

Minimax Adaptive Estimation for Finite Sets of Linear Systems

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Abstract

For linear time-invariant systems with uncertain parameters belonging to a finite set, we present a purely deterministic approach to multiple-model estimation and propose an algorithm based on the minimax criterion using constrained quadratic programming. The estimator tends to learn the dynamics of the system, and once the uncertain parameters have been sufficiently estimated, the estimator behaves like a standard Kalman filter.

1 INTRODUCTION

1.1 Problem Statement

In this article, we consider output prediction for linear systems of the form

$$\begin{aligned} x_{t+1} &= Fx_t + Gu_t + w_t \\ y_t &= Hx_t + v_t, \quad 0 \leq t \leq N-1, \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^p$ and $y_t \in \mathbb{R}^m$ are the states and the measured input and output at time-step t , respectively. $w_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^m$ are unmeasured

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process disturbance and measurement noise. The model, (F, H, G) is fixed but unknown, belonging to some finite set

$$\{(F_1, H_1, G_1), \dots, (F_K, H_K, G_K)\}.$$

consisting of triplets of real-valued matrices. In particular, we are interested in strictly causal estimation of y_N , such that the gain from disturbance trajectories $(w_t, v_t)_{t=0}^{N-1}$ to pointwise estimation error $(y_N - Hx_N)$ in some weighed ℓ_2 -norm is bounded by a constant $\gamma_N > 0$. This means that given positive definite matrices $P_0 \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$ and a nominal value of the initial state, \hat{x}_0 ,

$$\frac{|\hat{y}_N - Hx_N|^2}{|x_0 - \hat{x}_0|_{P_0^{-1}}^2 + \sum_{t=0}^{N-1} (|w_t|_{Q^{-1}}^2 + |v_t|_{R^{-1}}^2)} \leq \gamma_N^2, \quad (2)$$

should hold for all disturbances and models compatible with the measurement history $(y_t, u_t)_{t=0}^{N-1}$. This approach is different from the Bayesian approach to filtering where one takes the conditional expectation as the estimate \hat{y}_N . The interest in worst-case gain is motivated by robust feedback-control from estimates. In such settings instability or lack of performance due to model errors is a larger concern than robustness to outliers.

1.2 Background

Simultaneous estimation of states and parameters in linear systems is a bilinear estimation problem. The Maximum-likelihood approach leads to estimates which cannot be put in recursive form and must be obtained by iteration Bar-Shalom [1972]. A recursive method can be obtained by parametrizing the dynamical equations and the observer and learning the parameters using the *sequential prediction error* approach. Alternatively, one can augment the state vector with the uncertain parameters and apply nonlinear filtering methods such as the Extended Kalman filter Goodwin and Sin [1984]. Unfortunately, optimality guarantees for such methods are difficult to obtain. One exception is when the system can be modeled as a finite set of linear systems and the noise is Gaussian, then the Maximum-likelihood estimates can be put on a recursive form Crassidis and Junkins [2011].

Solutions based on the multiple-model approach have been tremendously successful in modeling and estimating complex engineering systems. In essence,

it consists of two parts: 1) design simpler models for a finite set of possible operating regimes. 2) Run a filter for each model and cleverly combine the estimates. Multiple-model adaptive estimation has been around since the '60s Magill [1965], Lainiotis [1976] and has been an active research field since. The estimation approach easily extends to systems where the active model can switch (hybrid systems) by matching a Kalman filter with each possible trajectory. In that case, the number of filters will grow exponentially, which has sparked research into more efficient methods. Notable numerically tractable and suboptimal algorithms for estimation in hybrid systems are the Generalized Pseudo Bayesian Ackerson and Fu [1970], Chang and Athans [1978], and the Interacting Multiple Model Blom and Bar-Shalom [1988]. The algorithms have been coupled with extended and unscented Kalman filters to deal with non-linear systems Akca and Önder Efe [2019], and Xiong et al. [2015] studied robustness to identification error. In Ronghua et al. [2008], the authors pointed out that methods based on Kalman filters are sensitive to noise distributions and proposed an Interactive Multiple Model algorithm based on particle filters to handle non-Gaussian noise at the expense of a 100 fold increase in computation. Recently, machine-learning approaches to classification have been combined with the Interacting Multiple Model estimator Li et al. [2021], Deng et al. [2020] and showed improved accuracy in simulations.

