Barrier Pairs for Safety Control of Uncertain Output Feedback Systems

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Abstract—The barrier function method for safety control typically assumes the availability of full state information. Unfortunately, in many scenarios involving uncertain dynamical systems, full state information is often unavailable. In this paper, we aim to solve the safety control problem for an uncertain single-input single-output system with partial state information. First, we develop a synthesis method that simultaneously creates a barrier function and a dynamic output feedback safety controller. This safety controller guarantees that the unit sublevel set of the barrier function is an invariant set under the uncertain dynamics and disturbances of the system. Then, we build an identifier-based estimator that provides a state estimate affine to the uncertain model parameters of the system. To detect the potential risks of the system, a fault detector uses the state estimate to find an upper bound for the barrier function. The fault detector triggers the safety controller when the system's original action leads to a potential safety issue and resumes the original action when the potential safety issue is resolved by the safety controller.

I. INTRODUCTION

Barrier functions [1], [2] are commonly used for safety control and verification of dynamical systems. However, the standard theory assumes that the full state information is available to compute the barrier function values. Unfortunately, full state information can only be estimated in many safety control scenarios involving disturbances and uncertainties, such as wearable robots [3] coupled with timevarying human dynamics and self-driving vehicles [4] in an uncertain environments.

There are multiple methods [5], [6], [7] for synthesizing a full state feedback controller that enforces a valid barrier function. Thus, safety control for a system without full state measurements naturally begins with finding a state observer and then builds a safety controller using the state estimate from the state observer. However, how to estimate the barrier function values using the state estimate from the state observer remains a question. Although some observer-based output feedback controllers can enforce safety constraints without knowing the barrier function values [8], [9], we cannot blindly use these controllers at all times. For example, a human operator may give a human assistive robot [3] an input that potentially violates some safety constraints of the

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human-robot coupled system. If we replace the original robot controller entirely with a safety controller, this robot will maintain safety but not assist the human operator. For this reason, we must estimate the barrier function value in real-time so that the safety controller only corrects the original system when necessary.

In this paper, we aim to solve the safety control problem for an uncertain single-input single-output (SISO) system with partial state information. The main contributions of this paper are summarized as follows.

- (1) In Sec. III, we develop a control synthesis method that simultaneously creates a barrier function and a dynamic output feedback safety controller. Using the controller parameter transformation scheme in [10], our dynamic output feedback safety controller guarantees that the unit sub-level set of our barrier function is an invariant set with bounded model uncertainty and disturbance. The proposed control synthesis method in this paper builds significantly upon the method in [7], which only focuses on full state feedback safety control.
- (2) In Sec. IV, we propose a robust fault detector that consists of an identifier-based estimator [11]. Similar to the method in [12], this identifier-based estimator provides us with a robust state estimate, which helps us find the upper bound for our vector norm barrier function using partial state information. However, unlike the estimator in [12], our fault detector in this paper does not require the system to be originally stable or have a stable static output feedback controller.
- (3) In Sec. V, we showcase our fault detector and safety controller, which work together to protect an uncertain SISO system from potential risks. In particular, we demonstrate how our fault detector helps us correctly trigger our safety controller when the system is about to violate the safety constraints and resume the original mission of the system when it is safe to do so.

Notation. We define $S_p \triangleq \begin{bmatrix} \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}$ and $S_k \triangleq \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \end{bmatrix}$ as two selection matrices, which extract the plant state x_p and the controller state x_k from the closed-loop state vector x_{CL} . Supposing $P \succ \mathbf{0}$,

$$\|\star\|_P \triangleq \sqrt{\star^\mathsf{T} P \star} \tag{1}$$

is a vector norm function based on P. In our LMIs, we define $He\{\star\} \triangleq \star + \star^T$.

II. PRELIMINARIES

In this section, we introduce models of an uncertain dynamical system Σ_p and a full-order dynamic output feedback

controller Σ_k . The uncertain dynamical system is described as a polytopic linear differential inclusion (PLDI) [13]. Then we give an overview of barrier pairs, which will be used for safety verification and control of the uncertain dynamical system. Based on the concept of barrier pairs, we present our formal problem statement.

A. State Model

In this paper, we consider a SISO system. We will explore how we can extend our study to a multiple-input multiple-output (MIMO) system in the future.

First, assume that the transfer function of our system Σ_p from input u to output y is expressed as

$$G_p(s) = \frac{y(s)}{u(s)} = \frac{\beta_1 s^{n-1} + \dots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_{n-1} s + \alpha_n}, \quad (2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ are the uncertain parameters of our plant. Let us define

$$A_0 \triangleq \begin{bmatrix} \mathbf{0}_{(n-1)\times 1}^\mathsf{T} & 0\\ \mathbf{I}_{(n-1)\times (n-1)} & \mathbf{0}_{(n-1)\times 1} \end{bmatrix} \text{ and } c_0 \triangleq \begin{bmatrix} \mathbf{0}_{1\times (n-1)} & 1 \end{bmatrix}.$$
(3)

