Throughput of Freeway Networks under Ramp Metering Subject to Vehicle Safety Constraints

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Abstract—Ramp metering is one of the most effective tools to combat traffic congestion. In this paper, we present a ramp metering policy for a network of freeways with arbitrary number of on- and off-ramps, merge, and diverge junctions. The proposed policy is designed at the microscopic level and takes into account vehicle following safety constraints. In addition, each on-ramp operates in cycles during which it releases vehicles as long as the number of releases does not exceed its queue size at the start of the cycle. Moreover, each onramp dynamically adjusts its release rate based on the traffic condition. To evaluate the performance of the policy, we analyze its throughput, which is characterized by the set of arrival rates for which the queue sizes at all on-ramps remain bounded in expectation. We show that the proposed policy is able to maximize the throughput if the merging speed at all the onramps is equal to the free flow speed and the network has no merge junction. We provide simulations to illustrate the performance of our policy and compare it with a well-known policy from the literature.

I. Introduction

Ramp metering, which controls the inflow of traffic to a freeway, is one of the most efficient tools to combat traffic congestion. In this paper, we propose a microscopic (vehicle)-level ramp metering policy for freeway networks with arbitrary number of on- and off-ramps, merge and diverge junctions. Our approach builds up on our previous work [1], where we considered a single freeway with no merge or diverge junctions.

Ramp metering policies are commonly designed using macroscopic traffic flow models, which use spatio-temporal averaging of vehicle interactions to model the traffic. Many ramp metering policies have been proposed in the literature using this approach [2], [3]. These policies are either: (i) fixed-time [4], where no real-time traffic measurement is used, or (ii) local traffic-responsive [5], where on-ramps use only local measurements, or (iii) coordinated traffic-responsive [6]–[8], where on-ramps also use measurements from other parts of the network. In spite of its generality, the macroscopic-level approach is not suitable in the context of Connected and Automated Vehicles (CAVs) as it does not capture some of their capabilities, such as the ability

to dissipate traffic disturbance and provide accurate measurements of traffic through Vehicle-to-Vehicle (V2V) and Vehicle-to-Infrastructure (V2I) communication systems [9]–[11]. Additionally, macroscopic-level ramp metering policies do not account for vehicle following safety constraints. As a consequence, the vehicles may find themselves in unsafe situations near an on-ramp, which can trigger congestion or even an accident. While adding safety filters on top of the ramp metering policy is a potential solution, it may adversely affect other performance criteria such as the total travel time [1].

A more systematic approach is to explicitly include vehicle safety, connectivity, and automation protocols in the ramp metering design by considering the vehicles at the microscopic level. Somewhat surprisingly, the literature on the microscopic-level ramp metering design is sparse and only limited to isolated on-ramps [12], [13]. Relatively little attention has been given to analyzing the overall performance of a freeway network. For example, are certain system-level performance metrics such as *throughput* optimized? To bridge this gap and expand our previous work for a single freeway [1], we consider a network of freeways to design and analyze a microscopic ramp metering policy that can maximize the throughput.

The freeway network is modeled as a directed acyclic graph, with arbitrary number of on- and off-ramps, merge, and diverge junctions. Vehicles arrive from outside the network to the on-ramps and join the on-ramp queues. The on-ramp arrivals are sampled from i.i.d Bernoulli processes that are independent across different on-ramps, and the vehicle routes are sampled independently from a routing matrix. Once released from an on-ramp, each vehicle follows standard safety and speed rules until it reaches its destination off-ramp. Vehicles in the network have the same acceleration and braking capabilities, and are equipped with V2V and V2I communication systems. We leverage these connectivity capabilities to design a coordinated traffic-responsive ramp metering policy. The proposed policy operates under vehicle following safety constraints, where new vehicles are released only if there is enough gap on the mainline. Additionally, the on-ramps operate under synchronous cycles during which each on-ramp does not release more vehicles than its queue size at the start of the cycle. Moreover, an on-ramp's release rate is adjusted during a cycle according to the traffic condition in the network.

We evaluate the performance of a policy in terms of its throughput, which is characterized by the set of arrival rates under which the queue sizes at every on-ramp remain

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bounded in expectation. We show that the proposed policy is able to maximize the throughput when the merging speed at all the on-ramps equals the free flow speed and the network has no merge junction, e.g., a single freeway.

The remainder of the paper is organized as follows: in Section II, we describe the network structure, the vehicles behavior in the network, the demand model, and the performance metric. We provide our ramp metering policy and its performance analysis in Section III. We provide simulation results in Section IV, and conclude the paper in Section V.

We collect key notations to be used throughout the paper. Let \mathbb{N} and \mathbb{N}_0 respectively denote the set of positive integers, and non-negative integers. For $m \in \mathbb{N}$, [m] denotes the set $\{1,\ldots,m\}$. For a set A, |A| denotes its cardinality.

II. PROBLEM FORMULATION

A. Network Topology

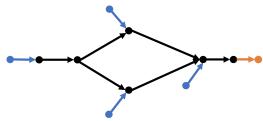


Fig. 1: An acyclic network with 4 on-ramps (blue), 1 off-ramp (orange), 6 freeway segments, 1 merge node, and 1 diverge node.