The Bayesian approach to the *Multiple-model* estimation problem involves assigning probability distributions to disturbances (w_t, v_t) and models (F, G, H) . The estimate is taken as the expected value of y_N conditioned on past measurements. If the disturbances are zero-mean and Gaussian, then the conditional expectation can be computed as the weighted average of Kalman filter estimates (one for each model), weighted by the conditional probability that its model is active.

It is evident in practice that the estimator's performance depends on the quality of the model set. The models must be distinguishable using measured signals, and the models should accurately describe the operating regimes. Since the estimates can be susceptible to non-Gaussian noise, it is surprising that deterministic approaches similar to those studied by the control community in the '80s and '90s have gathered little attention. Recent progress to minimax adaptive control of linear systems with uncertain parameters belonging to a finite set Rantzer [2021] under the assumption of *perfect measurements* has inspired this research into compatible estimation techniques.

1.3 Contribution

In this paper, we formulate the multiple-model estimation problem as a deterministic, two-player dynamic game. In particular, this formulation allows for online computation of the worst-case gain from disturbances to estimation error and tractable synthesis of suboptimal estimators that minimize the worst-case gain. Deterministic dynamic games have played a key role in solving and understanding \mathcal{H}_∞ filtering Shen and Deng [1997], Basar and Bernhard [1995]; our goal in this work has been to take a first step towards extending the advantages of that framework to the multiple model setting.

1.4 Outline

The outline is as follows: First, we introduce notation in Section 2, then we introduce minimax multiple-model filtering and the main results in Section 3. In Section 4, we present a simplified form for time-invariant systems. We illustrate the theory through a numerical example in Section 5. Section 6 contains concluding remarks, and supporting lemmata are given in the Appendix.

2 NOTATION

The set of $n \times m$ -dimensional matrices with real coefficients is denoted $\mathbb{R}^{n \times m}$. The transpose of a matrix A is denoted A^\top . For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, we write $A \succ (\succeq) 0$ to say that A is positive (semi)definite. Given $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, $|x|_A^2 := x^\top A x$. For a vector $x_t \in \mathbb{R}^n$ we denote the sequence of such vectors up to time t by $\mathbf{x}^t := (x_k)_{k=0}^t$.

3 MINIMAX MULTIPLE MODEL FILTERING

In contrast to the Bayesian approach, our approach is fully deterministic; similarly to Shen and Deng [1997], Basar and Bernhard [1995], we do not make explicit assumptions on the distribution of the noise trajectories \mathbf{w}^t and \mathbf{v}^t . We will instead construct a two-player dynamic game between a minimizing player that chooses the estimate, and a maximizing player that chooses dynamics and disturbances. Recall that we are interested in characterizing an

estimator \hat{y}_N such that the gain from disturbances to the pointwise estimation error is bounded by γ_N . I.e., (2) holds for all disturbances consistent with (1) and the data $(\mathbf{y}^{N-1}, \mathbf{u}^{N-1})$. Since the disturbances are unknown, we cannot evaluate (2) directly. However, define

$$J_N(\mathbf{y}^{N-1}, \mathbf{u}^{N-1}, \hat{y}_N) := \sup_{x_0, \mathbf{w}^{N-1}, \mathbf{v}^{N-1}, (F, G, H)} \left\{ |\hat{y}_N - Hx_N|^2 - \gamma_N^2 \left(|x_0 - \hat{x}_0|_{P_0^{-1}}^2 + \sum_{t=0}^{N-1} \left(|w_t|_{Q^{-1}}^2 + |v_t|_{R^{-1}}^2 \right) \right) \right\}, \quad (3)$$

where the maximization is performed subject to the constraints (1). Then (2) holds if and only if

$$J_N(y^{N-1}, u^{N-1}, \hat{y}_N) \leq 0.$$

In this setting, $w_t = x_{t+1} - Fx_t - Gu_t$ and $v_t = y_t - Hx_t$ are uniquely determined by the states, the measurements and the active model. Inserting into (3), we get

$$J_N(\mathbf{y}^{N-1}, \mathbf{u}^{N-1}, \hat{y}_N) = \sup_{\mathbf{x}^N, (F, G, H)} \left\{ |\hat{y}_N - Hx_N|^2 - \gamma_N^2 |x_0 - \hat{x}_0|_{P_0^{-1}}^2 - \gamma_N^2 \sum_{t=0}^{N-1} \left(|x_{t+1} - Fx_t - Gu_t|_{Q^{-1}}^2 + |y_t - Hx_t|_{R^{-1}}^2 \right) \right\}. \quad (4)$$