In Sec. IV-A, we will conduct a robust fault detection of Σ_p using an identifier-based estimator [11]. In order to build this identifier-based estimator, we need to use the following state space realization of Σ_p :

$$\dot{x}_p = A_0 x_p + \begin{bmatrix} b_y & b_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

$$y = c_0 x_p + w$$

$$(4)$$

where $b_y \triangleq [-\alpha_n \cdots -\alpha_2 -\alpha_1]^\mathsf{T}$ and $b_u \triangleq [\beta_n \cdots \beta_2 \beta_1]^\mathsf{T}$ contain all the uncertain plant parameters and w is a disturbance signal. Let us suppose that

$$b_{y} = \bar{b}_{y} + \sum_{i}^{n_{p}} \delta_{i} \cdot \theta_{i}^{y} \cdot \tilde{b}_{i}$$

$$b_{u} = \bar{b}_{u} + \sum_{i}^{n_{p}} \delta_{i} \cdot \theta_{i}^{u} \cdot \tilde{b}_{i}$$
(5)

where $\bar{b}_y \triangleq [-\bar{\alpha}_n \cdot \cdot \cdot -\bar{\alpha}_2 - \bar{\alpha}_1]^\mathsf{T}$ and $\bar{b}_u \triangleq [\bar{\beta}_n \cdot \cdot \cdot \bar{\beta}_2 \ \bar{\beta}_1]^\mathsf{T}$ are the nominal plant parameters, n_p is the number of dimensions of the parameter uncertainties, $\tilde{b}_i \in \mathbb{R}^{n \times 1}$, θ_i^y , $\theta_i^u \in \mathbb{R}$ describe the direction of the parameter uncertainties in each dimension, $\delta_i \in \mathbb{R}$ is a scalar variable such that $|\delta_i| \leq 1$ for all $i=1,2,\cdots,n_p$. Based on (5), $\begin{bmatrix} b_y & b_u \end{bmatrix}$ is in a polytopic region in $\mathbb{R}^{n \times 2}$ and (4) becomes a PLDI.

Next, let the state-space model of a full-order dynamic output feedback controller Σ_k be

$$\dot{x}_k = A_k x_k + b_k y \tag{6}$$

$$u = c_k x_k$$

where $A_k \in \mathbb{R}^{n \times n}$, $b_k \in \mathbb{R}^{n \times 1}$, and $c_k \in \mathbb{R}^{1 \times n}$ are the controller parameters to be determined.

Let us define

$$A_{\mathsf{CL}} \triangleq \begin{bmatrix} \hat{A}_p & \bar{b}_u c_k \\ b_k c_0 & A_k \end{bmatrix}, \quad B_{wp} \triangleq \begin{bmatrix} \bar{b}_y & \tilde{b}_p \\ b_k & \mathbf{0} \end{bmatrix}, \quad (7)$$

$$C_q \triangleq \begin{bmatrix} \Theta_y^\mathsf{T} c_0 & \Theta_u^\mathsf{T} c_k \end{bmatrix}, \quad D_{wp} \triangleq \begin{bmatrix} \Theta_y^\mathsf{T} & \mathbf{0} \end{bmatrix},$$

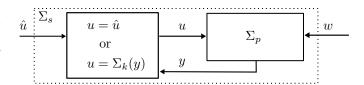


Fig. 1. In this paper, our switching system Σ_s chooses either an original input $u=\hat{u}$ of Σ_p or a known-to-be-safe input $u=\Sigma_k(y)$ based on a barrier function B.

where $\hat{A}_p \triangleq A_0 + \bar{b}_y c_0$, $\tilde{b}_p \triangleq [\tilde{b}_1 \ \tilde{b}_2 \ \cdots \ \tilde{b}_{n_p}]$, $\Theta_y \triangleq [\theta_1^y \ \theta_2^y \ \cdots \ \theta_{n_p}^y]$ and $\Theta_u \triangleq [\theta_1^u \ \theta_2^u \ \cdots \ \theta_{n_p}^u]$. Combining (4) and (6), our closed-loop system Σ_{CL} for $u = \Sigma_k(y)$ is described as

$$\dot{x}_{\mathsf{CL}} = A_{\mathsf{CL}} x_{\mathsf{CL}} + B_{wp} \begin{bmatrix} w \\ p \end{bmatrix}$$

$$q = C_q x_{\mathsf{CL}} + D_{wp} \begin{bmatrix} w \\ p \end{bmatrix}$$
(8)

where $x_{\text{CL}} \triangleq [x_p^{\mathsf{T}} \ x_k^{\mathsf{T}}]^{\mathsf{T}}, \ p = [p_1 \ p_2 \ \cdots \ p_{n_p}]^{\mathsf{T}}, \ q = [q_1 \ q_2 \ \cdots \ q_{n_p}]^{\mathsf{T}} \text{ and } p_i = \delta_i q_i \text{ for all } i = 1, 2, \cdots, n_p.$

B. Barrier Pairs

The concept of barrier pairs [7] describes the relationship between a barrier function and a feedback controller in a safety control problem. In this paper, we extend the definition of barrier pairs to output feedback systems.