The network is modeled as a directed acyclic graph. The nodes represent either junctions (merge or diverge), or sources/sinks; the edges represent either single-lane freeway segments that connect two junctions, or on- and off-ramps (VC1) The ego vehicle maintains the constant speed V_f if: that connect sources to merge nodes and diverge nodes to sinks, respectively; see Figure 1. We refer to any subset of the freeway segments as the mainline. For simplicity, we assume that each source has exactly one outgoing on-ramp, and each sink has exactly one incoming off-ramp. We distinguish between two types of merge node: on-ramp nodes, which have at least one incoming on-ramp, and regular merge nodes (or simply merge nodes), which have no incoming on-ramp. Similarly, we distinguish between an off-ramp node and a diverge node. We denote the set of on-ramp nodes by \mathcal{M} . For simplicity, we assume that each on-ramp node has exactly one incoming on-ramp edge. Thus, \mathcal{M} also specifies the set of on-ramp edges. For on-ramp $i \in \mathcal{M}$, we let \mathcal{M}_i^+ be the set of its successor on-ramps, i.e., the set of all on-ramps that are immediately downstream of on-ramp i. An element of \mathcal{M}_i^+ is denoted by i^+ . Similarly, we let \mathcal{M}_i^- be the set of its predecessor on-ramps. We say that two edges are adjacent if the head of one edge is the tail of the other edge. A route is a sequence of adjacent edges, with the first edge being an on-ramp and the last edge being an off-ramp. The collection of all routes is denoted by \mathcal{P} .

Vehicles arrive to on-ramps according to some exogenous stochastic process and join the on-ramp queues. For any on-ramp, we assume a point queue model with unlimited capacity. We also assume that vehicles that reach their destination off-ramp node, immediately leave the network without creating any obstruction. Once a vehicles is released from an on-ramp into the mainline, it follows standard speed and safety rules, which will be explained in the following section.

B. Vehicle-Level Objectives

Vehicles have the same length L, the same acceleration and braking capabilities, and are equipped with V2V and V2I communication systems. The exact information communicated by a vehicle will be specified in Section III, where we introduce our ramp metering policy. We use the term ego vehicle to refer to a specific vehicle under consideration, and denote it by e. Consider a vehicle following scenario and let v_e (resp. v_l) be the speed of the ego vehicle (resp. its leading vehicle), and S_e be the safety distance between the two vehicles required to avoid collision. We assume that S_e satisfies $S_e = hv_e + S_0 + \frac{v_e^2 - v_l^2}{2|a_{\min}|}$, which is calculated based on an emergency stopping scenario with details given in [14]. Here, h > 0 is a safe time headway constant, $S_0>0$ is an additional constant gap, and $a_{\min}<0$ is the minimum possible deceleration of the leading vehicle. We consider two general modes of operation for each vehicle: the speed tracking mode and the safety mode. The main objective in the speed tracking mode is to adjust and maintain the speed to the free flow speed V_f when the ego vehicle is far from any leading vehicle; the main objective in the safety mode is to avoid rear-end collision once the ego vehicle gets close to the leading vehicle.

We assume that vehicles obey the following rules:

- (a) it is at a safe distance with respect to its leading vehicle ¹, i.e., $y_e \ge S_e$, where y_e is the distance between the two vehicles. Note that if its leading vehicle is also at the constant speed V_f , then $y_e \ge$ $S_e = hV_f + S_0$. This distance is equivalent to a time headway of at least $\tau := h + (S_0 + L)/V_f$ between the front bumpers of the two vehicles.
- (b) it is near a merge node and predicts to be at a safe distance with respect to its leading vehicle after reaching the merging node, i.e., $\hat{y}_e(t_m) \geq \hat{S}_e(t_m) := h\hat{v}_e(t_m) + S_0 + \frac{\hat{v}_e^2(t_m) - \hat{v}_{\hat{l}}^2(t_m)}{2|a_{\min}|}$, where t_m is the time the ego vehicle predicts to reach the merge node, \hat{y}_e is the predicted distance between the two vehicles, and \hat{v} is the predicted speed. Each vehicle uses the following speed prediction rule to calculate its predicted time of merging, position, and speed: if in the speed tracking mode, it assumes that it remains in this mode in the future; if in the safety mode, it assumes a constant speed trajectory. These information are then communicated among all vehicles near a merge node via V2V communication.

¹At a diverge node, the leading vehicle is uniquely determined by the ego vehicle's route choice.

(VC2) there exists $T_{\rm free}$ such that for any initial condition, vehicles reach the *free flow equilibrium* after at most $T_{\rm free}$ time if no other vehicle is released from the onramps. The free flow equilibrium refers to a state where each vehicle is moving at the constant speed V_f and will maintain this speed in the future if no additional vehicle is released from the on-ramps.

Remark 1. Due to lack of space, we assume that the ego vehicle immediately joins the mainline after being released. In other words, we ignore the vehicle dynamics between release from an on-ramp and joining the mainline. We also assume the speed at which the ego vehicle joins the mainline, i.e., the merging speed, is V_f . The general case where these assumptions are relaxed can be treated similar to [1]. We should emphasize, however, that the merging speed assumption can affect the performance of a ramp metering policy; see Remark 3.

In order to have a complete description of the network, we also need to specify the demand at the microscopic level. We will do this in the next section.

C. Demand Model

For convenience in the analysis, we adopt a discrete time setting with time steps of size τ , where τ is the minimum safe time headway at the free flow speed defined in Section II-B. Let vehicles arrive to on-ramp $i \in \mathcal{M}$ according to an i.i.d. Bernoulli process with parameter $\lambda_i \in [0,1]$. The arrival processes are assumed to be independent across different on-ramps. That is, at any given time step, the probability that a vehicle arrives at the *i*-th on-ramp is λ_i independent of everything else. We let $\lambda := [\lambda_i]$. A vehicle arriving to on-ramp i, chooses to take route $p \in \mathcal{P}$ with probability $R_{ip} \in [0,1]$ independent of everything else. Naturally, for every on-ramp i, $\sum_{p} R_{ip} = 1$, and $R_{ip} = 0$ for any p that is not compatible with the network topology. We assume that the vehicles do not change their route choice while in the network. We stack the routing probabilities in a routing matrix $R=[R_{ip}]$. Let $\rho_i=\sum_j\sum_{p:i\in p}\lambda_jR_{jp}$ be the average induced load on on-ramp i, where the notation $i \in p$ means that on-ramp i's node is on route p. This is the average number of arrivals in the network that wants to cross on-ramp i's node in order to reach their destination. The performance of a ramp metering policy is quantified in terms of the maximum demand it can handle. We will formalize this in the next section.

