New Algorithms for Discrete-Time Parameter Estimation

Yingnan Cui¹, Joseph E. Gaudio², and Anuradha M. Annaswamy¹

¹Massachusetts Institute of Technology ²Aurora Flight Sciences

Abstract

We propose two algorithms for discrete-time parameter estimation, one for time-varying parameters under persistent excitation (PE) condition, another for constant parameters under no PE condition. For the first algorithm, we show that in the presence of time-varying unknown parameters, the parameter estimation error converges uniformly to a compact set under conditions of persistent excitation, with the size of the compact set proportional to the time-variation of unknown parameters. Leveraging a projection operator, the second algorithm is shown to result in boundedness guarantees when the plant has constant unknown parameters. Simulations show better convergence results compared to recursive least squares (RLS) and comparable results to RLS with forgetting factor.

1 Introduction

A central task in adaptive control is the compensation for online parametric uncertainties in the processes of adaptive filtering, prediction and control[1, 2, 3, 4, 5]. Over the past several decades, several discrete-time online estimation schemes have been designed, including conditions for parametric convergence[6, 7, 8]. Most of the methods in these works assume that the unknown parameters are constant. This paper focuses on the case where parameters are varying with time.

In many of the adaptive filtering, prediction, and control problems, the underlying structure consists of a linear regression relation between the prediction error and parameter error[4, 9, 10, 11, 12, 13]. A typical gradient-based algorithm, for example, minimizes a quadratic criterion in terms of the prediction error[3, 5]. In adaptive systems, a necessary and sufficient condition that is needed to ensure uniform asymptotic convergence of the estimates to their true values is noted as persistent excitation [14, 15]. The gradient descent method has a constant learning rate which often leads to extremely slow convergence of the estimates. The well-known recursive least squares (RLS) algorithm has a covariance matrix that is time-varying and is adjusted through the use of the outer product of the underlying regressor[3, 5]. One of the main inadequacies of the RLS algorithm is its inability to ensure estimation in the presence of time-varying parameters, which is due to the fact that the

learning rates themselves can converge rapidly to zero and as a result, unable to track the changes in the parameters.

The literature on online parameter estimation algorithms in discrete-time is vast (see for example, [6, 7, 15, 8, 14, 9, 10, 11, 16, 17, 18, 19, 20, 21]). In many of them, the learning rates are assumed to be constant. Of these, works such as [16, 17, 19, 20, 8] have focused on a RLS-type algorithm and address variations thereof. In [16, 17, 18], combined variable-rate and variable-direction forgetting is adopted to address some of the inadequacies of the RLS algorithm. In [19], a data-dependent forgetting rate is used in the update of the covariance matrix. A nonlinear weighting coefficient is devised in the update of the covariance matrix in [20]. Another interesting approach has been suggested in [8], which consists of a dynamic regressor extension and mixing (DREM) idea, and it tries to lift the limitation of the input signal by a linear, BIBO-stable operator. In most of these works, the unknown parameters are assumed to be constant. In addition, the focus in most of these papers is more in showing boundedness of the parameter estimates, rather than the speed of convergence of these estimates to their true values.

In this paper, we present two algorithms: the first assumes that persistent excitation conditions are met and the second assumes that they are not, both of which represent the two main contributions of this paper. Time-varying learning rate in the form of a time-varying gain matrix is included in both algorithms, but no covariance resetting or forgetting factor is introduced. It is shown that under persistent excitation, when the unknown parameter is time-varying, the first algorithm leads to convergence of the parameter error to a compact set that is proportional to the rate of variation of the unknown parameters. The convergence is uniform and asymptotic under persistent excitation. For the second algorithm, no assumptions of persistent excitation are made, and it is shown that when the unknown parameters are constants, the parameter estimation error is bounded. In contrast to RLS with forgetting based algorithms such as those in [21, 18, 16, 17], we provide a stability proof in this paper for this latter case when persistent excitation is absent, through the construction of a Lyapunov function. Simulation results show that our algorithms are comparable to RLS with forgetting, and result in much faster convergence compared to RLS algorithms.

Section 2 presents the definitions of discrete-time persistent excitation and some mathematical preliminaries. The underlying problems are presented in Section 3. Section 4 presents two classes of algorithms. Section 5 gives stability and convergence analysis for the first algorithm. Section 6 introduces projection in the update of the time-varying gain matrix and establishes stability for the second algorithm. Simulation results are shown in Section 7. Section 8 provides a summary and concluding remarks.

2 Preliminaries

We denote \mathbb{N}^+ as the set of positive integers, and $\|\cdot\|$ as the Euclidean norm.

Definition 1 ([22]). A bounded function $\phi_k : [k_0, \infty) \to \mathbb{R}^N$ is persistently exciting (PE) at a level α if there exists $\Delta T \in \mathbb{N}^+$ and $\alpha > 0$ such that

$$\sum_{i=k+1}^{k+\Delta T} \phi_i \phi_i^{\top} \succeq \alpha I, \ \forall k \ge k_0.$$

In Definition 1, the degree of excitation is given by α . It can be noted that this PE condition pertains to a property of ϕ that needs to hold over a moving window for all $k \geq k_0$. The following matrix inversion lemma is useful in deriving the results that follow.

Lemma 1. (Kailath Variant of the Woodbury Identity [23].)

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}$$

3 Problem Formulation

The class of discrete-time models we consider in this paper is of the form[3, 5]

$$y_k = -\sum_{i=1}^n a_{i,k}^* y_{k-i} + \sum_{j=1}^m b_{j,k}^* u_{k-j-d} + \sum_{\ell=1}^p c_{\ell,k}^* f_{\ell}(y_{k-1}, \dots, y_{k-n}, u_{k-1-d}, \dots, u_{k-m-d}),$$
 (1)

where $a_{i,k}^*$, $b_{j,k}^*$ and $c_{\ell,k}^*$ are unknown parameters that may vary with time and need to be identified, and d is a known time-delay. The function f_{ℓ} is an analytic function of its arguments and is assumed to be such that the system in (1) is bounded-input-bounded-output (BIBO) stable. Denote $z_{k-1} = [y_{k-1}, \ldots, y_{k-n}]^{\top}$ and $v_{k-d-1} = [u_{k-1-d}, \ldots, u_{k-m-d}]^{\top}$. We rewrite (1) in the form of a linear regression

$$y_k = \phi_k^\top \theta_k^*, \tag{2}$$

where $\phi_k = [z_{k-1}^\top, v_{k-d-1}^\top, f_1(z_{k-1}^\top, v_{k-d-1}^\top), \dots, f_p(z_{k-1}^\top, v_{k-d-1}^\top)]^\top$ is a regressor determined by exogenous signals and $\theta_k^* = [a_{1,k}^*, \dots, a_{n,k}^*, b_{1,k}^*, \dots, b_{m,k}^*, c_{1,k}^*, \dots, c_{\ell,k}^*]^\top$ is the underlying unknown parameter vector. We propose to identify the parameter θ_k^* as θ_k using an estimator

$$\hat{y}_k = \phi_k^\top \theta_k. \tag{3}$$

This leads to a prediction error

$$e_{y,k} = \phi_k^{\mathsf{T}} \tilde{\theta}_k, \tag{4}$$

where $e_{y,k} = \hat{y}_k - y_k$ is the output prediction error and the parameter error is $\tilde{\theta}_k = \theta_k - \theta_k^*$. We define the time variation in θ_k^* as $\Delta \theta_k^* = \theta_k^* - \theta_{k-1}^*$.