We will call an estimator \hat{y}_N^* a minimax estimator if

$$\inf_{\hat{y}_N} J_N(y^{N-1}, u^{N-1}, \hat{y}_N) = J_N(y^{N-1}, u^{N-1}, \hat{y}_N^*) =: J_N^*(y^{N-1}, u^{N-1}), \quad (5)$$

holds, where \hat{y}_N are functions of past data \mathbf{y}^{N-1} and u^{N-1} . This constitutes a two-player dynamic game and would be linear quadratic if not for the model being chosen by the maximizing player. The intuition behind (5) makes sense in the following way. The minimizing player is penalized for deviating from the true (noiseless) output, and the maximizing player is penalized for selecting a model which requires large disturbances w and v to be compatible with the data. As N increases, the penalty for selecting a model different from the truth grows too large, resulting in a learning mechanism. It turns

out that the cost associated with the disturbance trajectories required to explain each model corresponds to the accumulated prediction errors from a corresponding Kalman filter and that the minimax estimate is a weighted interpolation between the Kalman filter estimates.

Theorem 1. Consider matrices $F_1, \dots, F_K \in \mathbb{R}^{n \times n}$, $H_1, \dots, H_K \in \mathbb{R}^{m \times n}$, $G_1, \dots, G_K \in \mathbb{R}^{n \times p}$ and positive definite $Q, P_0 \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$. Define $P_{t,i}$ according to

$$\begin{aligned} P_{0,i} &= P_0 \\ P_{t+1,i} &= Q + F_i(P_{t,i} - P_{t,i}H_i^\top(R + H_iP_{t,i}H_i^\top)^{-1}H_iP_{t,i})F_i^\top, \end{aligned}$$

and assume that $H_iP_{N,i}H_i^\top \prec \gamma_N^2 I$. Then the cost (4) is equivalent to

$$J_N(\mathbf{y}^{N-1}, u^{N-1}, \hat{y}_N) = \max_i \left\{ |\hat{y}_N - H_i \check{x}_{N,i}|_{(I - \gamma_N^{-2} H_i P_{N,i} H_i^\top)^{-1}}^2 - \gamma_N^2 c_{N,i} \right\}. \quad (6)$$

$\check{x}_{N,i}$ is the Kalman filter estimate of x_N using the i th model, and $c_{N,i}$ are generated according to

$$\begin{aligned} \check{x}_{0,i} &= x_0 \\ \check{x}_{t+1,i} &= F_i \check{x}_{t,i} + K_{t,i}(y_t - H_i \check{x}_{t,i}) + G_i u_t \\ K_{t,i} &= F_i P_{t,i} H_i^\top (R + H_i P_{t,i} H_i^\top)^{-1} \\ c_{0,i} &= 0 \\ c_{t+1,i} &= |H_i \check{x}_{t,i} - y_t|_{(R + H_i P_{t,i} H_i^\top)^{-1}}^2 + c_{t,i}. \end{aligned}$$

Proof. We will perform the maximization over state-trajectories in (4) in two steps. First over past trajectories (\mathbf{x}^{N-1}) and then over the future state x_N . The right-hand side of (4) becomes

$$\sup_{x_N, i} \left\{ |\hat{y}_N - H_i x_N|^2 - \gamma_N^2 \inf_{\mathbf{x}^{N-1}} \left\{ |x_0 - \hat{x}_0|_{P_0^{-1}}^2 + \sum_{t=0}^{N-1} (|x_{t+1} - F_i x_t - G_i u_t|_{Q^{-1}}^2 + |y_t - H_i x_t|_{R^{-1}}^2) \right\} \right\},$$