D. Ramp Metering and Performance Metric

To conveniently track vehicle locations in discrete time, we introduce the notion of slot. A slot is associated with a particular point on the mainline at a particular time. Consider the maximum number of distinct slots that can be placed on a freeway segment, such that the distance between adjacent slots is $hV_f + S_0 + L$, i.e., the minimum safety distance at the free flow speed as per (VC1) plus the vehicle length. Consider a configuration of these slots at t=0. At the end of each time step τ , each slot replaces the next slot with the last slot replacing the first slot of that segment. Without

loss of generality, we let the location of the first and last slots of each segment coincide with the tail and head of that segment at the end of each time step. Thus, if the ego vehicle is released from an on-ramp at the end of a given time step, its location coincides with a slot.

For $i \in \mathcal{M}$, let $Q_i(t)$ be the set of the route choices of the vehicles waiting at on-ramp i, arranged in the order of their arrival, at time t. We use $|Q_i(t)|$ to denote the queue size at on-ramp i at time t. Let $|Q(t)| = \lceil |Q_i(t)| \rceil$ be the vector of queue sizes at all the on-ramps at time t. A ramp metering policy is a function $\pi(t) = [\pi_i(t)]$, where $\pi_i(t) =$ 1 if a vehicle is released from on-ramp i at time t, and $\pi_i(t) = 0$ otherwise. The key metric we use to evaluate the performance of a ramp metering policy is its throughput, which is defined as follows: for a given ramp metering policy π and routing matrix R, let $U_{\pi,R}$ be the set of λ 's for which $\limsup_{t\to\infty} \mathbb{E}[|Q_i(t)|] < \infty$ for all $i \in \mathcal{M}$. We say that the network is under-saturated if $\lambda \in U_{\pi,R}$; otherwise, it is called saturated. The throughput is the boundary of the set $U_{\pi,R}$. We are interested in finding a ramp metering policy π such that for every R and any other policy π' , we have $U_{\pi',R} \subseteq U_{\pi,R}$. We say such a policy maximizes the throughput.

III. MAIN RESULTS

A. Dynamic Release Rate Policy with Rate Allocation

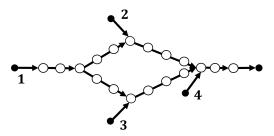


Fig. 2: A configuration of slots (empty circles) at time t=0. For this configuration and by choosing $b_i=3$, $a_i=1$, $i\in[3]$, and $b_4=a_4=1$ the release times can be made conflict-free.

In this section we present and analyze our ramp metering policy. The policy presented is coordinated, in the sense that each on-ramp may require information from other parts of the network in addition to the information obtained from its vicinity. However, it does not require any information about the demand or the route choice of the vehicles, i.e., it is reactive [15]. The policy operates under synchronous cycles of fixed length during which each on-ramp does not release more vehicles than its queue size at the beginning of the cycle. The synchronization of cycles is done once offline. This policy allocates a fixed maximum release rate to each on-ramp. In addition, it imposes dynamic minimum time gap criterion between release of successive vehicles from the same on-ramp. Changing the time gap between release of successive vehicles by an on-ramp is akin to changing its release rate. An inner approximation to the throughput of this policy is provided, and is then compared to an outer approximation.

Definition 1. (Dynamic Release Rate policy with rate Allocation (DRRA)) The policy works in cycles of fixed length $T\tau$, where $T\in\mathbb{N}$. At the beginning of the k-th cycle at $t_k=(k-1)T\tau$, each on-ramp allocates itself a "quota" equal to the queue size at that on-ramp at t_k . At time $t\in[t_k,t_{k+1}]$ during the k-th cycle, on-ramp i releases the ego vehicle if:

- (M1) $t = (b_i m + n)\tau$, where $m \in \mathbb{N}_0$, $n \in \{n_1, \dots, n_{a_i}\} \subseteq [b_i]$, and $a_i, b_i \in \mathbb{N}$, $a_i \leq b_i$, are such that the release times at all the on-ramps are *conflict-free*, meaning that no two vehicles that move at the constant speed V_f after being released can simultaneously reach a merging node; see Figure 2. The term a_i/b_i is the maximum release rate allocated to on-ramp i.
- (M2) the on-ramp has not reached its quota.
- (M3) the ego vehicle is at a safe distance with respect to its leading and following vehicles at the moment of release.
- (M4) at least g(t) time has passed since the release of the last vehicle from on-ramp i.