Assumption 1. The time variation $\Delta \theta_k^*$ is bounded, i.e.,

$$\|\Delta \theta_k^*\| \le \Delta^*. \tag{5}$$

Assumption 2. There exists a $\theta_{\max}^* \geq 0$ such that

$$\|\theta_k^*\| \le \theta_{\max}^*. \tag{6}$$

In the following sections, we will consider two different cases: In the first case, there is persistent excitation in the input signal while Assumption 1 and Assumption 2 are assumed; In the second case, no persistent excitation is assumed and we only consider the simpler case where $\theta_k^* = \theta^*$, a constant.

4 Adaptive Laws with a Time-Varying Learning Rate: The Algorithms

4.1 Algorithm 1: For Inputs with Persistent Excitation

We propose the following recursive algorithm for estimating θ_k^* as θ_k , for all $k \geq 0$:

$$\theta_{k+1} = \theta_k - \frac{\lambda_\Gamma \kappa \Gamma_k \phi_k e_k}{1 + \|\phi_k\|^2},\tag{7}$$

$$\Omega_k = (1 - \lambda_{\Omega})\Omega_{k-1} + \frac{\phi_k \phi_k^{\mathsf{T}}}{1 + \|\phi_k\|^2},\tag{8}$$

$$\Gamma_k = \Gamma_{k-1} + \lambda_{\Gamma} (\Gamma_{k-1} - \kappa \Gamma_{k-1} \Omega_k \Gamma_{k-1}), \tag{9}$$

where λ_{Γ} , κ are positive scalars and $\Gamma_k \in \mathbb{R}^{N \times N}$ is the time-varying gain matrix. The initial value Γ_0 is a symmetric positive definite matrix chosen so that $\Gamma_0 \leq \Gamma_{\max} I$, where $\Gamma_{\max} \in \mathbb{R}^+$ is the maximum learning rate that is assumed to be given. The information matrix $\Omega_k \in \mathbb{R}^{N \times N}$ captures any excitation that may be present in the regressor ϕ_k . The initial value of Ω_k is a symmetric positive definite matrix with $0 \prec \Omega_0 \leq I$. The parameter $0 < \lambda_{\Omega} < 1$ is used for the adjustment of the evolution of Ω_k and is directly related to the upper bound of Ω_k . The scalars λ_{Γ} , κ and λ_{Ω} represent various weights of the proposed algorithm.

The main innovation in the algorithm above is in the evolution of Γ_k . Unlike the recursive least squares algorithm with forgetting (see [3] for example), the algorithm in (9) automatically ensures that Γ_k stays within certain bounds, through a combination of linear and nonlinear components. As will be shown in the following section, stability and exponential decrease of parameter error towards a compact set are established under appropriate excitation conditions. In what follows, we assume that the regressors are persistently exciting, satisfying Definition 1.

4.2 Algorithm 2: For Inputs with No Excitation

When there is no persistent excitation in the input signal, we propose the following update of Γ_k as a replacement of (9) in (7)-(9):

$$\Gamma_k = \operatorname{Proj}_{\Gamma_{\max}} [\Gamma_{k-1} + \lambda_{\Gamma} (\Gamma_{k-1} - \kappa \Gamma_{k-1} \Omega_k \Gamma_{k-1})], \tag{10}$$

where $\text{Proj}(\cdot)$ is defined as in Algorithm 3. The stability and convergence properties of these two algorithms are discussed in the following sections.

5 Stability and Convergence Analysis for Algorithm 1

Before addressing the boundedness of the parameter estimates and their convergence, we establish the boundedness of the information matrix Ω_k in (8) and time-varying gain matrix Γ_k defined in (9).

Algorithm 3 The Projection Operator $\Gamma_k = \text{Proj}(\Gamma'_k)$

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\begin{aligned} & \textbf{Require:} \ \Gamma_{\max}, \Gamma_k' > 0 \\ & \textbf{if} \ \|\Gamma_k'\|_2 > \Gamma_{\max} \ \textbf{then} \\ & UDU^\top \leftarrow \Gamma_k' \\ & \textbf{for} \ i \leftarrow 1 : N \ \textbf{do} \\ & \textbf{if} \ D_{i,i} > \Gamma_{\max} \ \textbf{then} \\ & D_{i,i} \leftarrow \Gamma_{\max} \\ & \textbf{end} \ \textbf{if} \\ & \textbf{end for} \\ & \Gamma_k \leftarrow UDU^\top \\ & \textbf{else} \\ & \Gamma_k = \Gamma_k' \\ & \textbf{end if} \end{aligned}
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5.1 Boundedness of Ω_k

We first show that for any arbitrary regressor ϕ_k , Ω_k is a positive semi-definite matrix with an upper bound.

Lemma 2. For the dynamic system in (8), if $0 < \lambda_{\Omega} < 1$ and Ω_0 is such that $0 \leq \Omega_0 \leq I$, then $0 \leq \Omega_k \leq \Omega_{\max}I$, for all $k \geq 1$, where $\Omega_{\max} = 1/\lambda_{\Omega}$.

Proof. We consider the lower bound and the upper bound of Ω_k separately.

1. First we prove $\Omega_k \succeq 0$ by induction. Let $v \in \mathbb{R}^N$ and $v \neq 0$. Since $0 < \lambda_{\Omega} < 1$ and $\Omega_0 \succeq 0$,

$$v^{\top} \Omega_1 v = (1 - \lambda_{\Omega}) v^{\top} \Omega_0 v + \frac{\|\phi_1^{\top} v\|^2}{1 + \|\phi_1\|^2} \ge 0$$

Therefore, $\Omega_1 \succeq 0$. Similarly if $0 \preceq \Omega_{k-1} \preceq \frac{1}{\lambda_{\Omega}}I$, it follows that

$$v^{\top} \Omega_k v = (1 - \lambda_{\Omega}) v^{\top} \Omega_{k-1} v + \frac{\|\phi_k^{\top} v\|^2}{1 + \|\phi_k\|^2} \ge 0.$$

- 2. The lower bound $\Omega_k \succeq 0$ is immediate when applying the definition of positive-semidefinite matrices.
- 3. Next we prove $\Omega_k \leq \Omega_{\text{max}}I$. Using convolution, Ω_k in (8) can be written as

$$\Omega_k = (1 - \lambda_{\Omega})^k \Omega_0 + \sum_{i=0}^{k-1} (1 - \lambda_{\Omega})^{k-1-i} \frac{\phi_i \phi_i^{\top}}{1 + \|\phi_i\|^2}$$

$$\leq (1 - \lambda_{\Omega})^k \Omega_0 + I \sum_{i=0}^{k-1} (1 - \lambda_{\Omega})^i.$$

Since $\Omega_0 \leq I$, using simple algebra, we can then show that $\Omega_k \leq \Omega_{\max} I$, for all $k \geq 0$.

Lemma 2 guarantees that Ω_k in (8) is a BIBO-stable system and determines an explicit upper bound of Ω_k . In the next lemma, we focus on its lower bound, and show that Ω_k is positive definite, under persistent excitation (PE) conditions.

Lemma 3. If $0 < \lambda_{\Omega} < 1$, and ϕ_k is persistently exciting over a period ΔT at a level α for all $k \geq k_1$, then $\Omega_k \succeq \Omega_{PE}I$, $\forall k \geq k_2$, where $\Omega_{PE} = (1 - \lambda_{\Omega})^{\Delta T - 1} \alpha/d$, $d = \max_{i \geq k_1} \{1 + \|\phi_i\|^2\}$ and $k_2 = k_1 + \Delta T$.