Definition 1. A barrier pair is a pair (\mathbf{B}, Σ_k) consisting of a barrier function $\mathbf{B}: x_{\mathsf{CL}} \to \mathbb{R}$ and a controller Σ_k satisfying the following conditions:

(a)
$$\varepsilon \leq \mathbf{B}(x_{\mathsf{CL}}) \leq 1, \ u = \Sigma_k(y) \implies \dot{\mathbf{B}}(x_{\mathsf{CL}}) < 0,$$

(b)
$$\mathbf{B}(x_{\mathsf{CL}}) \leq 1 \implies x_p \in \mathcal{X}_s, \ \Sigma_k(y) \in \mathcal{U}.$$

In particular, $x_p \in \mathcal{X}_s$ and $u \in \mathcal{U}$ are the state and input constraints. Intuitively, (a) and (b) mean the invariance and constraint satisfaction properties of a barrier pair.

C. Problem Statement

In this paper, we consider an uncertain dynamical system Σ_p described in (4) and (5) with assumption $|w| \leq \bar{w}$ and safety constraints $x_p \in \mathcal{X}_s$ and $u \in \mathcal{U}$.

Problem. Assuming that y and u are the only available measurements of Σ_p and $x_p=0$ at t=0, find a barrier pair (B,Σ_k) and a switching system Σ_s where (B,Σ_k) satisfies conditions (a) and (b) in Definition 1 for Σ_p and where Σ_s switches Σ_p from its original input $u=\hat{u}$ to $u=\Sigma_k(y)$ whenever B is close to 1 and switches back to $u=\hat{u}$.

Fig. 1 shows a block diagram of Σ_s that is similar to the switching systems proposed in [2], [7]. This type of switching system seeks to maintain the safety of Σ_p through a minimum intervention in its original input $u=\hat{u}$. However, unlike the previous works, the full state of Σ_p is not available in our problem. Therefore, our switching system cannot switch between $u=\hat{u}$ and $u=\Sigma_k(y)$ based on the exact value of our barrier function \mathbf{B} .

To address this limitation, we will solve our problem in two steps. In Sec. III, we show a synthesis method that creates a barrier pair for Σ_p . In Sec. IV, we propose a robust fault detector that finds an upper bound for B using only the measurements of y and u. Based on this upper bound for B, we will construct Σ_s in Sec. IV-C.

III. BARRIER PAIR SYNTHESIS

In this section, we focus on the barrier pair synthesis for our system Σ_p described in (4) and (5). First we formulate conditions (a) and (b) in Definition 1 as linear matrix inequality (LMI) constraints. Then we introduce our LMI optimization problems for barrier pair synthesis.

A. LMIs for Invariance Property

In this paper, we define our barrier function as a vector norm function

$$\mathbf{B}(x_{\mathsf{CL}}) \triangleq \|x_{\mathsf{CL}}\|_{P}.\tag{9}$$

Let us partition P and $Q \triangleq P^{-1}$ as

$$P = \begin{bmatrix} X & V \\ V^\mathsf{T} & \star \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Y & W \\ W^\mathsf{T} & \star \end{bmatrix}, \qquad (10)$$

where $X, Y, V, W \in \mathbb{R}^{n \times n}$. In addition, we define

$$\Pi_1 \triangleq \begin{bmatrix} \mathbf{I} & X \\ \mathbf{0} & V^{\mathsf{T}} \end{bmatrix} \quad \text{and} \quad \Pi_2 \triangleq \begin{bmatrix} Y & \mathbf{I} \\ W^{\mathsf{T}} & \mathbf{0} \end{bmatrix}, \qquad (11)$$

where $\Pi_1 = P\Pi_2$ and $\Pi_2 = Q\Pi_1$. In Proposition 1, we will use Π_1 and Π_2 to perform a controller parameter transformation [10] and derive the LMI constraints for condition (a) in Definition 1.

Proposition 1. Supposing that $|w| \leq \bar{w}$, there exist a barrier function $\mathbf{B}(x_{\mathsf{CL}}) = \|x_{\mathsf{CL}}\|_P$ and a controller Σ_k in the form of (6) such that $(\mathbf{B}(x_{\mathsf{CL}}), \Sigma_k)$ satisfies condition (a) in Definition 1, if there exist $X \succ \mathbf{0}$, $Y \succ \mathbf{0}$, $E \in \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times 1}$, $G \in \mathbb{R}^{1 \times n}$, and μ_w , μ_1 , μ_2 , \cdots , $\mu_{n_p} \geq 0$ such that

$$\begin{bmatrix} Y & \mathbf{I} \\ \mathbf{I} & X \end{bmatrix} \succ 0. \tag{12}$$

and

$$\begin{bmatrix} H_A & \star & \star \\ H_B^{\mathsf{T}} & -\mathbf{M}_{wp} & \star \\ \mathbf{M}_p H_C & \mathbf{M}_p D_{wp} & -\mathbf{M}_p \end{bmatrix} \prec 0, \tag{13}$$

where $\mathbf{M}_p \triangleq \operatorname{diag}(\mu_1, \mu_2, \cdots, \mu_{n_p}), \mathbf{M}_{wp} \triangleq \operatorname{diag}(\mu_w, \mathbf{M}_p),$

$$H_{A} \triangleq \operatorname{He} \left\{ \begin{bmatrix} \hat{A}_{p}Y + \bar{b}_{u}G & \hat{A}_{p} \\ E & X\hat{A}_{p} + Fc_{0} \end{bmatrix} \right\} + \mu_{w} \frac{\bar{w}^{2}}{\varepsilon^{2}} \begin{bmatrix} Y & \mathbf{I} \\ \mathbf{I} & X \end{bmatrix}, \tag{14}$$

