convex formulation to the verification and control synthesis for navigation with probabilistic safety constraints leading to a

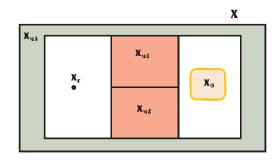


Fig. 3: Schematic diagram of unsafe sets in Example 2.

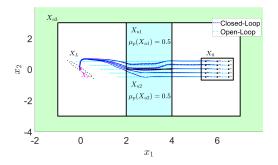


Fig. 4: State trajectories of Example 2 avoiding unsafe sets X_{u1} and X_{u2} both with probability measure of 0.5.

infinite dimensional convex optimization problem. The finite dimensional approximation of the optimization problem relies on the data-driven approximation of the linear P-F operator. Simulation examples are presented to show the feasibility of the proposed framework.

REFERENCES

- [1] P. Fiorini and Z. Shiller, "Motion planning in dynamic environments using velocity obstacles," The International Journal of Robotics Research, vol. 17, no. 7, pp. 760-772, 1998.
- [2] H. Yu, J. Moyalan, D. Tellez-Castro, U. Vaidya, and Y. Chen, "Convex optimal control synthesis under safety constraints," in Accepted in 60th IEEE Conference on Decision and Control 2021. IEEE, 2021.
- [3] H. Yu, J. Moyalan, U. Vaidya, and Y. Chen, "Data-driven optimal control of nonlinear dynamics under safety constraints," in Submitted for publication. IEEE, 2021.
- [4] N. E. Du Toit and J. W. Burdick, "Probabilistic collision checking with chance constraints," IEEE Transactions on Robotics, vol. 27, no. 4, pp. 809-815, 2011.
- [5] A. Majumdar and M. Pavone, "How should a robot assess risk? towards an axiomatic theory of risk in robotics," in Robotics Research. Springer, 2020, pp. 75-84.
- D. D. Fan, K. Otsu, Y. Kubo, A. Dixit, J. Burdick, and A.-A. [6] Agha-Mohammadi, "Step: Stochastic traversability evaluation and planning for risk-aware off-road navigation," in Robotics: Science and Systems. RSS Foundation, 2021, pp. 1–21.

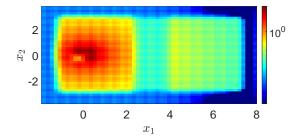


Fig. 5: $\rho(\mathbf{x})$ in Example 2 with respect to probability measure of 0.5 in the unsafe sets \mathbf{X}_{u1} and \mathbf{X}_{u2} .

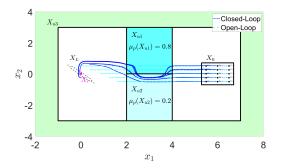


Fig. 6: State trajectories of Example 2 avoiding unsafe sets X_{u1} and X_{u2} with probability measure of 0.8 and 0.2 respectively.

- [7] S. Hacohen, S. Shoval, and N. Shvalb, "Probability navigation function for stochastic static environments," *International Journal of Control, Automation and Systems*, vol. 17, no. 8, pp. 2097–2113, 2019.
- [8] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise:* Stochastic Aspects of Dynamics. New York: Springer-Verlag, 1994.
- [9] U. Vaidya, "Optimal motion planning using navigation measure," *International Journal of Control*, vol. 91, no. 5, pp. 989–998, 2018.
- [10] B. Huang and U. Vaidya, "A convex approach to data-driven optimal control via perron-frobenius and koopman operators," *arXiv preprint arXiv:2010.01742*, 2020.
- [11] J. Moyalan, H. Choi, Y. Chen, and U. Vaidya, "Sum of squares based convex approach for optimal control synthesis," in 2021 29th Mediterranean Conference on Control and Automation (MED). IEEE, 2021, pp. 1270–1275.
- [12] R. Rajaram, U. Vaidya, M. Fardad, and B. Ganapathysubramanian, "Stability in the almost everywhere sense: A linear transfer operator approach," *Journal of Mathematical analysis* and applications, vol. 368, no. 1, pp. 144–156, 2010.
- [13] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley, "A datadriven approximation of the koopman operator: Extending dynamic mode decomposition," *Journal of Nonlinear Science*, vol. 25, no. 6, pp. 1307–1346, 2015.
- [14] M. Korda and I. Mezić, "On convergence of extended dynamic mode decomposition to the koopman operator," *Journal of Nonlinear Science*, vol. 28, no. 2, pp. 687–710, 2018.
- [15] S. Klus, I. Schuster, and K. Muandet, "Eigendecompositions of transfer operators in reproducing kernel hilbert spaces,"

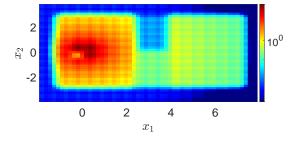


Fig. 7: $\rho(\mathbf{x})$ in Example 2 with respect to probability measure of 0.8 and 0.2 in the unsafe sets \mathbf{X}_{u1} and \mathbf{X}_{u2} respectively.

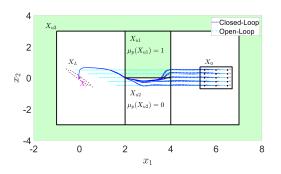


Fig. 8: State trajectories of Example 2 avoiding unsafe sets X_{u1} and X_{u2} with probability measure of 1 and 0 respectively.

Journal of Nonlinear Science, vol. 30, no. 1, pp. 283-315, 2020.

[16] A. Rantzer and S. Prajna, "On analysis and synthesis of safe control laws," in *Proceedings of the Allerton Conference on Communication, Control, and Computing*, 2004.

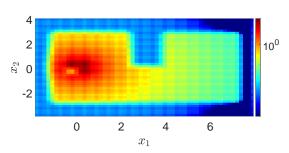


Fig. 9: $\rho(\mathbf{x})$ in Example 2 with respect to probability measure of 1 and 0 in the unsafe sets \mathbf{X}_{u1} and \mathbf{X}_{u2} respectively.