Proof. Using convolution, Ω_k can be written as

$$\Omega_k = (1 - \lambda_{\Omega})^k \Omega_0 + \sum_{i=0}^{k-1} (1 - \lambda_{\Omega})^{k-i-1} \frac{\phi_i \phi_i^{\top}}{1 + \|\phi_i\|^2}$$

Since ϕ_k is persistently exciting,

$$\Omega_k \succeq (1 - \lambda_{\Omega})^{k-1} \sum_{i=k-\Delta T}^{k-1} (1 - \lambda_{\Omega})^{-i} \frac{\phi_i \phi_i^{\top}}{1 + \|\phi_i\|^2}$$
$$\succeq \Omega_{PE} I$$

The effects of persistent excitation on the regressor, Ω_k and Γ_k are shown in Table 1.

 $\begin{array}{c|cccc} k \in & [k_1, k_2] & k_2 & [k_2, \infty) \\ \hline \|\phi_k \phi_k^\top\| & \geq 0 \\ \|\sum_{k=k_1}^{k_2} \phi_k \phi_k^\top\| & \geq \alpha \\ \|\Omega_k\| & \geq \Omega_{PE} \\ \|\Gamma_k\| & \geq \Gamma_{PE} \end{array}$

Table 1: Norm of Signals in Different Phases of PE Propagation

5.2 Boundedness of Γ_k

One of the major contributions of this paper is the adjustment of the time-varying learning rate matrix Γ_k in (9). The update in (9) allows the linear part and the quadratic part to automatically balance each other so that Γ_k is always bounded under conditions of persistent excitation. In this section we will show this result formally.

We first consider the scalar case of (9), given by

$$\gamma_k = \gamma_{k-1} + \lambda_{\Gamma}(\gamma_{k-1} - \kappa \omega_k \gamma_{k-1}^2), \tag{11}$$

where

$$0 < \omega_{\min} \le \omega_k \le \omega_{\max}. \tag{12}$$

Let $\gamma_{\text{max}} > 0$ and

$$\gamma_{\min} = \min \left\{ \frac{1}{\kappa \omega_{\max}}, \gamma_{\max} + \lambda_{\Gamma} (\gamma_{\max} - \kappa \omega_{\max} \gamma_{\max}^2) \right\}$$
 (13)

where $\kappa_{\min} \leq \kappa < \kappa_{\max}$, and κ_{\min} , κ_{\max} are defined as

$$\kappa_{\min} = \frac{1}{\gamma_{\max}\omega_{\min}}, \quad \kappa_{\max} = \frac{1 + \lambda_{\Gamma}}{\lambda_{\Gamma}\omega_{\max}\gamma_{\max}}.$$
(14)

In the update of (11), γ_{max} and λ_{Γ} are chosen arbitrarily by the user. The gain κ is then chosen using the bounds in (14). We now state the following lemma that establishes bounds of γ_k (See Fig. 1 and Fig. 2 for an illustration of the bounds) which in turn depend on $\lambda_{\Gamma}, \gamma_{\text{max}}$ and κ .

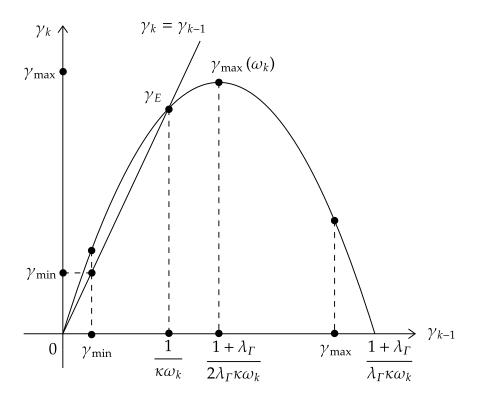


Figure 1: A parabola for illustrating the bounds of γ_k whose update is in (11) for the case when $\gamma_{\max} > \frac{1+\lambda_{\Gamma}}{2\lambda_{\Gamma}\kappa\omega_k}$. For a given ω_k , the line $\gamma_k = \gamma_{k-1}$ intersects the parabola at γ_E , giving $\gamma_k = \gamma_{k-1} = \frac{1}{\kappa\omega_k}$. When the parabola is above the straight line, $\gamma_k \geq \gamma_{k-1}$; Otherwise $\gamma_k \leq \gamma_{k-1}$. The point $\gamma_{\max}(\omega_k)$ is the maximum obtainable value for γ_k and it will be no larger than γ_{\max} . When $\gamma_{k-1} = \gamma_{\max}$, the resulting γ_k will be no less than γ_{\min} .

Lemma 4. If $0 < \lambda_{\Gamma} < \min\left\{\frac{1}{\omega_{\max}/\omega_{\min}-1}, 1\right\}$, and $\gamma_{\min} \leq \gamma_0 \leq \gamma_{\max}$, then the solutions of (11) are such that $0 < \gamma_{\min} \leq \gamma_k \leq \gamma_{\max}$, $\forall k \geq 0$.

Proof. Defining the right hand side of Eq. (11) as $f(\gamma_{k-1}, \omega_k)$, it is easy to see that f(.,.) is quadratic in γ_{k-1} , with one maximum and two minima (see Fig. 1 and Fig. 2). Suppose ω^* is such that

$$\frac{1+\lambda_{\Gamma}}{2\lambda_{\Gamma}\kappa\omega^*} = \gamma_{\text{max}}.$$
 (15)

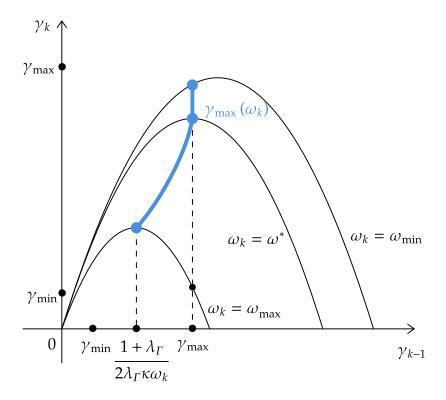


Figure 2: Illustration of the upper bounds of γ_k as ω_k varies. The blue curve shows how $\gamma_{\max}(\omega_k)$ changes. The value of ω^* is defined such that (15) is satisfied. Under the assumptions for Lemma 4, $\gamma_{\max}(\omega_k)$ gets larger as ω_k decreases. The global maximum can be obtained when $\omega_k = \omega_{\min}$ and $\gamma_{k-1} = \gamma_{\max}$.

It is easy to check that $\omega^* \geq \omega_{\min}$. If $\omega_k \geq \omega^*$,

$$f(\gamma_{k-1}, \omega_k) \le \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega^* \gamma_{\max}^2)$$

$$\le \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega_{\min} \gamma_{\max}^2).$$

If $\omega_k \leq \omega^*$,

$$f(\gamma_{k-1}, \omega_k) \leq \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega_k \gamma_{\max}^2)$$

$$\leq \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega_{\min} \gamma_{\max}^2).$$

Noticing the bounds of λ_{Γ} and κ , simple algebraic manipulations show that

$$\max_{\gamma_k,\omega_k} f(\gamma_{k-1},\omega_k) = \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega_{\min} \gamma_{\max}^2)$$

$$< \gamma_{\max}.$$

Similarly, it can be shown by inspection that

$$\min_{\omega_k} f(\gamma_{k-1}, \omega_k) = \gamma_{k-1} + \lambda_{\Gamma}(\gamma_{k-1} - \kappa \omega_{\max} \gamma_{k-1}^2).$$

Since the largest decrease of f occurs when γ_k is at its maximum value, it follows that

$$\min_{\gamma_k,\omega_k} f(\gamma_{k-1},\omega_k) = \gamma_{\max} + \lambda_{\Gamma}(\gamma_{\max} - \kappa \omega_{\max} \gamma_{\max}^2).$$

In addition, $\kappa \omega_{\text{max}} > 0$. Therefore, the lower bound of γ_k and that it is positive are ensured through the choice of γ_{min} as in (13).