$$H_B \triangleq \begin{bmatrix} \bar{b}_y & \tilde{b}_p \\ F + X \bar{b}_y & X \tilde{b}_p \end{bmatrix}, \quad H_C \triangleq \begin{bmatrix} \Theta_u^\mathsf{T} G + \Theta_y^\mathsf{T} c_0 Y & \Theta_y^\mathsf{T} c_0 \end{bmatrix},$$

$$E \triangleq V A_k W^{\mathsf{T}} + F c_0 Y + X \bar{b}_u G + X \hat{A}_p Y,$$

$$F \triangleq V b_k, \qquad G \triangleq c_k W^{\mathsf{T}}.$$

$$(15)$$

Proof. Notice that (13) is obtained by performing a congruence transformation with $diag(\Pi_1, \mathbf{I}, \mathbf{I})$ on

$$\begin{bmatrix} \Phi_Q + \mu_w \frac{\bar{w}^2}{\varepsilon^2} Q & \star & \star \\ B_{wp}^\mathsf{T} & -\mathbf{M}_{wp} & \star \\ \mathbf{M}_p C_q Q & \mathbf{M}_p D_{wp} & -\mathbf{M}_p \end{bmatrix} \prec \mathbf{0}, \tag{16}$$

where $\Phi_Q \triangleq A_{\mathsf{CL}}Q + QA_{\mathsf{CL}}^\mathsf{T}$. Thus, we focus on the following two steps to complete the proof. First, we will show that condition (a) in Definition 1 holds for $(\mathbf{B}(x_{\mathsf{CL}}), \Sigma_k)$ if (16) holds. Then, we will show that (16) holds if (12) and (13) hold.

In the first step, we consider $\mathbf{B}^2(x_{\mathsf{CL}}) = x_{\mathsf{CL}}^\mathsf{T} P x_{\mathsf{CL}}$ as a quadratic Lyapunov function candidate for our closed-loop system with $u = \Sigma_k(y)$. Then, $(\mathbf{B}(x_{\mathsf{CL}}), \Sigma_k)$ satisfies condition (a) in Definition 1 for Σ_p in (4) and (5) under the assumption $|w| \leq \bar{w}$ if and only if $\frac{\mathsf{d}\, B^2(x_{\mathsf{CL}})}{\mathsf{d}\, t} < 0$, or equivalently

$$\begin{bmatrix} x_{\mathsf{CL}} \\ w \\ p \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A_{\mathsf{CL}}^{\mathsf{T}} P + P A_{\mathsf{CL}} & \star & \star \\ B_{w}^{\mathsf{T}} P & \mathbf{0} & \star \\ B_{p}^{\mathsf{T}} P & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_{\mathsf{CL}} \\ w \\ p \end{bmatrix} < 0, \quad (17)$$

for all x_{CL} , w, and p that

$$x_{\mathsf{CL}}^{\mathsf{T}} P x_{\mathsf{CL}} \geq \varepsilon^2, \quad w^2 \leq \bar{w}^2, \quad \text{and} \quad p_i^2 \leq q_i^2, \ \forall i = 1, \cdots, n_p.$$
 (18)

Using the S-procedure, (17) holds under the conditions in (18) if there exist μ_{CL} , μ_w , μ_1 , \cdots , $\mu_{n_p} \geq 0$ such that for all x_{CL} , w, and p that

$$\begin{bmatrix} x_{\mathsf{CL}} \\ w \\ p \end{bmatrix}^{\mathsf{T}} H_P \begin{bmatrix} x_{\mathsf{CL}} \\ w \\ p \end{bmatrix} + \mu_w \bar{w}^2 - \mu_{\mathsf{CL}} < 0, \tag{19}$$

where

$$H_{P} \triangleq \begin{bmatrix} \Phi_{P} + \frac{\mu_{\mathsf{CL}}}{\varepsilon^{2}} P + C_{q}^{\mathsf{T}} \mathbf{M}_{p} C_{q} & \star \\ B_{wp}^{\mathsf{T}} P & -\mathbf{M}_{wp} + D_{wp}^{\mathsf{T}} \mathbf{M}_{p} D_{wp} \end{bmatrix}$$

and $\Phi_P \triangleq A_{\mathsf{CL}}^{\mathsf{T}} P + P A_{\mathsf{CL}}$. Then, (19) holds if $H_P \prec \mathbf{0}$ and $\mu_{\mathsf{CL}} = \mu_w \bar{w}^2$. Through a congruence transformation with $\mathrm{diag}(Q, \mathbf{I})$ on H_P and the Schur complement, $H_P \prec \mathbf{0}$ for $\mu_{\mathsf{CL}} = \mu_w \bar{w}^2$ is equivalent to (16).