Once an on-ramp reaches its quota, it does not release a vehicle during the rest of the cycle. The minimum time gap g(t) is piecewise constant, updated periodically at $t=T_{\rm per},2T_{\rm per},\ldots$, as described in Algorithm 1. In Algorithm 1, X_f is defined as follows: for the ego vehicle, let $x_e^T=((y_e-S_e)\mathbb{1}_{\{y_e< S_e\}},v_e-V_f,a_e)$, where $\mathbb{1}_{\{y_e< S_e\}}$ is the indicator function, and a_e is the acceleration. The state of all the vehicles is collectively denoted as $X^T=(x_e^T)_{e\in[n]}$, where n is the total number of vehicles on the mainline. We let $X_{f_1}(t):=\|X(t)\|$. Moreover, for $t\geq T_{\rm per}$, let $X_{f_2}(t):=\sum |\hat{y}_e(t_m)-\hat{S}_e(t_m)|$, where the sum is over all the vehicles in the time interval $[t-T_{\rm per},t]$ that predicted to violate the safety distance after reaching a merging node. We let $X_f(t)=X_{f_1}(t)+X_{f_2}(t)$. Roughly, $X_f(t)$ measures the distance to the free flow equilibrium state at time t.

Algorithm 1 Update rule for the minimum time gap between release of successive vehicles

Input: design constants:
$$T_{\mathrm{per}} > 0, \gamma_1 > 0, \ \gamma_2 > 0, \ \theta^\circ > 0, \ \beta > 1$$
 initial condition: $g(0) = 0, \ \theta(0) = \theta^\circ, \ X_f(0) = X_{f_1}(0)$ for $t = T_{\mathrm{per}}, 2T_{\mathrm{per}}, \cdots$ do if $X_f(t) \leq \max\{X_f(t-T_{\mathrm{per}}) - \gamma_1, 0\}$ then $\theta(t) \leftarrow \theta(t-T_{\mathrm{per}})$ $g(t) \leftarrow \max\{g(t-T_{\mathrm{per}}) - \gamma_2, 0\}$ else $\theta(t) \leftarrow \beta\theta(t-T_{\mathrm{per}})$ $g(t) \leftarrow g(t-T_{\mathrm{per}}) + \theta(t)$ end if end for

Theorem 1. For any initial condition, $T \in \mathbb{N}$, design constants in Algorithm 1, and $a_i, b_i \in \mathbb{N}$ such that $a_i \leq b_i$, $i \in \mathcal{M}$, and the release times are conflict-free, the DRRA policy keeps the network under-saturated if $\rho_i < a_i/b_i$ for all $i \in \mathcal{M}$.

Proof. See Appendix II.

V2I communication requirements: This policy uses information about |Q| (if T>1) and X_f . After a finite time, the mainline vehicles only communicate X_{f_1} to all the onramps at the end of each update period $T_{\rm per}$. The number of vehicles that constitute X_{f_1} is no more than the total number of slots. Furthermore, at the end of each time step during a cycle, the vehicles near an on-ramp communicate their state to that on-ramp for the safety gap evaluation.

Remark 2. A non-reactive version of the DRRA policy with better throughput: suppose that the first vehicle in every on-ramp's queue also communicates its route choice to that on-ramp. Consider a version of the DRRA policy, where, in addition to the releasing times in (M1), on-ramp i releases the ego vehicle at $t=m\tau, m\in\mathbb{N}_0$, if the ego vehicle's route contains no merge node. This policy is non-reactive since it uses the route choice of the vehicles, but gives a better throughput since more vehicles are released compared to the DRRA policy. We compare the performance of these two policies via simulations in Section IV.

Remark 3. Recall from Section II-B that we made the simplifying assumption that the merging speed at all the on-ramps is V_f . One can relax this assumption, but the throughput would suffer due to the larger safety distance required at the moment of merging. However, the main steps of the proof of Theorem 1 remain the same and a similar bound for the throughput can be obtained; see [1, Theorem 2].

B. A Necessary Condition

We now provide a necessary condition for undersaturation, against which we benchmark the sufficient condition in Theorem 1. Let $\bar{D}_{\pi,p}(m\tau)$ be the cumulative number of vehicles that has crossed point p on the mainline up to time $m\tau, m \in \mathbb{N}_0$, under the ramp-metering policy π . Then, the crossing rate at point p is defined as $\bar{D}_{\pi,p}(m\tau)/m$ and the "long-run" crossing rate is $\limsup_{m\to\infty} \bar{D}_{\pi,p}(m\tau)/m$. Recall the definition of ρ_i for on-ramp i's node. We can extend this definition to every node in the network that is not a source or sink. Let $\rho:=\max_i \rho_i$, where i is any node that is not a source or sink.

Theorem 2. Suppose that the long-run crossing rate is no more than one vehicle per τ seconds for at least one point on each freeway segment. If the network is under-saturated under the policy π , then $\rho < 1$.

Proof. The proof is similar to [1, Theorem 4]. \Box

Remark 4. The long-run crossing rate condition in Theorem 2 is very mild in practice. In particular, if the traffic flow at some point on each freeway segment is less than the mainline capacity, then the long-run crossing rate condition is also satisfied [1].

Remark 5. Consider a network with no merge node, and note that for any node j that is not a source or sink, we have $\rho_j \leq \rho_i$ for some $i \in \mathcal{M}$. For such a network, one can set

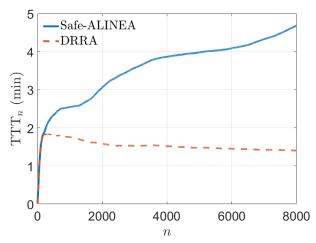


Fig. 3: The average total travel time (TTT_n) under the Safe-ALINEA and DRRA policies, where n is the number of completed trips.

 $a_i=b_i=1$ for all $i\in\mathcal{M}$ in the DRRA policy. The sufficient condition of Theorem 1 then becomes $\max_{i\in\mathcal{M}}\rho_i=\rho<1$. Combining with Theorem 2, it follows for any routing matrix R and any other ramp metering policy π' that $U_{\pi',R}\subseteq U_{\mathrm{DRRA},R}$, except, maybe at the boundary of $U_{\mathrm{DRRA},R}$. Since the boundary of $U_{\mathrm{DRRA},R}$ has zero volume (measure), we can conclude that the DRRA policy maximizes the throughput for all practical purposes.