Remark 1. It should also be noted that (12) implies that ω_k is persistently exciting, i.e., we consider the period $k \in [k_2, \infty)$ as shown in Table 1.

Remark 2. As we will show next, the properties we obtain in Lemma 4 for the scalar γ_k will be directly utilized in showing the properties of the matrix Γ_k . Specifically, it will be shown that each eigenvalue of Γ_k has an update similar to (11).

We now proceed to the matrix case. First we consider the case when ϕ_k is persistently exciting and thus $\Omega_{PE}I \leq \Omega_k \leq \Omega_{\max}I$, $\forall k \geq k_2$, as shown in Lemma 3. Let $\Gamma_{\max} > 0$, $\kappa_{\min} = \frac{1}{\Gamma_{\max}\Omega_{PE}}$, $\kappa_{\max} = \frac{1+\lambda_{\Gamma}}{\lambda_{\Gamma}\Omega_{\max}\Gamma_{\max}}$, $\Gamma_{PE} = \min\left\{\frac{1}{\kappa\Omega_{\max}}, \Gamma_{\max} + \lambda_{\Gamma}(\Gamma_{\max} - \kappa\Omega_{\max}\Gamma_{\max}^2)\right\}$, the following Lemma gives the bounds on Γ_k for $k \geq k_2$.

Lemma 5. If ϕ_k is persistently exciting, $\Omega_{PE}I \preceq \Omega_k \preceq \Omega_{\max}I$, $\forall k \geq k_2$, learning rate gain $0 < \lambda_{\Gamma} < \min\left\{\frac{1}{\Omega_{\max}/\Omega_{PE}-1}, 1\right\}$, gain $\kappa_{\min} \leq \kappa < \kappa_{\max}$, and initial value $\Gamma_{PE}I \preceq \Gamma_{k_2} \preceq \Gamma_{\max}I$, then $\Gamma_{PE}I \preceq \Gamma_k \preceq \Gamma_{\max}I$, $\forall k > k_2$ for some $k_2 > k_1$.

Proof. We first establish the lower bound Γ_{PE} using induction: Let $\Gamma_{PE}I \leq \Gamma_{k-1} \leq \Gamma_{\max}I$. It is easy to check that $\Gamma_{PE} > 0$. Since Γ_{k-1} is positive definite, we can factorize Γ_{k-1} as [24]

$$\Gamma_{k-1} = Q_{k-1} \Lambda_{k-1} Q_{k-1}^{\top} \tag{16}$$

where Λ_{k-1} is diagonal and all of its entries are the eigenvalues of Γ_{k-1} , and Q_{k-1} is orthogonal. Using Eq. (9) and the fact that Q_{k-1} is orthogonal, we obtain that

$$\begin{split} \Gamma_k &= \Gamma_{k-1} + \lambda_{\Gamma} (\Gamma_{k-1} - \kappa \Gamma_{k-1} \Omega_k \Gamma_{k-1}) \\ &\succeq \Gamma_{k-1} + \lambda_{\Gamma} (\Gamma_{k-1} - \kappa \Omega_{\max} \Gamma_{k-1}^2) \\ &= Q_{k-1} \left[\Lambda_{k-1} + \lambda_{\Gamma} (\Lambda_{k-1} - \kappa \Omega_{\max} \Lambda_{k-1}^2) \right] Q_{k-1}^{\top}, \end{split}$$

since

$$x^{\top}(\Omega_{\max}I - \Omega_k)x \ge 0, \ \forall x \ne 0.$$

Since Γ_{k-1} is positive definite, it follows that

$$v^{\top} \Gamma_{k-1} (\Omega_{\max} I - \Omega_k) \Gamma_{k-1} v \ge 0, \ \forall v \ne 0.$$

Thus

$$\Gamma_{k-1}(\Omega_{\max}I - \Omega_k)\Gamma_{k-1} \succeq 0. \tag{17}$$

Denote the *i*th diagonal element of Γ_{k-1} as $\lambda_{k-1,i}$, $i=1,\cdots,N$. Since $\Gamma_{PE}I \leq \Gamma_{k-1} \leq \Gamma_{\max}I$, we have $\Gamma_{PE} \leq \lambda_{k-1,i} \leq \Gamma_{\max}$, $\forall i$. Applying Lemma 4, we obtain $\Gamma_{PE} \leq \lambda_{k,i} \leq \Gamma_{\max}$, $\forall i$ and the lower bound of Γ_k is proved. Since the lower bound holds for $k=k_2$, proof by induction guarantees that the lower bound holds for all $k \geq k_2$.

To prove the upper bound, similar to Eq. (17), using Lemma 3 we can show that

$$\Gamma_{k-1}(\Omega_{PE}I - \Omega_k)\Gamma_{k-1} \le 0. \tag{18}$$

Therefore

$$\Gamma_k \leq Q_{k-1} \left[\Lambda_{k-1} + \lambda_{\Gamma} (\Lambda_{k-1} - \kappa \Omega_{PE} \Lambda_{k-1}^2) \right] Q_{k-1}^{\top}.$$

Using the same procedure as above, it can be shown that the upper bound holds for Γ_k if it holds for Γ_{k-1} , and using induction that the upper bound holds for all k, Lemma 5 is proved.

Remark 3. The condition $\Omega_{PE} > \frac{1}{4}\Omega_{\text{max}}$ in Lemma 5 indicates that the minimum eigenvalue and the maximum eigenvalue should not be too far apart. This may seem to be restricted, but it should be noted that, when they are very far apart, there will be a similar problem of the disparage of convergence rates for methods like covariance resetting: Some parts of the covariance matrix are decreasing to zero quickly due to the large eigenvalues, while other parts will decrease slowly.

Remark 4. Note that Lemma 5 is essentially an extension of Lemma 4. The extension to the matrix Γ_k from scalar γ_k has been carried out by diagonalization and applying Lemma 4. In Lemma 5, it should be noted that Ω_{PE} is corresponding to ω_{\min} in Lemma 4.

5.3 Stability and Parameter Convergence

We now return to the main algorithm in (7), (8) and (9). Note that in Section 5.1 and Section 5.2 we have shown that both Ω_k and Γ_k are bounded under persistent excitation. In this section we consider the boundedness of θ_k as well as its convergence.

We first introduce an intermediate variable in the evolution of Γ_k in (9) in the form of a matrix $\bar{\Gamma}_k$,

$$\bar{\Gamma}_k = \Gamma_{k-1} - \lambda_{\Gamma} \kappa \Gamma_{k-1} \frac{\phi_{k-1} \phi_{k-1}^{\mathsf{T}}}{1 + \|\phi_{k-1}\|^2} \Gamma_{k-1}, \tag{19}$$

and consider a Lyapunov function candidate

$$V_k = \tilde{\theta}_k^{\top} \bar{\Gamma}_k^{-1} \tilde{\theta}_k. \tag{20}$$

Under conditions of excitation, V_k will be shown to be a Lyapunov function and converge to a compact set that scales with Δ^* .