Now, let us establish the second step. Since (13) is obtained by performing a congruence transformation with $\operatorname{diag}(\Pi_1, \mathbf{I}, \mathbf{I})$ on (16), (16) holds if there exists nonsingular $\operatorname{diag}(\Pi_1, \mathbf{I}, \mathbf{I})$ such that (13) holds. Obviously, $\operatorname{diag}(\Pi_1, \mathbf{I}, \mathbf{I})$ is non-singular if and only if V is nonsingular. Since $PQ = \mathbf{I}$, we have $VW^T = \mathbf{I} - XY$. Then, there exists non-singular V and W if $\mathbf{I} - XY$ is non-singular. $\mathbf{I} - XY$ is non-singular if $Y \succ \mathbf{0}$ and $X - Y^{-1} \succ \mathbf{0}$. Using the Schur complement, $Y \succ \mathbf{0}$ and $X - Y^{-1} \succ \mathbf{0}$ if (12) holds. Consequently, we obtain that there exists non-singular $\operatorname{diag}(\Pi_1, \mathbf{I}, \mathbf{I})$ if (12) holds.

Therefore, condition (a) in Definition 1 holds for $(\mathbf{B}(x_{\mathsf{CL}}), \Sigma_k)$ if there exist $X \succ \mathbf{0}, Y \succ \mathbf{0}, E \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{n \times 1}, G \in \mathbb{R}^{1 \times n}, \text{ and } \mu_w, \mu_1, \mu_2, \cdots, \mu_{n_p} \geq 0 \text{ such that (12) and (13) hold.}$

If we define the scalar variables $\mu_w, \mu_1, \mu_2, \cdots, \mu_{n_n}$ a

priori, (13) becomes an LMI in (X, Y, E, F, G). Even though (13) is not an LMI with respect to these variables jointly, we can consider methods such as D-K iteration [14] for searching the values of these scalar variables.

B. LMIs for State and Input Limits

Now, let us focus on condition (b) in Definition 1 for barrier function $\mathbf{B}(x_{\mathsf{CL}}) = \|x_{\mathsf{CL}}\|_P$ and safety controller Σ_k in the form of (6). In this paper, let us define \mathcal{X}_s and \mathcal{U} as

$$\mathcal{X}_s \triangleq \{x_p : |f_i^\mathsf{T} x_p| \le 1, \ i = 1, 2, \dots, n_f\},$$
 (21)

$$\mathcal{U} \triangleq \{u : |u| \le \bar{u}\}. \tag{22}$$

Since $f_i^\mathsf{T} x_p = f_i^\mathsf{T} S_p x_\mathsf{CL}$ and $u = c_k S_k x_\mathsf{CL}$, condition (b) in Definition 1 holds for $(\mathbf{B}(x_\mathsf{CL}), \Sigma_k)$ if

$$f_i^{\mathsf{T}} S_p Q S_p^{\mathsf{T}} f_i \le 1$$
, for $i = 1, 2, \dots, n_f$, (23)

$$c_k S_k Q S_k^\mathsf{T} c_k^\mathsf{T} \le \bar{u}^2. \tag{24}$$

Since Σ_k is undetermined (i.e., A_k , b_k and c_k are also decision variables), (24) is not an LMI. We address this issue as follows. Since $Q = P^{-1}$, we can use the Schur complement to obtain that (24) holds if

$$\begin{bmatrix} P & \star \\ c_k S_k & \bar{u}^2 \end{bmatrix} \succeq \mathbf{0}. \tag{25}$$

By performing a congruence transformation with $diag(\Pi_2, \mathbf{I})$ on (25), we obtain that (24) holds if

$$\begin{bmatrix} Y & \star & \star \\ \mathbf{I} & X & \star \\ G^{\mathsf{T}} & \mathbf{0} & \bar{u}^2 \end{bmatrix} \succeq \mathbf{0},\tag{26}$$

where (26) is an LMI in our new variable set (X, Y, E, F, G) introduced in Proposition 1. In addition, we can use the Schur complement to obtain that (26) also implies condition (12) in Proposition 1.

C. Barrier Pair Construction

Through a convex optimization

$$\begin{array}{ll} \underset{X,Y,E,F,G}{\text{maximize}} & \log(\det(Y)) \\ \text{subject to} & X \succ 0, \, Y \succ 0, \\ & (13), \, (23), \, (26), \end{array} \tag{27}$$

we obtain a solution of (X, Y, E, F, G) that maximize the volume of the x_p space projection $\{x_p : x_p = S_p x_{CL}, \mathbf{B}(x_{CL}) \leq 1\}$ of the unit sub-level set of $\mathbf{B}(x_{CL})$.

There are multiple methods to construct a controller Σ_k based on a solution of (X, Y, E, F, G). For example, Ref. [15, Lemma 7.9] defines $VV^{\mathsf{T}} = X - Y^{-1}$ and W = -YV. After obtaining V and W, we can construct our controller parameters A_k , b_k , and c_k according to (15).

IV. ROBUST FAULT DETECTION

Although barrier pair (\mathbf{B}, Σ_k) is obtained in the previous section, the calculation of \mathbf{B} relies on knowing the full state x_{CL} of the closed-loop system. Since x_p is not available, the true value of \mathbf{B} is unknown. In this section, we will introduce a robust fault detector that provides an upper bound for \mathbf{B} using only the measurements of y and y.