IV. SIMULATIONS

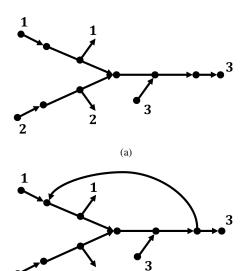


Fig. 4: The two networks used in Sections IV-B and IV-C. Figure (a) is a three-legged merge junction, and figure (b) is constructed by making the first network cyclic.

We simulate the DRRA policy and compare its performance with its non-reactive version introduced in Remark 2 and a well-known policy from the literature. All the simulations were performed in MATLAB. We use the following

setup in all the simulations: let h=1.5 [s], $S_0=4$ [m], L=4.5 [m], and $V_f=15$ [m/s]. The initial queue size at all the on-ramps is set to zero, and vehicles arrive at the onramps according to i.i.d Bernoulli processes with the same rate λ .

A. Total Travel Time

In this section, we compare the total travel time under the DRRA policy and ALINEA [5], which is a macroscopic ramp metering policy. We also add the safety filter (M3) on top of ALINEA, and use the name Safe-ALINEA to refer to the resulting policy. An on-ramp's outflow in the ALINEA policy is determined according to the following equation:

$$r(k) = r(k-1) + K_r(\hat{o} - o(k)).$$

Here, $r(\cdot)$ is the on-ramp's outflow, K_r is a positive design constant, $o(\cdot)$ is the occupancy of the mainline downstream of the on-ramp, and \hat{o} is the desired occupancy. We use time steps of size 60 [s], $K_r = 70$ [veh/h] [5], and $\hat{o} = 13$ % corresponding to the capacity flow. For the DRRA policy, we use the following parameters: T=13, $T_{\rm per}=2\tau$, $\gamma_1 = 50, \ \gamma_2 = 10, \ \theta^{\circ} = 0.1, \ \beta = 1.01, \ \text{and} \ a_i = b_i = 1,$ i = 1, 2, 3. The network is a ring road with perimeter 1860 [m] and 3 on- and off-ramps. The on-ramps are located symmetrically at locations 0, 600, and 1200; the off-ramps are also located symmetrically 155 [m] upstream of each on-ramp. We consider a congested initial condition where the initial number of vehicles on the mainline is 100; each vehicle is at the constant speed of 6.7 [m/s], and is at the distance $h \times 6.7 + S_0 \approx 14.1$ [m] from its leading vehicle. The details of the vehicle following behavior is given in [1]. We also let $\lambda = 0.54$ and

$$R_1 = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

where columns specify the off-ramps.

We evaluate the average Total Travel Time (TTT_n), which is computed by averaging the total travel time of the first n vehicles that complete their trips. The total travel time of a vehicle equals its waiting time in an on-ramp queue plus the time it spends on the freeway to reach its destination. Figure 3 shows the simulation result. From the figure, we can see that the DRRA policy provides significant improvement in the TTT_n compared to the Safe-ALINEA policy. In fact, the network becomes saturated under the Safe-ALINEA policy, which implies that the TTT_n will grow steadily with

B. Acyclic Network with Merge Junction

The second network is a symmetric three-legged merge junction; see Figure 4a. The total length of freeway segments on each leg is 310 [m]. on-ramps 1 and 2 are located at the beginning, while on-ramp 3 is located in the middle of their corresponding legs. Similarly, off-ramps 1 and 2 are in the

middle, while off-ramp 3 is at the end of their corresponding legs. The mainline is initially empty and

$$R_2 = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{pmatrix}.$$

The on-ramps use the DRRA policy with T=1 and other parameters chosen similar to Section IV-A. In addition, the release times of on-ramps 1 and 2 are set to $t=(2m+1)\tau$ and $t=(2m+2)\tau$, respectively, while $t=m\tau$ for on-ramp 3, $m\in\mathbb{N}_0$ (thus, $a_i=1,\,b_i=2,\,i\in[2]$, and $a_3=b_3=1$). For the given routing matrix, the inner and outer estimates of the under-saturation region given by Theorems 1 and 2 are $\lambda<1/2$ and $\lambda<5/9$, respectively. From simulations, the under-saturation regions of the DRRA policy and its non-reactive version are estimated to be $\lambda<1/2$ and $\lambda<5/9$, respectively. This suggest that there is a gap between the inner and outer estimates of the under-saturation region in the DRRA policy, but not in its non-reactive version.

C. Cyclic Network with Merge Juction

The third network is constructed by making the second network cyclic; see Figure 4b. The length of the additional segment in this network is 610 [m]. The mainline is initially empty and

$$R_3 = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

The on-ramps again use the DRRA policy with the same parameters as in the previous section. The inner and outer estimates of the under-saturation region are $\lambda < 1/3$ and $\lambda < 5/9$, respectively. From simulations, the under-saturation regions of the DRRA policy and its non-reactive version are estimated to be $\lambda < 0.4$ and $\lambda < 5/9$, respectively. Therefore, Theorem 1 holds for this cyclic network. Based on the simulation results and the results in [1] given for a ring road network, we conjecture that Theorem 1 holds for general cyclic networks.

V. CONCLUSION AND FUTURE WORK

In this paper, we provided a microscopic ramp metering policy for freeway networks with arbitrary number of on- and off-ramps, merge, and diverge junctions. We analyzed the throughput of our policy for acyclic networks, and showed via simulations that it achieves a similar throughput for cyclic networks. We plan to confirm this observation analytically in the future. Moreover, we plan to extend our analysis to policies with tighter estimates of the throughput.