In the case of persistent excitation, as proved in Lemma 3, $\Omega_{PE}I \leq \Omega_k$, $\forall k \geq k_2$ for some $k_2 > 0$. Let λ_{Ω} satisfy

$$\alpha \lambda_{\Omega} (1 - \lambda_{\Omega})^{\Delta T - 1} > d, \tag{21}$$

where ΔT , α and d are defined in Lemma 3. Assume that λ_{Γ} satisfies

$$0 < \lambda_{\Gamma} < \min \left\{ \frac{1}{\Omega_{\text{max}}/\Omega_{PE} - 1}, 1, \Omega_{PE} \right\}. \tag{22}$$

We define

$$\kappa_{\min} = \frac{1}{\Gamma_{\max} \Omega_{PE}},\tag{23}$$

$$\kappa_{\text{max}} = \min \left\{ \frac{1 + \lambda_{\Gamma}}{\lambda_{\Gamma} \Omega_{\text{max}} \Gamma_{\text{max}}}, \frac{1}{\lambda_{\Gamma} \Gamma_{\text{max}}}, \frac{1}{(1 - \lambda_{\Omega}) \Omega_{\text{max}} \Gamma_{\text{max}}} \right\}, \tag{24}$$

and choose any κ that satisfies

$$\kappa_{\min} \le \kappa < \kappa_{\max}.$$
(25)

According to Lemma 5, $\Gamma_{PE}I \leq \Gamma_k \leq \Gamma_{\max}I$, and Γ_{PE} is defined as

$$\Gamma_{PE} = \min \left\{ \frac{1}{\kappa \Omega_{\text{max}}}, \Gamma_{\text{max}} + \lambda_{\Gamma} (\Gamma_{\text{max}} - \kappa \Omega_{\text{max}} \Gamma_{\text{max}}^2) \right\}.$$
 (26)

Furthermore, let

$$\bar{\Gamma}_{\min} = \min \left\{ \Gamma_{\max} - \lambda_{\Gamma} \kappa \Gamma_{\max}^2, \Gamma_{PE} - \lambda_{\Gamma} \kappa \Gamma_{PE}^2 \right\}. \tag{27}$$

We give the expression of the upper-bound for the decay rate as

$$\mu_1 = \Psi_{\min} \bar{\Gamma}_{\min}, \tag{28}$$

and consider a parameter μ_2 that satisfies $0 < \mu_2 < \mu_1$. In (28), $\Psi_{\min} = (1 + \lambda_{\Gamma} \bar{\Gamma}_{\min}^{-1} \Gamma_{\max})^{-1}$ $\Upsilon_{\min} \Gamma_{\max}^{-2}$, where $\Upsilon_{\min} = \min \left\{ \lambda_{\Gamma} \Gamma_{\max} \left[1 - \kappa (1 - \lambda_{\Omega}) \Omega_{\max} \Gamma_{\max} \right], \lambda_{\Gamma} \Gamma_{PE} \left[1 - \kappa (1 - \lambda_{\Omega}) \Omega_{\max} \Gamma_{PE} \right] \right\}$. Finally, a compact set is defined as

$$D_{PE} = \left\{ V | V \le V_{PE} \right\},\tag{29}$$

where

$$V_{PE} = \left[\frac{\Delta^* \left(c_1 + \sqrt{c_1^2 + 4c_2(\mu_1 - \mu_2)} \right)}{2(\mu_1 - \mu_2)} \right]^2, \tag{30}$$

$$c_1 = \bar{\Gamma}_{\text{max}}^{1/2} \bar{\Gamma}_{\text{min}}^{-1},\tag{31}$$

$$c_2 = \bar{\Gamma}_{\min}^{-1}.\tag{32}$$

We now state the stability result with persistent excitation.

Theorem 1. Let ϕ_k be persistently exciting. For the ARMA model in (1), under the assumptions in 1 and 2, subject to the bounds in (5), if the hyperparameters λ_{Ω} , λ_{Γ} and κ satisfy (21), (22) and (25), respectively, the adaptive algorithm in (7), (8) and (9) guarantees that in D_{PE}^c

$$V_k \le \exp\left[-\mu_2(k-k_2)\right] V_{k_2}, \ \forall k \ge k_2$$

Proof. Starting from (7), we get

$$\theta_k - \theta_{k-1}^* = \left(I - \frac{\lambda_\Gamma \kappa \Gamma_{k-1} \phi_{k-1} \phi_{k-1}^\top}{1 + \|\phi_{k-1}\|^2} \right) \tilde{\theta}_{k-1}. \tag{33}$$

Using (19), Γ_k can be written as

$$\Gamma_k = \bar{\Gamma}_k + \underbrace{\lambda_{\Gamma} \Gamma_{k-1} - \lambda_{\Gamma} \kappa (1 - \lambda_{\Omega}) \Gamma_{k-1} \Omega_{k-1} \Gamma_{k-1}}_{\Upsilon_k}$$
(34)

Now we show that under the conditions in Theorem 1, $\bar{\Gamma}_k \succeq \bar{\Gamma}_{\min}I \succ 0$. With simple manipulations, it can be shown that

$$\bar{\Gamma}_k \succeq Q_{k-1} \left(\Lambda_{k-1} - \lambda_{\Gamma} \kappa \Lambda_{k-1}^2 \right) Q_{k-1}^{\top}. \tag{35}$$

It is easy to see that $\Gamma_{PE} \leq \lambda_{k-1,i} \leq \Gamma_{\max}$, $\forall i$, where $\lambda_{k-1,i}$ is the *i*th diagonal element of Γ_{k-1} . From the inequality in (35), $\bar{\Gamma}_k$ assumes its lower bound either when $\lambda_{k-1,i} = \Gamma_{PE}$ or when $\lambda_{k-1,i} = \Gamma_{\max}$. It follows therefore that $\bar{\Gamma}_k \succeq \bar{\Gamma}_{\min} I$, where $\bar{\Gamma}_{\min} > 0$ is defined in (27). It is also obvious that $\bar{\Gamma}_k \preceq \bar{\Gamma}_{\max} I$, where $\bar{\Gamma}_{\max} = \Gamma_{\max}$. Using the same approach, we can also show that $0 \prec \Upsilon_{\min} I \preceq \Upsilon_k \preceq \Upsilon_{\max} I$, where Υ_k is defined in (34) and $\Upsilon_{\max} = \lambda_{\Gamma} \Gamma_{\max}$. With these bounds established, we consider the Lyapunov function in (20). Since $\tilde{\theta}_k = \theta_{k-1}^* - \theta_k^* + \bar{\Gamma}_k \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1}$, we obtain that

$$\begin{split} V_k &= \left(\theta_{k-1}^* - \theta_k^* + \bar{\Gamma}_k \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1}\right)^\top \bar{\Gamma}_k^{-1} \\ & \left[\theta_{k-1}^* - \theta_k^* + \left(I - \lambda_{\Gamma} \kappa \frac{\Gamma_{k-1} \phi_{k-1} \phi_{k-1}^\top}{1 + \|\phi_{k-1}\|^2}\right) \tilde{\theta}_{k-1}\right] \\ &\leq \tilde{\theta}_{k-1}^\top \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} + 2\tilde{\theta}_{k-1}^\top \Gamma_{k-1}^{-1} (\theta_{k-1}^* - \theta_k^*) \\ &+ (\Delta^*)^2 \bar{\Gamma}_{\min}^{-1}. \end{split}$$