A. Identifier-Based Estimator

In [11], the concept of an identifier-based estimator was developed for the purpose of model identification and adaptive control. However, as a byproduct, it also provides us with a robust state estimate \hat{x}_p to the model uncertainty of Σ_p . The identifier-based estimator for our system Σ_p is a pair of sensitivity function filters

$$\dot{z}_y = A_z^{\mathsf{T}} z_y + c_0^{\mathsf{T}} y
\dot{z}_u = A_z^{\mathsf{T}} z_u + c_0^{\mathsf{T}} u$$
(28)

where $A_z \triangleq A_0 - b_z c_0$ and $b_z \in \mathbb{R}^{n \times 1}$ is defined by the user such that A_z is a Hurwitz matrix. Let us define

$$E_y \triangleq \mathcal{C}_y \mathcal{C}_0^{-1}, \quad E_u \triangleq \mathcal{C}_u \mathcal{C}_0^{-1}$$
 (29)

where \mathcal{C}_0 is the controllability matrix of $(A_z^\mathsf{T}, c_0^\mathsf{T})$, \mathcal{C}_y is the controllability matrix of (A_z^T, z_y) , and \mathcal{C}_u is the controllability matrix of (A_z^T, z_u) .

Lemma 1. z_y and z_u are the states of the identifier-based estimator in (28). E_y and E_u are defined as (29). If we define a state estimate \hat{x}_p for Σ_p in (4) as

$$\hat{x}_p = E_y^{\mathsf{T}}(b_y + b_z) + E_u^{\mathsf{T}}b_u, \tag{30}$$

the state estimation error $e \triangleq x_p - \hat{x}_p$ follows

$$\dot{e} = A_z e - b_z w. \tag{31}$$

Proof. This is similar to the proofs of [11, Lemma 2] and [12, Lemma 1]. Subtracting (31) from (4), we obtain that

$$\dot{\hat{x}}_p = A_z \hat{x}_p + (b_y + b_z)y + b_u u. \tag{32}$$

Since C_0 is the controllability matrix of $(A_z^\mathsf{T}, c_0^\mathsf{T})$, we can derive from (28) that

$$\dot{E}_y = A_z^{\mathsf{T}} E_y + y \cdot \mathbf{I},
\dot{E}_u = A_z^{\mathsf{T}} E_u + u \cdot \mathbf{I}.$$
(33)

Notice that A_z is in a canonical form, which leads to $E_y^\mathsf{T} A_z = A_z E_y^\mathsf{T}$ and $E_u^\mathsf{T} A_z = A_z E_u^\mathsf{T}$. By taking the transpose of (33), we obtain $\dot{E}_y^\mathsf{T} = A_z E_y^\mathsf{T} + y \cdot \mathbf{I}$ and (27) $\dot{E}_u^\mathsf{T} = A_z E_u^\mathsf{T} + u \cdot \mathbf{I}$. Therefore, (32) holds if $\hat{x}_p = E_y^\mathsf{T} (b_y + b_z) + E_u^\mathsf{T} b_u$.

Notice that in (30), \hat{x}_p is affine in b_y and b_u , which are defined in (5). Substituting (5) into (30), we have

$$\hat{x}_p = \bar{x}_p + \sum_{i=1}^{n_p} \delta_i \cdot \tilde{x}_i^p$$

$$\bar{x}_p \triangleq E_y^{\mathsf{T}}(\bar{b}_y + b_z) + E_u^{\mathsf{T}}\bar{b}_u, \quad \tilde{x}_i^p \triangleq \theta_i^y \cdot E_y^{\mathsf{T}}\tilde{b}_i + \theta_i^u \cdot E_u^{\mathsf{T}}\tilde{b}_i \tag{34}$$

where $\delta \triangleq [\delta_1 \ \delta_2 \ \cdots \ \delta_{n_p}]$ and $|\delta_i| \leq 1$ for all $i = 1, 2, \cdots, n_p$. Let us define a set

$$\bar{\Delta} \triangleq \{ [\delta_1 \ \delta_2 \ \cdots \ \delta_{n_p}] : |\delta_i| = 1, i = 1, 2, \cdots, n_p \},$$
 (35)

which consists of all the 2^{n_p} extreme values of δ . Then, we have

$$\hat{x}_p(\delta) \in \mathsf{Co}\big\{\bar{x}_p + \sum_{i}^{n_p} \delta_i \cdot \tilde{x}_i^p, \, \forall \, \delta \in \bar{\Delta}\big\}. \tag{36}$$

Although we do not know the exact value of \hat{x}_p due to the uncertainty in b_y and b_u , (36) shows that the possible value of $\hat{x}_p(\delta)$ is in a polytopic region in \mathbb{R}^n .

B. Barrier Function Estimation

Next, we will explain how (31) and (36) help us find a computable upper bound for our barrier function **B**.

Proposition 2. Let us define

$$\bar{\mathbf{B}} \triangleq \max_{\bar{\delta} \in \bar{\Delta}} \mathbf{B}(\hat{x}_{\mathsf{CL}}(\bar{\delta})) + r_e, \tag{37}$$

where $\hat{x}_{\mathsf{CL}}(\bar{\delta}) \triangleq [\hat{x}_p(\bar{\delta})^\mathsf{T} \ x_k^\mathsf{T}]^\mathsf{T}$ and $r_e > 0$. Supposing that $|w| \leq \bar{w}, \ \mathbf{B}(x_{\mathsf{CL}}) \leq \bar{\mathbf{B}}$ if there exists $\mu_e \geq 0$ such that

$$\begin{bmatrix} A_z^{\mathsf{T}} X + X A_z + \mu_e \cdot X & \star \\ -b_z^{\mathsf{T}} X & -\mu_e \cdot \frac{r_e^2}{\bar{w}^2} \end{bmatrix} \prec \mathbf{0}. \tag{38}$$