¹ $\max_{\mathbf{x}^N} \{\dots\} = \max_{x_N} \{\max_{\mathbf{x}^{N-1}} \{\dots\}\}.$

where $i = 1, \dots, K$ is an index for the active model (F_i, H_i, G_i) . Apply Lemma 4 to get

$$\begin{aligned} J_N(y^{N-1}, u^{N-1}, \hat{y}_N) &= \sup_{x_N, i} \{ |\hat{y}_N - H_i x_N|^2 - \gamma_N^2 V_{N,i}((x_N, y^{N-1})) \} \\ &= \sup_{i, x_N} \left\{ |\hat{y}_N - H_i x_N|^2 - \gamma_N^2 \left(|x_N - \check{x}_N|_{P_{N,i}^{-1}}^2 + c_{N,i} \right) \right\}. \end{aligned}$$

For fix \hat{y}_N and i , the assumption $H_i P_{N,i} H_i^\top \prec \gamma_N^2 I$ guarantees that we maximize a concave function of x_N and we apply Lemma 5 with $A = H_i$, $X = I$, $Y = P_{N,i}$ to conclude²,

$$J_N(y^{N-1}, u^{N-1}, \hat{y}_N) = \max_i |\hat{y}_N - H_i \check{x}_{N,i}|_{(I - \gamma_N^{-2} H_i P_{N,i} H_i^\top)^{-1}}^2 - \gamma_N^2 c_{N,i}.$$

□

Remark 1. *Theorem 1 holds also for time-varying systems, if F_i and H_i are replaced by $F_{t,i}$ and $H_{t,i}$. Further, P_0 , Q and R can be time-varying and differ between models.*

Remark 2. *Equation (6) is monotonically increasing in γ_N and the smallest γ_N^* such that $J_N(y^{N-1}, u^{N-1}, \hat{y}_N) \leq 0$ can be found efficiently through bisection.*

The below Corollary follows from Theorem 1 and describes how to compute the minimax estimator as a convex quadratic program.

Corollary 2. *With assumptions as in Theorem 1, consider the convex program*

$$\begin{aligned} &\underset{\hat{y}_N, t}{\text{minimize}} && t \\ &\text{subject to:} && |\hat{y}_N - H_i \check{x}_{N,i}|_{(I - \gamma_N^{-2} H_i P_{N,i} H_i^\top)^{-1}}^2 - \gamma_N^2 c_{N,i} \leq t \\ &&& \forall i = 1 \dots K. \end{aligned}$$

The minimizing argument \hat{y}_N^ satisfies (5).*

Remark 3. *If the model set is a singleton, then $\hat{y}_N^* = H x_N^* = H \check{x}_N$ is the estimate generated by the Kalman filter, which is a well known result Basar and Bernhard [1995].*

²The maximizing argument is given by $x_N^*(\hat{y}_N, i) = (H_i^\top H_i - \gamma_N^2 P_{N,i}^{-1})^{-1} (H_i^\top \hat{y}_N - P_{N,i}^{-1} \gamma_N^2 \check{x}_{N,i})$

3.1 On $c_{N,i}$ and the relation to conditional probability.

It is known (see for instance Crassidis and Junkins [2011]) that if w_t and v_t are uncorrelated Gaussian white noise with covariances Q and R , the conditional probability that the measured output \mathbf{y}^N has been generated by the model (F_i, G_i, H_i) and the input \mathbf{u}^N can be expressed as

$$p(i|\mathbf{y}^N, \mathbf{u}^N) = \frac{\alpha_N e^{-|y_N - H_i \check{x}_{N,i}|_{\tilde{R}_{N,i}}^2}}{\det(2\pi \tilde{R}_{N,i})^{1/2}} p(i|\mathbf{y}^{N-1}, \mathbf{u}^{N-1}).$$

α_N is some normalization constant independent of i , and

$$\tilde{R}_{N,i} = R + H_i P_{N,i} H_i^\top,$$

with $P_{N,i}$ as in Theorem 1. Taking $c_{N,i}$ as in Theorem 1 we see that the conditional probability is proportional to $e^{-c_{N+1,i}}$,

$$p(i|\mathbf{y}^{N-1}, \mathbf{u}^{N-1}) \propto e^{-c_{N+1,i}} \prod_{t=1}^N \det(2\pi \tilde{R}_{t,i})^{-1/2}.$$

4 STATIONARY SOLUTION

For a set of time-invariant systems, we summarize a simple version of the filter in the below theorem.