Since $\Upsilon_k \succ 0$, Υ_k can be decomposed as $\Upsilon_k = \Upsilon_k^{1/2} \Upsilon_k^{1/2}$. Applying Lemma 1 to (34), we have

$$\Gamma_k^{-1} = \bar{\Gamma}_k^{-1} - \underbrace{\bar{\Gamma}_k^{-1} \Upsilon_k^{1/2} \left(I + \Upsilon_k^{1/2} \bar{\Gamma}_k^{-1} \Upsilon_k^{1/2} \right)^{-1} \Upsilon_k^{1/2} \bar{\Gamma}_k^{-1}}_{\Psi_k}.$$
 (36)

Since $\Upsilon_k^{1/2} \succ 0$ and $\bar{\Gamma}_k^{-1} \succ 0$, applying the definition of positive definite matrices we obtain that $\Upsilon_k^{1/2}\bar{\Gamma}_k^{-1}\Upsilon_k^{1/2} \succ 0$. Also, it follows that $\Upsilon_k^{1/2}\bar{\Gamma}_k^{-1}\Upsilon_k^{1/2} \preceq \lambda_{\Gamma}\bar{\Gamma}_{\min}^{-1}\Gamma_{\max}I$. Therefore $\left(I + \Upsilon_k^{1/2}\bar{\Gamma}_k^{-1}\Upsilon_k^{1/2}\right)^{-1} \succeq (1 + \lambda_{\Gamma}\bar{\Gamma}_{\min}^{-1}\Gamma_{\max})^{-1}I$. Applying the definition of positive definite matrices again, we get $\Psi_k \succeq \Psi_{\min}I \succ 0$. Substitute Γ_k^{-1} in (36) into V_k , we get

$$V_{k} \leq \tilde{\theta}_{k-1}^{\top} \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} + \Delta^{*} \bar{\Gamma}_{\max}^{1/2} \bar{\Gamma}_{\min}^{-1} \sqrt{V_{k-1}} + (\Delta^{*})^{2} \bar{\Gamma}_{\min}^{-1}$$

$$\leq (1 - \underbrace{\Psi_{\min} \bar{\Gamma}_{\min}}_{\mu_{1}}) V_{k-1} + \Delta^{*} \underbrace{\bar{\Gamma}_{\max}^{1/2} \bar{\Gamma}_{\min}^{-1}}_{c_{1}} \sqrt{V_{k-1}}$$

$$+ (\Delta^{*})^{2} \underbrace{\bar{\Gamma}_{\min}^{-1}}_{c_{2}},$$
(37)

where we have replaced $\tilde{\theta}_{k-1}$ with an upper bound involving V_{k-1} . From (36) and $\Gamma_k^{-1} > 0$, it follows that $\Psi_{\min}\bar{\Gamma}_{\min} < 1$. Therefore, the increment of V_k , ΔV_k can be expressed as

$$\Delta V_k = V_k - V_{k-1}$$

$$\leq -\mu_1 V_{k-1} + c_1 \Delta^* \sqrt{V_{k-1}} + c_2 (\Delta^*)^2,$$

where μ_1 is defined in (28). Choosing any $0 < \mu_2 < \mu_1$, it can be noted that $\Delta V_k < 0$ in D_{PE}^c , where the compact set D_{PE} is defined in (29). For any k such that $V_k > V_{PE}$, it follows that

$$V_k \le (1 - \mu_2) V_{k-1}$$

We therefore obtain that when $k \geq k_2$, for all $V_k > V_{PE}$

$$V_k \le (1 - \mu_2)^{k - k_2} V_{k_2}$$

$$\le \exp\left[-\mu_2(k - k_2)\right] V_{k_2}.$$

Remark 5. The key steps that enabled the proof of Theorem 1 was 1) to express $\Gamma_k = \bar{\Gamma}_k + \Upsilon_k$, and 2) to define V_k only using $\bar{\Gamma}_k$ as in (19) rather than Γ_k . This provided the necessary tractability as it avoided matrix inversions involving Γ_k . From (19), it can be seen that $\bar{\Gamma}_k$ is similar to the standard gain matrix used in recursive least squares [3] and therefore retains the desired positive-definiteness property with suitable choice of the hyperparameters in (26). This in turn allowed the proof of Theorem 1.

Remark 6. Theorem 1 guarantees convergence of V_k to a compact set D_{PE} . For static parameters, i.e., $\theta_k^* \equiv \theta^*$ and $\Delta^* = 0$, from (30) it can be shown that under PE, $\tilde{\theta}_k$ converges exponentially fast towards the origin.

Remark 7. Note that under assumptions 1 and 2, (30) can also be written as

$$V_{PE} = 2\Delta^* \theta_{\text{max}}^* \left[\frac{c_1 + \sqrt{c_1^2 + 4c_2(\mu_1 - \mu_2)}}{2(\mu_1 - \mu_2)} \right]^2.$$
 (38)

From (38), note that the compact set scales linearly with both the bounds of the unknown parameter θ_{\max}^* and the time variation of the unknown parameter Δ^* .

Remark 8. A few comments about the choice of the hyperparameters in (7), (8) and (9) are in order. We assume that Γ_{max} is given. Parameter λ_{Ω} is chosen according to (21) and Ω_{max} is available according to Lemma 2. Then λ_{Γ} and κ will in turn be chosen from the bounds in (22), (25). Thus all hyperparameters can be determined and Γ_{PE} can be derived from (26).

Remark 9. Note that in practice, since we generally do not know the excitation level α , the period ΔT and the maximum normalization signal d, it is hard to use (21) as a guideline to choose λ_{Ω} . Thus careful tuning of the hyperparameters is required.

6 Stability and Convergence Analysis for Algorithm 2

As mentioned earlier, we assume that the unknown parameter θ^* is a constant and does not vary with k. When there is no persistent excitation in the input signal, the matrix Γ_k can exhibit bursting behavior and in fact drift to infinity[3, 25]. To avoid this, and prove stability, we introduce a projection operator in the update of Γ_k as a replacement of (9),

$$\Gamma_k = \operatorname{Proj}_{\Gamma_{\max}}(\Gamma_k'), \tag{39}$$

where

$$\Gamma_k' = \Gamma_{k-1} + \lambda_{\Gamma}(\Gamma_{k-1} - \kappa \Gamma_{k-1} \Omega_k \Gamma_{k-1}).$$

The function $\operatorname{Proj}_{\Gamma_{\max}}(\cdot)$ is defined as in Algorithm 3.

The following Lemma establishes boundedness of Γ_k in (39) with no assumptions on PE.

Lemma 6. If $\Gamma_{\min}I \leq \Gamma_0 \leq \Gamma_{\max}I$ and $\kappa \leq \frac{\lambda_{\Gamma}}{\lambda_{\Omega}}[(1+\lambda_{\Gamma})\Gamma_{\max} - \Gamma_{\min}]\Gamma_{\max}$, the update of Γ_k according to (39) ensures that $\Gamma_{\min}I \leq \Gamma_k \leq \Gamma_{\max}I$ for all $k \geq 0$.