Proof. Through the triangle inequality of $\|\star\|_P$, we have

$$\mathbf{B}(x_{\mathsf{CL}}) = \|x_{\mathsf{CL}}\|_{P} \le \|\hat{x}_{\mathsf{CL}}(\delta)\|_{P} + \|S_{p}^{\mathsf{T}}e\|_{P},\tag{39}$$

where $e \triangleq x_p - \hat{x}_p$. According to (36),

$$\|\hat{x}_{\mathsf{CL}}(\delta)\|_{P} \le \max_{\bar{\delta} \in \bar{\Delta}} \|\hat{x}_{\mathsf{CL}}(\bar{\delta})\|_{P}. \tag{40}$$

Based on the definition of P in (10),

$$||S_p^{\mathsf{T}}e||_P \le ||e||_X.$$
 (41)

Therefore, $\mathbf{B}(x_{\mathsf{CL}}) \leq \bar{\mathbf{B}}$ for all w that $|w| \leq \bar{w}$ if $\max_{\bar{\delta} \in \bar{\Delta}} \|\hat{x}_{\mathsf{CL}}(\bar{\delta})\|_c + \|e\|_X \leq \bar{\mathbf{B}}$, or equivalently $\|e\|_X \leq r_e$, for all w that $|w| \leq \bar{w}$.

At t=0, $e \triangleq x_p - \hat{x}_p = 0$ for $x_p = z_y = z_u = 0$. Since e=0 at t=0 and $\dot{e} = A_z e - b_z w$, $\|e\|_X \leq r_e$ for all w that $|w| \leq \bar{w}$ if and only if $\frac{\mathsf{d} \|e\|_X^2}{\mathsf{d} t} < 0$, or equivalently

$$\begin{bmatrix} e \\ w \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A_z^{\mathsf{T}} X + X A_z & \star \\ -b_z^{\mathsf{T}} X & \mathbf{0} \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} < 0, \tag{42}$$

for all e and w that

$$e^{\mathsf{T}} X e \ge r_e^2$$
 and $w^2 \le \bar{w}^2$. (43)

Using the S-procedure, we obtain that (42) holds under the conditions in (43) if there exists $\mu_e \geq 0$ such that (38) holds. Therefore, $\mathbf{B}(x_{\mathsf{CL}}) \leq \bar{\mathbf{B}}$ for all w that $|w| \leq \bar{w}$ if there exists $\mu_e \geq 0$ such that (38) holds.

Notice that X is already obtained from optimization (27). If we fix the value of μ_e , (38) becomes an LMI in b_z . Through convex optimization

minimize
$$r_e^2$$
 (44) subject to (38)

 b_z in (28) can be defined for minimizing r_e . Since $\hat{x}_{\mathsf{CL}}(\bar{\delta})$ for all $\bar{\delta} \in \bar{\Delta}$ can be obtained from our identifier-based estimator, $\bar{\mathbf{B}}$ is available to us.

C. Switching Logic of Σ_s

Based on the value of $\bar{\mathbf{B}}$ (Fig. 2), we can now define the switching logic of Σ_s . Let us define two thresholds $\underline{\varepsilon}$ and $\bar{\varepsilon}$ (with $\varepsilon < \underline{\varepsilon} < \bar{\varepsilon} \le 1$). Σ_s switches from the original input

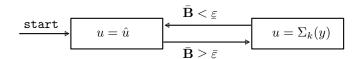


Fig. 2. Σ_s switches from the original input $u=\hat{u}$ to $u=\Sigma_k(y)$ if $\bar{\mathbf{B}}\geq\bar{\varepsilon}$ and switches back to $u=\hat{u}$ if $\bar{\mathbf{B}}<\underline{\varepsilon}$.

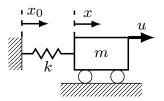


Fig. 3. In our example, we consider a simplified 1-DOF physical human-robot interaction system, which a mass-spring system with an uncertain human stiffness k and a robot inertia m.

 $u=\hat{u}$ to $u=\Sigma_k(y)$ if $\bar{\mathbf{B}}\geq\bar{\varepsilon}$ and switches back to $u=\hat{u}$ if $\bar{\mathbf{B}}<\underline{\varepsilon}$. According to Proposition 1, x_{CL} converges to residual set $\{x_{\mathsf{CL}}:\mathbf{B}(x_{\mathsf{CL}})\leq\varepsilon\}$ when $u=\Sigma_k(y)$. Therefore, when Σ_k is in control of the system, the true value of $\mathbf{B}(x_{\mathsf{CL}})$ goes below ε in finite time. By setting the values of ε and ε closer to 1, we reduce the intervention from Σ_k in the original operation of Σ_p .

V. EXAMPLE

In this section, we provide an example to illustrate the robust fault detection and safety control of our proposed method.