Theorem 3. *Consider matrices $F_1, \dots, F_K \in \mathbb{R}^{n \times n}$, $H_1, \dots, H_K \in \mathbb{R}^{m \times n}$ and positive definite $Q, P_0 \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$. Assume that the algebraic Riccati equations*

$$P_i = Q + F_i(P_i - P_i H_i^\top (R + H_i P_i H_i^\top)^{-1} H_i P_i) F_i^\top,$$

have solutions $H_i P_i H_i^\top \prec \gamma_N^2 I$. Then a minimax strategy \hat{y}_N^ for the game defined by*

$$\min_{\hat{y}_N} \max_{\mathbf{x}^{N,i}} \left\{ |\hat{y}_N - H_i x_N|^2 - \gamma_N^2 |x_0 - \hat{x}_0|_{P_i^{-1}}^2 - \gamma_N^2 \sum_{t=0}^{N-1} \left(|x_{t+1} - F_i x_t - G_i u_t|_{Q^{-1}}^2 + |y_t - H_i x_t|_{R^{-1}}^2 \right) \right\},$$

and (1), is the minimizing argument of

$$\min_{\hat{y}_N} \max_i \left\{ |\hat{y}_N - H_i \check{x}_{N,i}|^2_{(I - \gamma_N^{-2} H_i P_i H_i^\top)^{-1}} - \gamma_N^2 c_{N,i} \right\}.$$

$\check{x}_{N,i}$ is the Kalman filter estimate of x_N using the i th model, and $c_{N,i}$ are generated according to

$$\begin{aligned} \check{x}_{0,i} &= x_0 \\ \check{x}_{t+1,i} &= F_i \check{x}_{t,i} + K_i (y_t - H_i \check{x}_{t,i}) + G_i u_t \\ K_i &= F_i P_i H_i^\top (R + H_i P_i H_i^\top)^{-1} \\ c_{0,i} &= 0 \\ c_{t+1,i} &= |H_i \check{x}_{t,i} - y_t|_{(R + H_i P_i H_i^\top)^{-1}}^2 + c_{t,i}. \end{aligned}$$

Proof. This is a special case of Theorem 1, by replacing P_0 with P_i . \square

5 EXAMPLE

In this example, we compare a minimax estimator synthesized using Corollary 2, bisecting over γ_N , to find the estimator \hat{y}_N^* such that (2) is satisfied for the smallest possible γ_N . We compare this to a Bayesian multiple-model estimator Crassidis and Junkins [2011] and calculate the corresponding bound γ_N using Theorem 1 and bisection. Consider the uncertain linear system

$$\begin{aligned} x_{t+1} &= F x_t + w_t \\ y_t &= x_t + v_t \end{aligned}, \quad F \in \{-1, 1\}.$$

The weights in (2) are chosen to be $Q = R = P_0 = 1$. We generate data \mathbf{y}^{N-1} by simulating the system with $F = 1$ and w_t, v_t as independent Gaussian white noise with intensity 1. For $N = 5$ we find

$$\begin{aligned} P_{5,1} &= P_{5,-1} = 1.62, \\ \check{x}_{5,1} &= -2.34, \quad \check{x}_{5,-1} = 1.50, \\ c_{5,1} &= 3.56, \quad c_{5,-1} = 8.11. \end{aligned}$$

In Fig. 1, we illustrate (6) for $N = 5$ and the estimates. Note that $\gamma = 1.51$ can be guaranteed for the minimax estimator, but not the Bayesian. Fig. 2 contains a comparison between the smallest γ_N so that (2) can be guaranteed for the minimax estimator and the Bayesian estimator when $N = 1 \dots 20$.