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APPENDIX I PERFORMANCE ANALYSIS TOOL

Consider an initial condition where the vehicles are in the free flow equilibrium, and the location of each vehicle coincides with a slot. Thus, their location coincide with a slot in the future. Thereafter, during each time step, the following sequence of events occurs: (i) the slots on each segment replace the next slot with the last slot replacing the first slot; the numbering of slots is reset with the new first slot numbered 1, and the rest numbered in an increasing order in the direction of the flow, (ii) vehicles that reach their destination off-ramp exit the network, (iii) if permitted by the ramp metering policy, a vehicle is released into a slot at the free flow speed. Since the initial location of all vehicles coincide with a slot and the released times are conflict-free, the location of the released vehicle will also coincide with a slot in the future.

For the aforementioned initial condition, we can cast the network dynamics as a discrete-time Markov chain, with time steps of size τ . Let M(n) be the set of the routes of the occupants of all the slots: $M_{\ell}(n) = p$ if the vehicle occupying slot ℓ at time n has to take route p, and $M_{\ell}(n) = 0$

if slot ℓ is empty at time n. Consider the following discrete-time Markov chain with the state $Z_{\triangle}(n) := Y(n\triangle), \ n \ge 0$, where Y(n) := (Q(n), M(n)), and $\triangle \in \mathbb{N}$ will be specified for the policy being analyzed. The transition probabilities are determined by the ramp metering policy being analyzed, but will not be specified explicitly for brevity. For the ramp metering policy considered in this paper, the state Y(n) = (0,0) is reachable from all other states, and $\mathbb{P}(Y(n+1) = (0,0) \mid Y(n) = (0,0)) > 0$. Hence, the Markov chain Z_{\triangle} is irreducible and aperiodic.

APPENDIX II PROOF OF THEOREM 1

For any initial condition, we first show that there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $X_f(kT_{\text{per}}) = 0$. This and (VC1) imply that every vehicle moves at the free flow speed for all $t \geq t_0$. Hence, after a finite time after t_0 , $g(\cdot) = 0$, and the location of every vehicle coincides with a slot thereafter. We can then use the Markov chain setting from Appendix I.

Using an argument similar to [1], one can easily show that $X_f(kT_{per})$ is uniformly bounded for all $k \in \mathbb{N}$. To show $X_f(\cdot) = 0$ after a finite time, we use a proof by contradiction. Suppose that $X_f(kT_{per}) \neq 0$ for infinitely many $k \in \mathbb{N}$. Since $X_f(kT_{per})$ is uniformly bounded, there exists an infinite sequence $\{k_n\}_{n\geq 1}$ such that $X_f(k_nT_{per})>$ $X_f((k_n-1)T_{per})-\gamma_1$ for all $n\geq 1$. This implies that $\theta(k_n T_{\text{per}}) = \beta \theta((k_n - 1) T_{\text{per}})$ for all $n \geq 1$. Since $\theta(\cdot)$ is non-decreasing and $\beta > 1$, it follows that $\lim_{t \to \infty} \theta(t) = \infty$, which in turn implies that $\limsup_{t\to\infty} g(t) = \infty$. Let k_f be such that $g(k_f T_{per}) > |\mathcal{M}|(T_{free} + T_{empty})(1 + \gamma_2/T_{per}),$ where T_{empty} is an upper bound on the additional time required for all vehicles on the mainline to leave the network after reaching the free flow state, if no additional vehicle is released. Let $t_f := k_f T_{per}$. Note that for all $t \in [t_f, t_f + |\mathcal{M}|(T_{\text{free}} + T_{\text{empty}})], g(t) > |\mathcal{M}|(T_{\text{free}} + T_{\text{empty}}).$ Thus, each on-ramp releases at most one vehicle during the interval $[t_f, t_f + |\mathcal{M}|(T_{\text{free}} + T_{\text{empty}})]$. Hence, there exists a time interval of length at least $(T_{\mathrm{free}} + T_{\mathrm{empty}})$ in $[t_f, t_f +$ $|\mathcal{M}|(T_{\text{free}} + T_{\text{empty}})|$ during which no on-ramp releases a vehicle. Condition (VC2) then implies that all vehicles reach the free flow state after at most T_{free} time, and leave the network after at most an additional T_{empty} time, at the end of which (say at time t_0), the network becomes empty. Thus, for all $k \geq k_0 := \lceil t_0/T_{\rm per} \rceil$, $X_f(kT_{\rm per}) = 0$, which is a contradiction to the assumption that $X_f(kT_{per}) \neq 0$ for infinitely many k.

We assume that the vehicles are initially in the free flow equilibrium state, and the location of each vehicle coincides with a slot at t=0 as in Appendix I. For the sake of readability, we present proofs of intermediate claims at the end. We adopt the Markov chain setting from Appendix I with $\{Z_{\triangle}(n)\}_{n\geq 0}$ as the Markov chain, where $\Delta=kT$ and $k\in\mathbb{N}$ is to be determined. For on-ramp $i\in\mathcal{M}$, let $N_i(n)$ be the degree of on-ramp i representing the number of vehicles in the network at time n that need to cross on-ramp i's node in order to reach their destination. We recursively define the