Proof. Due to the projection in Algorithm 3, $\Gamma_k \leq \Gamma_{\max} I$, $\forall k \geq 0$. Since Γ_{k-1} is positive definite, Γ_{k-1} can be factorized as (16), thus

$$\begin{split} \Gamma_k &= \operatorname{Proj}_{\Gamma_{\max}}[\Gamma_{k-1} + \lambda_{\Gamma}(\Gamma_{k-1} - \kappa \Gamma_{k-1} \Omega_k \Gamma_{k-1})] \\ &\succeq \operatorname{Proj}_{\Gamma_{\max}}[\Gamma_{k-1} + \lambda_{\Gamma}(\Gamma_{k-1} - \kappa \Omega_{\max} \Gamma_{k-1}^2)] \\ &= \operatorname{Proj}_{\Gamma_{\max}}\{Q_{k-1}[\Lambda_{k-1} \\ &+ \lambda_{\Gamma}(\Lambda_{k-1} - \kappa \Omega_{\max} \Lambda_{k-1}^2)]Q_{k-1}^{\top}\} \end{split}$$

Denote the *i*th diagonal element of Λ_{k-1} as $\lambda_{k-1,i}$, $i=1,\ldots,N$. The smallest $\lambda_{k,i}$ will be obtained when $\lambda_{k-1,i}=\Gamma_{\max}$. From the upper bound of κ , $\lambda_{k,i}\geq\Gamma_{\min}$. Using induction, $\Gamma_k\succeq\Gamma_{\min}I$, $\forall k$.

Remark 10. Lemma 6 shows that with projection, Γ_k will be bounded even without persistent excitation. The projection in Algorithm 3 ensures that the upper bound will always be preserved. The lower bound of Γ_k is possible because Ω_k always has an upper bound.

Before stating the main stability results, we first define a few quantities. Let

$$0 < \lambda_{\Omega} < 1, \tag{40}$$

$$\lambda_{\Gamma} > 0, \tag{41}$$

$$\Gamma_{\min}I \le \Gamma_0 \le \Gamma_{\max}I. \tag{42}$$

Parameter κ satisfies

$$0 < \kappa < \min \left\{ \frac{1}{\Omega_{\max} \Gamma_{\min}}, \frac{\lambda_{\Gamma}}{\lambda_{\Omega}} [(1 + \lambda_{\Gamma}) \Gamma_{\max} - \Gamma_{\min}] \Gamma_{\max}, \frac{1}{\lambda_{\Gamma} \Gamma_{\max}}, \frac{1}{(1 - \lambda_{\Omega}) \Gamma_{\max} \Omega_{\max}} \right\},$$

$$(43)$$

and Γ_{\min} is defined as

$$\Gamma_{\min} = \min \left\{ \Gamma_{\max} + \lambda_{\Gamma} (\Gamma_{\max} - \kappa \Omega_{\max} \Gamma_{\max}^2), \frac{1}{\kappa \Omega_{\max}} \right\}, \tag{44}$$

where $\Omega_{\text{max}} = 1/\lambda_{\Omega}$. Finally, we define

$$\bar{\Gamma}_{\min} = \min \left\{ \Gamma_{\max} - \lambda_{\Gamma} \kappa \Gamma_{\max}^2, \Gamma_{\min} - \lambda_{\Gamma} \kappa \Gamma_{\min}^2 \right\}, \tag{45}$$

$$\Upsilon_{\min} = \lambda_{\Gamma} \min \left\{ \Gamma_{\min} - \kappa (1 - \lambda_{\Omega}) \Gamma_{\min}^{2} \Omega_{\max}, \right.
\left. \Gamma_{\max} - \kappa (1 - \lambda_{\Omega}) \Gamma_{\max}^{2} \Omega_{\max} \right\}.$$
(46)

Theorem 2. For the ARMA model in (1), if the parameters in θ^* are constant, and the hyperparameters λ_{Ω} , λ_{Γ} and κ satisfy (40), (41) and (43), respectively, the adaptive algorithm in (7), (8) and (39) guarantees that

$$V_k \leq V_{k-1}$$
.

Proof. From (7) and (19), the parameter error can be written as

$$\tilde{\theta}_k = \bar{\Gamma}_k \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1},\tag{47}$$

where $\tilde{\theta}_k$ is defined as $\tilde{\theta}_k = \theta_k - \theta^*$. Consider again a decomposition of Γ_k as in (34). Following the approach to proving that $\bar{\Gamma}_k$ and Υ_k are bounded in (34), it is easy to show that $0 \prec \bar{\Gamma}_{\min} I \preceq \bar{\Gamma}_k \preceq \bar{\Gamma}_{\max} I$ and $0 \prec \Upsilon_{\min} I \preceq \Upsilon_k \preceq \Upsilon_{\max} I$ by considering the hyperparameters in (40)-(43), where $\bar{\Gamma}_{\min}$, Υ_{\min} are defined in (45) and (46), respectively, and $\bar{\Gamma}_{\max} = \Gamma_{\max}$, $\Upsilon_{\max} = \lambda_{\Gamma} \Gamma_{\max}$. Substitute (47) into the Lyapunov function in (20) and expand it, we obtain

$$V_{k} = \tilde{\theta}_{k-1}^{\top} \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} - \lambda_{\Gamma} \kappa \frac{\tilde{\theta}_{k-1}^{\top} \phi_{k-1} \phi_{k-1}^{\top} \tilde{\theta}_{k-1}}{1 + \|\phi_{k-1}\|^{2}}.$$
 (48)

Since $\bar{\Gamma}_k \succ 0$ and $\Upsilon_k \succ 0$, the update of Γ_k in (39) indicates that $\Gamma_k \succeq \bar{\Gamma}_k$. Therefore,

$$V_{k} \leq \tilde{\theta}_{k-1}^{\top} \bar{\Gamma}_{k-1}^{-1} \tilde{\theta}_{k-1} - \lambda_{\Gamma} \kappa \frac{\tilde{\theta}_{k-1}^{\top} \phi_{k-1} \phi_{k-1}^{\top} \tilde{\theta}_{k-1}}{1 + \|\phi_{k-1}\|^{2}}$$

$$(49)$$

$$= V_{k-1} - \lambda_{\Gamma} \kappa \frac{\tilde{\theta}_{k-1}^{\top} \phi_{k-1} \phi_{k-1}^{\top} \tilde{\theta}_{k-1}}{1 + \|\phi_{k-1}\|^2}.$$
 (50)

Thus
$$\Delta V_k \leq 0$$
.

Remark 11. The second main contribution of the paper is Theorem 2. While Algorithm 2 is fairly similar to RLS with forgetting factor, Theorem 2 guarantees that stability can be ensured with Algorithm 2. To our knowledge, RLS with forgetting factor has not been shown to be stable under non-PE conditions. Extension to the case of θ_k^* is a topic for future work, which will require a combined examination of (36) and Algorithm 3.

7 Numerical Simulations

A discretized and linearized model of F-16 longitudinal dynamics with a sample rate of 10 Hz is given by [26],

$$G(\mathbf{q}) = \frac{-0.6213\mathbf{q} + 0.5839}{\mathbf{q}^2 - 1.8403\mathbf{q} + 0.8591}.$$
(51)

The problem is that the coefficients of (51) are unknown and we propose the use of Algorithms 1 and 2 to identify them using an input

$$u(k) = \sin\left(\frac{3\pi k}{4}\right) + \sin\left(\frac{2\pi k}{5}\right) + \sin\left(\frac{\pi k}{5}\right). \tag{52}$$

We compare our algorithms with the standard RLS algorithm

$$\theta_k = \theta_{k-1} + \frac{\Gamma_{k-1}\phi_{k-1}e_{k-1}}{1 + \phi_{k-1}^{\mathsf{T}}\Gamma_{k-2}\phi_{k-1}},\tag{53}$$

$$\Gamma_{k-1} = \Gamma_{k-2} - \frac{\Gamma_{k-2}\phi_{k-1}\phi_{k-1}^{\top}\Gamma_{k-2}}{1 + \phi_{k-1}^{\top}\Gamma_{k-2}\phi_{k-1}},\tag{54}$$

and the RLS with forgetting chosen as

$$\theta_k = \theta_{k-1} + \frac{\Gamma_{k-1}\phi_{k-1}e_{k-1}}{\alpha + \phi_{k-1}^{\mathsf{T}}\Gamma_{k-2}\phi_{k-1}},\tag{55}$$

$$\Gamma_{k-1} = \frac{1}{\alpha} \left[\Gamma_{k-2} - \frac{\Gamma_{k-2} \phi_{k-1} \phi_{k-1}^{\top} \Gamma_{k-2}}{\alpha + \phi_{k-1}^{\top} \Gamma_{k-2} \phi_{k-1}} \right], \tag{56}$$

where $\alpha = 0.99$. In all simulations, we choose the initial values as $\Gamma_0 = 3I$ and $\theta_0 = 0$.