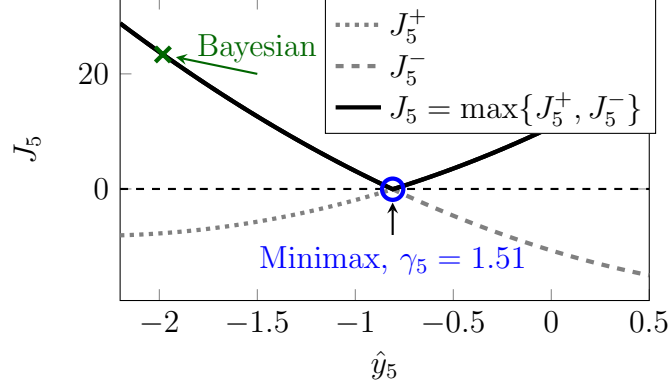


Figure 1: Illustration of the optimization problem (6) for $N = 5$, together with the minimax solution and the one given by a Bayesian multiple model estimator for $\gamma_N = 1.51$. The minimax estimate has a guaranteed worst-case gain bound from disturbances to observer error lower than 1.51, whereas the Bayesian estimator does not. Here $J_5^+ = |\hat{y}_5 - \check{x}_{5,1}|_{(I - \gamma_5^{-2} P_{5,1})^{-1}}^2 - c_{5,1}$ corresponds to $F = 1$, whereas J_5^- (defined similarly) corresponds to $F = -1$. $J_5 = J_5(\mathbf{y}^5, 0, \hat{y}_5)$ is then equivalent to (6).

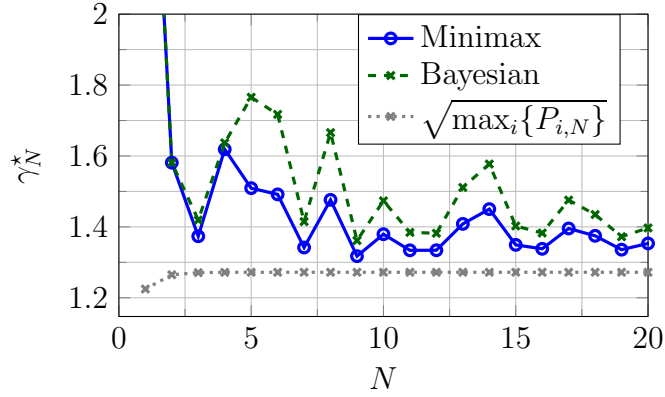


Figure 2: The smallest γ_N such that $J_N(\mathbf{y}^{N-1}, 0, \hat{y}_N) \leq 0$ for the minimax estimator (blue) compared to the Bayesian multiple-model adaptive estimator (green) for one realization.

6 CONCLUSIONS

We stated the minimax criterion for output prediction, where the dynamics belong to a finite set of linear systems and proposed a minimax estimation strategy. The strategy can be implemented as a convex program, and the resulting estimate is a weighted interpolation of Kalman filter estimates. We showed in a numerical example how to apply the theoretical results to compute the worst-case gain from disturbances to error for any multi-model estimation algorithm online and how to generate estimates that minimize the said gain.

By running a minimax estimator in parallel to another estimator, we can measure the worst-case performance level of the other estimator. A large difference in performance levels indicates that the nominal estimator may be highly sensitive to errors in the noise model.

Predetermining the smallest achievable gain from disturbances to estimation errors is still an open research problem, that is, finding necessary and sufficient conditions such that

$$\sup_{\mathbf{y}^{N-1}} J_N^*(\mathbf{y}^{N-1}, \mathbf{u}^{N-1}) \leq 0.$$

In future work, we plan to develop a Multiple-model adaptive estimator with a prescribed ℓ_2 -gain bound from disturbance to error and methods for infinite sets of linear systems.

APPENDIX — SUPPORTING LEMMATA

Lemma 4. *The cost function*

$$V_{N,i}(x_N, \mathbf{y}^{N-1}) = \min_{\mathbf{x}^{N-1}} \left\{ |x_0 - \hat{x}_0|_{P_0^{-1}}^2 + \sum_{k=1}^{N-1} (|x_{t+1} - F_i x_t - G_i u_t|_{Q^{-1}}^2 + |y_t - H_i x_t|_{R^{-1}}^2) \right\} \quad (7)$$

under the dynamics (1), is of the form

$$V_{t,i}(x, \mathbf{y}^{t-1}) = |x - \check{x}_{t,i}|_{P_{t,i}}^2 + c_{t,i},$$

where $P_{t,i}$ and $c_{t,i}$ are generated as

$$\begin{aligned}
P_{0,i} &= P_0 \\
P_{t+1,i} &= Q + F_i P_{t,i} F_i^\top \\
&\quad - F_i P_{t,i} H_i^\top (R + H_i P_{t,i} H_i^\top)^{-1} H_i P_{t,i} F_i^\top \\
\check{x}_{0,i} &= x_0 \\
\check{x}_{t+1,i} &= F_i \check{x}_{t,i} + K_{t,i} (y_t - H_i \check{x}_{t,i}) + G_i u_t \\
K_{t,i} &= F_i P_{t,i} H_i^\top (R + H_i P_{t,i} H_i^\top)^{-1} \\
c_{0,i} &= 0 \\
c_{t+1,i} &= |H_i \check{x}_{t,i} - y_t|_{(R + H_i P_{t,i} H_i^\top)^{-1}}^2 + c_{t,i}.
\end{aligned}$$