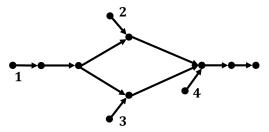


Fig. 5: For on-ramp 4, $U_4^0=\{\{4\}\},$ $U_4^1=\{\{2,3\}\},$ and $U_4^2=\{\{1,1\},\{1,3\},\{1,2\}\}.$

following collections of index sets:

$$U_i^0 = \{\{i\}\}, \ U_i^1 = \{\mathcal{M}_i^-\},$$

and for k > 2,

$$U_i^k = \{ \bigcup_{j \in I} I_j : I \in U_i^{k-1}, I_j = \{j\} \text{ or } \mathcal{M}_i^- \} \setminus U_i^{k-1},$$

where each $\cup_{j\in I}I_j$ is treated as a *multiset*; see Figure 5. In words, $I\in U_i^1$, i.e., $I=\mathcal{M}_i^-$, is the set of all on-ramps that precede on-ramp i; if $I\in U_i^2\cup U_i^1$, then for any on-ramp j that precedes on-ramp i, either $j\in I$ or all on-ramps that precede j belong to I; and so on. Note that since the network is acyclic, the number of sets U_i^k is finite for every i. Let $U=\cup_{i\in\mathcal{M},k\geq 0}U_i^k$. Consider the candidate Lyapunov function $V:\mathcal{S}\to [0,\infty)$ defined as:

$$V(n) \equiv V(Z_{\triangle}(n)) := N^2(n\triangle), \tag{1}$$

where $\mathcal S$ is the range of values of Y in Appendix I, and $N(m)=\max_{I\in U}\sum_{j\in I}N_j(m).$

Note that (1) implies $V(n+1) - V(n) = N^2((n+1)\Delta) - N^2(n\Delta)$. We show at the end of the proof that if V(n) is large enough, specifically if $V(n) > L := ((C_1+1)\Delta^2 + C_2)^2$, where $C_1 := (|\mathcal{M}| + \rho + \delta)|\mathcal{M}||\mathcal{P}|$ and $C_2 := (2T + \max_{j \in \mathcal{M}} (r_j + a_j))|\mathcal{M}||\mathcal{P}|$, then

$$N((n+1)\triangle) \le N(n\triangle) + \tilde{A}(\triangle) + C_2, \tag{2}$$

where $\tilde{A}(\triangle)$ satisfies the following: it is independent of $N(n\triangle)$, and there exists $\triangle_1, \varepsilon, C_3 > 0$ such that for all $\triangle > \triangle_1$ we have

$$\mathbb{E}\left[\tilde{A}(\triangle)\right] < -\varepsilon\triangle, \quad \mathbb{E}\left[\tilde{A}^2(\triangle)\right] < C_3\triangle^2. \tag{3}$$

The inequality (2) implies that

$$N^{2}((n+1)\triangle) \leq N^{2}(n\triangle) + 2N(n\triangle) \left(\tilde{A}(\triangle) + C_{2}\right) + \tilde{A}^{2}(\triangle) + 2C_{2}\tilde{A}(\triangle) + C_{2}^{2}.$$
(4)

By choosing $\triangle \ge \triangle_1$ and taking conditional expectation from both sides of (4) we obtain

$$\mathbb{E}\left[V(n+1) - V(n) \mid V(n) > L\right] \le -N(n\Delta)$$

$$+ 2N(n\Delta)\left(-\varepsilon\Delta + C_2 + \frac{1}{2}\right) + C_3\Delta^2 - 2\varepsilon C_2\Delta + C_2^2.$$

Since for any $i \in \mathcal{M}$, $|Q_i(n\triangle)| \leq N_i(n\triangle) \leq N(n\triangle)$, we have $||Q(n\triangle)||_{\infty} := \max_{i \in \mathcal{M}} |Q_i(n\triangle)| \leq N(n\triangle)$. Hence,

choosing $\triangle > \max\{\triangle_1, \triangle_2 := \frac{C_2 + 1/2}{\varepsilon}\}$ gives

$$\mathbb{E}\left[V(n+1) - V(n) \mid V(n) > L \right] \le -\|Q(n\triangle)\|_{\infty} \\ -2\varepsilon(C_1+1)\triangle^3 + ((C_1+1)(2C_2+1) + C_3)\triangle^2 \\ -4\varepsilon C_2\triangle + 3C_2^2 + 1.$$

If \triangle_3 is chosen such that for all $\triangle > \max\{\triangle_1, \triangle_2, \triangle_3\}$,

$$-2\varepsilon(C_1+1)\triangle^3 + ((C_1+1)(2C_2+1) + C_3)\triangle^2 - 4\varepsilon C_2\triangle + 3C_2^2 + 1 < 0,$$
(5)

then $\mathbb{E}\left[V\left(n+1\right)-V\left(n\right)|V\left(n\right)>L\right]\leq -\|Q(n\triangle)\|_{\infty}$. Such a \triangle_3 always exists because the $-2\varepsilon(C_1+1)\triangle^3$ term in (5) dominates for sufficiently large \triangle . We let $\tilde{\Delta}>\max\{\triangle_1,\triangle_2,\triangle_3\}$ be the minimum natural number that is divisible by T.

Finally, if $V(n) \leq L$, then $\|Q(n\triangle)\|_{\infty} \leq \sqrt{L}$. Also, $V(n+1) \leq V(n) + (\sqrt{L} + |\mathcal{M}|\tilde{\triangle})^2$ because the total number of arrivals at each time step does not exceed $|\mathcal{M}|$. Combining this with the previously considered case of V(n) > L, we get

$$\mathbb{E}\left[V(n+1) - V(n) \mid Z_{\tilde{\triangle}}(n) \right] \leq -\|Q(n\triangle)\|_{\infty} + \left((\sqrt{L} + |\mathcal{M}|\tilde{\triangle})^2 + \sqrt{L} + 1 \right) \mathbb{1}_B,$$

where $B = \{Z_{\tilde{\Delta}}(n) \in \mathcal{S} : V(n) \leq L\}$ (a finite set). The result then follows from the well-known Foster-Lyapunov drift criterion [16, Theorem 14.0.1].