A. System Model

Here we consider an uncertain mass-spring system Σ_p (Fig. 3) with a transfer function

$$G_p(s) = \frac{y(s)}{u(s)} = \frac{k}{m \cdot s^2 + k}$$
 (45)

where the system has a unit mass m=1 and a spring stiffness k, the output $y=k(x-x_0)$ measures the spring force, and the input u represents an adjustable force exerting to the mass. The spring stiffness is uncertain and defined as $k=\hat{k}+\delta\cdot\bar{k}$, where $\hat{k}=10,\,\bar{k}=1,$ and $|\delta|\leq 1.$ The state model of Σ_p can be expressed as

$$\dot{x}_{p} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_{p} + \begin{bmatrix} -\frac{k}{m} \\ 0 \end{bmatrix} y + \begin{bmatrix} \frac{k}{m} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x_{p} + w$$
(46)

where $x_p = [\dot{x} \ x]^\mathsf{T}$ and w is an unknown exogenous input. If we consider this system as a simplified 1-DOF human-robot interaction model in a wearable robot control problem [3], then m is the robot inertia, k is an uncertain human joint stiffness, and $x_0 = \frac{w}{k}$ is a desired joint position where the human operator tends to move to.

The safe regions \mathcal{X}_s and \mathcal{U} are defined as

$$\mathcal{X}_s \triangleq \{ [\dot{x} \ x]^\mathsf{T} : |\dot{x}| \le 2, \ |x| \le 2 \},\tag{47}$$

$$\mathcal{U} \triangleq \{ u : |u| \le 10 \}. \tag{48}$$

Assuming that $|w| \le \bar{w} = 0.05$, we aim to achieve a residual set $\{x_{\mathsf{CL}} : \mathbf{B}(x_{\mathsf{CL}}) \le \varepsilon\}$ of our barrier function with $\varepsilon = 0.5$.

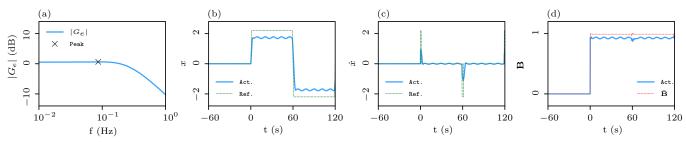


Fig. 4. Simulation Results of Example—(a) shows the frequency response of $G_e(s)$, where the peak value is at $f_e=0.09\,\mathrm{Hz}$. (b)-(d) show the results of our safety control test under the disturbance signals $w(t)=0.05\sin(2\pi f_e t)$. In particular, (b)-(c) show the actual (Act.) and reference (Ref.) values of x and \dot{x} and (d) shows the actual value (Act.) and upper bound ($\bar{\mathbf{B}}$) of \mathbf{B} .

B. Safety Control

Through optimizations (27) and (44), we obtain a barrier pair (\mathbf{B}, Σ_k) and a state estimator in the form of (28), which provides us with a state estimate \hat{x}_p to calculate a barrier function upper bound $\bar{\mathbf{B}}$ with $r_e = 0.05$.

In our safety control test, we want to demonstrate the robustness of our switching system Σ_s with respect to the bounded model uncertainty and disturbance in the test. Unfortunately, it is difficult to implement the exact worst case of the bounded disturbance signal w(t) that maximizes the peak value of the estimation error residue $\|e\|_X$. Instead, we implement w(t) as a sinusoidal signal in this example. Let us define a transfer function

$$G_e(s) \triangleq \frac{\|e(s)\|_X}{w(s)} \tag{49}$$

from the disturbance w(s) to the estimation error residue $\|e(s)\|_X$. In the frequency domain (Fig. 4a), $\|G_e(s)\|_{\infty}=1.09$ at $f_e=0.09\,\mathrm{Hz}$. Therefore, we define the disturbance as a sinusoidal signal $w(t)=\bar{w}\cdot\sin(2\pi f_e t)$, which gives us the maximum sinusoidal response of $G_e(s)$.

To test the fault detection, we implement the original input \hat{u} as a reference tracking controller, which lets x follow a trapezoidal reference (Fig. 4b-c). The reference is designed to violate the safety limits of x and \dot{x} on purpose in such a way that \hat{u} can cause potential risks. Fig. 4b-d show that the safety controller Σ_k is correctly triggered when the system is about to violate the constraints of x and \dot{x} . Fig. 4d shows that the true value of \mathbf{B} is strictly lower than $\dot{\mathbf{B}}$ at all times.

VI. DISCUSSION

In our problem statement, we assume that the initial state of x_p is 0. Similar to Ref. [12], our fault detector can also be extended to cases where the initial states of x_p are not 0. The Hurwitz matrix A_z in our identifier-based estimator in (28) guarantees that the state estimation error due to an unknown initial state of x_p converges to 0 exponentially.

Fig. 4d in our example shows a slight over-conservatism of the barrier function upper bound $\bar{\mathbf{B}}$. This is partly because the worst case of the disturbance input w(t) is difficult to find. In our example, we implement the disturbance input w(t) as a sinusoidal signal, which only leads to the worst case of the sinusoidal response of $G_e(s)$ in (49). Note that as long as $w \leq \bar{w}$, constraint (38) guarantees $\|e(t)\|_X \leq r_e$ no matter what type of signal we implement w(t) as.

In this paper, we considered the safety control problem for an uncertain SISO system with partial state information. By knowing the limits of the uncertain model parameters and disturbance a priori, our fault detector and safety controller work together to protect the uncertain SISO system from potential risks. In the future, we will extend our safety control method to MIMO systems.

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