Proof. The proof builds on forward dynamic programming Cox [1964], and is similar to one given in Goodwin et al. [2005] but differ in the assumption that F_i is not invertible. Further, the constant terms $c_{t,i}$ are explicitly computed. The cost function V_N^3 can be computed recursively

$$V_1(x, \mathbf{y}^0) = |x - x_0|_{P_0^{-1}}^2 \quad (8)$$

$$\begin{aligned}
V_{t+1}(x, \mathbf{y}^t) &= \min_{\xi} |x - F\xi - Gu_t|_{Q^{-1}}^2 \\
&\quad + |y_t - H\xi|_{R^{-1}}^2 + V_t(\xi, \mathbf{y}^{t-1}).
\end{aligned} \quad (9)$$

With a slight abuse of notation, we assume a solution of the form $V_t(x) = |x - \check{x}_t|_{P_t^{-1}}^2 + c_t$ and solve for the minimum

$$\begin{aligned}
V_{t+1}(x) &= \min_{\xi} |x - Gu_t|_{Q^{-1}}^2 + |\xi|_{F^\top Q^{-1}F + H^\top R^{-1}H + P_t^{-1}}^2 \\
&\quad - 2(F^\top Q^{-1}(x - Gu_t) + H^\top R^{-1}y_t + P_t^{-1}\check{x}_t)^\top \xi + |y_t|_{R^{-1}}^2 + |\check{x}_t|_{P_t^{-1}}^2.
\end{aligned}$$

Assume at this stage $S_t := F^\top Q^{-1}F + H^\top R^{-1}H + P_t^{-1} \succ 0$, then the minimizing ξ^* is a stationary point

$$\xi^* = S_t^{-1}(F^\top Q^{-1}(x - Gu_t) + H^\top R^{-1}y_t + P_t^{-1}\check{x}_t)$$

and the resulting partial cost

$$\begin{aligned}
|x - \check{x}_{t+1}|_{P_{t+1}^{-1}}^2 + c_{t+1} &= |x - Gu_t|_{Q^{-1}}^2 + |y_t|_{R^{-1}}^2 + |\check{x}_t|_{P_t^{-1}}^2 \\
&\quad - |F^\top Q^{-1}(x - Gu_t) + H^\top R^{-1}y_t + P_t^{-1}\check{x}_t|_{S_t^{-1}}^2 + c_t.
\end{aligned} \quad (10)$$

³We relax the index i in this proof

Since this should hold for arbitrary x and

$$x - \check{x}_{t+1} = (x - Gu_t) - (\check{x}_{t+1} - Gu_t),$$

we get

$$\begin{aligned} P_{t+1}^{-1} &= Q^{-1} - Q^{-1}FS_t^{-1}F^\top Q^{-1} \\ \check{x}_{t+1} - Gu_t &= P_{t+1}Q^{-1}FS_t^{-1}(H^\top R^{-1}y_t + P_t^{-1}\check{x}_t) \end{aligned}$$

The expression for calculating P_{t+1} can be further simplified using the Woodbury identity,

$$\begin{aligned} P_{t+1}^{-1} &= (Q + F(H^\top R^{-1}H + P_t^{-1})^{-1}F^\top)^{-1} \\ P_{t+1} &= Q + FP_tF^\top - FP_tH^\top(R + HP_tH^\top)^{-1}HP_tF^\top, \end{aligned}$$

where we used the Woodbury matrix identity twice. Inserting these expressions into (10), applying the Woodbury matrix identity to $S_t^{-1}F^\top(Q - FS_t^{-1}F^\top)^{-1}S_t^{-1} + S_t^{-1} = (S_t - F^\top Q^{-1}F)^{-1} = (H^\top R^{-1}H + P_t^{-1})^{-1}$ gives

$$\begin{aligned} c_{t+1} &= -|H^\top R^{-1}y_t + P_t^{-1}\check{x}_t|_{(H^\top R^{-1}H + P_t^{-1})^{-1}}^2 + |y_t|_{R^{-1}}^2 + |\check{x}_t|_{P_t^{-1}}^2 + c_t \\ &= |H\hat{x}_t - y_t|_{(R + HP_tH^\top)^{-1}}^2 + c_t \end{aligned}$$