7.1 Experiment 1

In experiment 1, we choose the following parameters for (7)-(9):

$$\lambda_{\Omega} = 0.1, \ \lambda_{\Gamma} = 0.4, \ \kappa = 15.$$

Fig. 3 shows the output and parameter error over 100 seconds for Algorithm 1. It can be noted that both RLS with forgetting and Algorithm 1 are faster than the standard RLS algorithm, with Algorithm 1 producing somewhat larger learning rates. Fig. 4 shows the evolution of time varying gain matrices in RLS with forgetting algorithm and our algorithm.

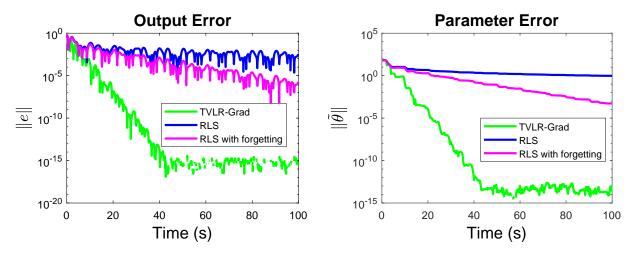


Figure 3: Experiment 1: Output error and parameter error of the three algorithms.

7.2 Experiment 2

In experiment 2, the following parameters are chosen in Algorithm 2.

$$\lambda_{\Omega} = 0.05, \ \lambda_{\Gamma} = 0.4, \ \kappa = 15, \ \Gamma_{\text{max}} = 15.$$

Fig. 5 shows the resulting output error and parameter error. In addition to the obvious effect of a lower learning rate due to the projection, we also note that compared to Fig. 3, the convergence of our algorithm becomes slower. Fig. 6 shows the evolution of the time-varying gain matrix in RLS with forgetting and our algorithm. Compared to Fig. 4, the values in Γ_k in our algorithm become a lot smaller and are comparable to those of RLS with forgetting. The advantage of Algorithm 2 over RLS with forgetting, as mentioned in the introduction, is a stability guarantee established above in Theorem 2.

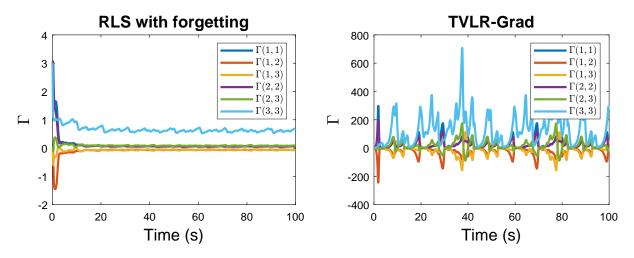


Figure 4: Experiment 1: Histories of the learning rate matrix.

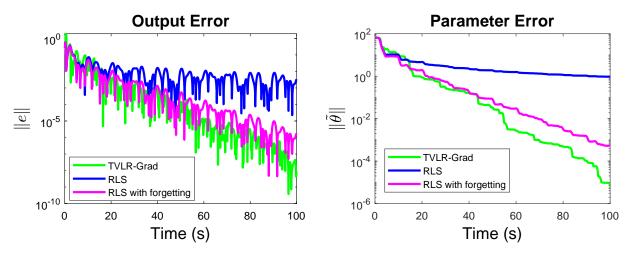


Figure 5: Experiment 2: Output error and parameter error of the three algorithms.

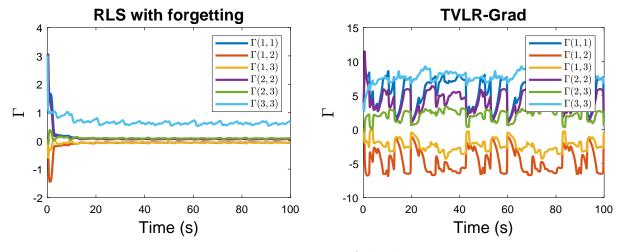


Figure 6: Experiment 2: Histories of the learning rate matrix.

7.3 Experiment 3

In experiment 3, we make the underlying true parameters time-varying through multiplying the true parameters by a sinusoidal wave. The time-varying parameters with become

$$\theta_k^* = [1 - 0.01 * \sin(0.02k)] \cdot \theta^*,$$

where θ^* has the same parameters we try to identify in experiment 1 and experiment 2. We adopt the same parameters for Algorithm 2 as in experiment 2. Fig. 7 shows the output and parameter error. All parameters in the three algorithms converge to a compact set. The performance of Algorithm 2 is comparable to that of RLS with forgetting. Fig. 8 shows the evolution of the time-varying gain matrix in RLS with forgetting and our Algorithm 2. Again, due to the projection operator, the magnitudes are comparable to those of RLS with forgetting.

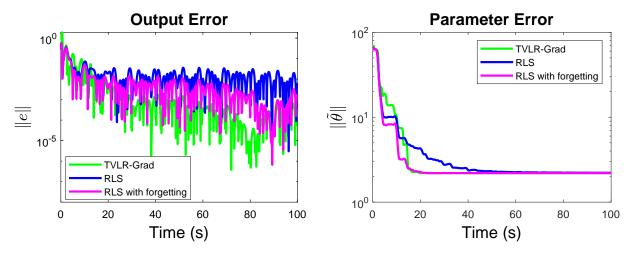


Figure 7: Experiment 3: Output error and parameter error of the three algorithms.

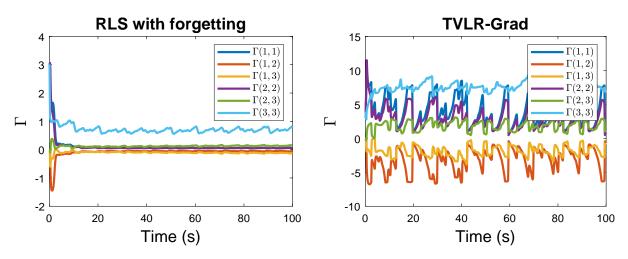


Figure 8: Experiment 3: Histories of the learning rate matrix.

8 CONCLUSIONS

We have presented new adaptive algorithms, one for discrete-time parameter estimation of a class of time-varying plants when PE conditions are met, and another when no PE conditions are assumed and the plant parameters are constants. We showed that in the presence of time-varying parameters, the parameter estimation error converges uniformly to a compact set under conditions of persistent excitation, with the size of the compact set proportional to the time-variation of the unknown parameters. The bounds for the compact set are shown to scale linearly with both the bounds of the unknown parameter and the time variation of the unknown parameter. The second algorithm leverages a projection operator, which is then shown to result in boundedness guarantees through the use of a Lyapunov function. While this algorithm is similar to RLS with forgetting factor, the important distinction is the proof of stability.

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