Next we show that \check{x} can be formulated as a state-observer

$$\begin{aligned} \check{x}_{t+1} - Gu_t &= P_{t+1}Q^{-1}FS_t^{-1}(H^\top R^{-1}y_t + P_t^{-1}\check{x}) \\ &= P_{t+1}Q^{-1}FS_t^{-1}H^\top R^{-1}(y_t - H\check{x}_t) \\ &\quad + P_{t+1}Q^{-1}FS_t^{-1}(H^\top R^{-1}H + P_t^{-1})\check{x}_t \end{aligned}$$

Use the matrix inversion lemma $(A + BCD)^{-1}BC = A^{-1}B(C + DA^{-1}B)^{-1}$.

$$\begin{aligned} P_{t+1}Q^{-1}FS_t^{-1} &= -(-Q^{-1} + Q^{-1}FS_t^{-1}F^\top Q^{-1})^{-1}Q^{-1}FS_t^{-1} \\ &= -(-Q^{-1})^{-1}(Q^{-1}F)(S_t - F^\top Q^{-1}F)^{-1} \\ &= F(H^\top R^{-1}H + P_t^{-1})^{-1}. \end{aligned}$$

Insert in to the previous expression and conclude

$$\check{x}_{t+1} = F\check{x}_t + K_t(y_t - H\check{x}) + Gu_t,$$

where

$$K_t = FP_tH^\top(R + HP_tH^\top)^{-1}$$

□

Lemma 5. For $x \in \mathbb{R}^n$, $v, y \in \mathbb{R}^m$, a non-zero matrix $A \in \mathbb{R}^{n \times m}$, positive-definite matrices $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times m}$, and a positive real number $\gamma_N > 0$ such that

$$A^\top X^{-1} A - \gamma_N^2 Y^{-1} \prec 0,$$

it holds that

$$\begin{aligned} \max_v \{ |x - Av|_{X^{-1}}^2 - \gamma_N^2 |y - v|_{Y^{-1}}^2 \} \\ = |x - Ay|_{(X - \gamma_N^{-2} AY A^\top)^{-1}}^2. \end{aligned} \quad (11)$$

Proof. Expanding the left-hand side of (11) and equating the gradient with 0 we get

$$\begin{aligned} & \max_v \{ |x - Av|_{X^{-1}}^2 - \gamma_N^2 |y - v|_{Y^{-1}}^2 \} \\ &= \max_v \left\{ |v|_{A^\top X^{-1} A - \gamma_N^2 Y^{-1}}^2 + |x|_{X^{-1}}^2 - \gamma_N^2 |y|_{Y^{-1}}^2 - 2v^\top (A^\top X^{-1} x - \gamma_N^2 Y^{-1} y) \right\} \\ &= |x|_{X^{-1}}^2 - \gamma_N^2 |y|_{Y^{-1}}^2 - |A^\top X^{-1} x - \gamma_N^2 Y^{-1} y|_{(A^\top X^{-1} A - \gamma_N^2 Y^{-1})^{-1}}^2 \\ &= |x|_{X^{-1} - X^{-1} A^\top (A^\top X^{-1} A - \gamma_N^2 Y^{-1})^{-1} A^\top X^{-1}}^2 \\ &\quad + |y|_{-\gamma_N^2 Y^{-1} - \gamma_N^2 Y^{-1} (A^\top X^{-1} A - \gamma_N^2 Y^{-1})^{-1} Y^{-1} \gamma_N^2}^2 \\ &\quad - 2x^\top X^{-1} A (A^\top X^{-1} A - \gamma_N^2 Y^{-1})^{-1} (-\gamma_N^2 Y^{-1} y) \\ &= |x|_{(X - \gamma_N^{-2} AY A^\top)^{-1}}^2 + |Ay|_{(X - \gamma_N^{-2} AY A^\top)^{-1}}^2 - 2x^\top (X - \gamma_N^{-2} AY A^\top)^{-1} Ay \\ &= |x - Ay|_{(X - \gamma_N^{-2} AY A^\top)^{-1}}^2. \end{aligned}$$

□

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