Proof of (2)

Consider on-ramp $i\in\mathcal{M}$. If for every interval $[b_im,b_i(m+1))$ in $[n\triangle,(n+1)\triangle-1],\ a_i$ vehicles cross on-ramp i, then

$$N_{i}((n+1)\triangle) \leq N_{i}(n\triangle) + A_{i}(n\triangle + 1, (n+1)\triangle) - \lfloor \frac{\triangle}{b_{i}} \rfloor a_{i}$$

$$\leq N(n\triangle) + A_{i}(n\triangle + 1, (n+1)\triangle) - \frac{a_{i}}{b_{i}}\triangle + a_{i},$$

where $A_i(m_1,m_2)$ is the total number of arrivals in the network during the time interval $[m_1,m_2]$ that want to cross on-ramp i's node in order to reach their destination. If not, then there exist some time step in $[n\triangle,(n+1)\triangle-1]$ during which a valid slot that is empty crosses on-ramp i's node. Let $m_i \in [n\triangle,(n+1)\triangle-1]$ be the last time step at which this happens, i.e., $N_i(m_i+1)=N_i(m_i)+A_i(m_i+1,m_i+2)$. Hence,

$$N_i((n+1)\triangle) \le N_i(m_i+1) + A_i(m_i+2, (n+1)\triangle) + a_i - \frac{a_i}{b_i}((n+1)\triangle - m_i - 1).$$

By the assumption of the theorem, i.e., $\rho_i < \frac{a_i}{b_i}$ for all $i \in \mathcal{M}$, there exists $\delta > 0$ such that for all $i \in \mathcal{M}$,

$$\rho_i + \delta < \frac{a_i}{b_i}.$$

Thus, (6) can be rewritten as

$$N_i((n+1)\triangle) \le N_i(m_i+1) + A_i(m_i+2, (n+1)\triangle) - (\rho_i+\delta)((n+1)\triangle - m_i-1) + a_i.$$
(7)

Furthermore, the queue size at on-ramp i at time m_i+1 cannot exceed T, i.e., $|Q_i(m_i+1)| \leq T$. This is because, a valid slot that is empty crosses on-ramp i at time step m_i . Therefore, the quotas of on-ramp i at time m_i must be zero. This implies that $Q_i(m_i+1)$ only consists of vehicles arriving after the start of the most recent cycle before m_i+1 . Since the number of arrivals to on-ramp i is no more than one per time step, it follows that $|Q_i(m_i+1)| \leq T$. Hence, $N_i(m_i+1) \leq \sum_{j \in \mathcal{M}_i^-} N_j(m_i+1) + |Q_i(m_i+1)| + r_i \leq \sum_{j \in \mathcal{M}_i^-} N_j(m_i+1) + T + r_i$, where r_i is the total number of slots between on-ramp i and any on-ramp $i \in \mathcal{M}_i^-$. Combining this with (7) gives

$$N_{i}((n+1)\triangle) \leq \sum_{j \in \mathcal{M}_{i}^{-}} N_{j}(m_{i}+1) + A_{i}(m_{i}+2, (n+1)\triangle) - (\rho_{i}+\delta)((n+1)\triangle - m_{i}-1) + T + r_{i} + a_{i}.$$
(8)

Repeating the above steps for any on-ramp $j \in \mathcal{M}_i^-$ gives

$$N_j(m_i+1) \le \sum_{k \in \mathcal{M}_j^-} N_k(m_j+1) + A_j(m_j+2, m_i+1) - (\rho_j + \delta)(m_i - m_j) + T + r_j + a_j,$$

where $m_j \in [n\triangle, m_i + 1]$ is defined similar to m_i if at least one valid slot that is empty crosses on-ramp j, and $m_j = n\triangle - 1$ otherwise. This process can be repeated until we find a multiset $J_i \in U$ of on-ramps, such that for each $j \in J_i$ with multiplicity $\operatorname{mult}(j)$, either $m_j^c = n\triangle - 1$, or $m_j^c > n\triangle - 1$ and $\mathcal{M}_j^- = \emptyset$, $c \in [\operatorname{mult}(j)]$. Indeed, such a J_i always exist since the network is acyclic. Let \tilde{J}_i be a multiset containing all the on-ramps visited in this process (for example, if $J_i = \mathcal{M}_i^-$, then $\tilde{J}_i = \{i\} \cup \mathcal{M}_i^-$). For simplicity of notation, we drop the superscript c in m_j^c with the understanding that m_j could take different values for on-ramps with multiplicity greater than one. We also use j^+ to denote an on-ramp that succeeds on-ramp j. By combining all the inequalities in this process we obtain

$$N_i((n+1)\Delta) \le \sum_{j \in J_i} N_j(m_j + 1)$$

$$+ \sum_{j \in \tilde{J}_i} (A_j(m_j + 2, m_{j+} + 1)$$

$$-(\rho_j + \delta)(m_{j+} - m_j) + T + r_j + a_j).$$

Note that if $m_j > n\triangle - 1$ for some $j \in J_i$, then $N_j(m_j + 1) \le T \le N_j(n\triangle) + T$, which implies $\sum_{j \in J_i} N_j(m_j + 1) \le \sum_{j \in J_i} N_j(n\triangle) + |\mathcal{M}||\mathcal{P}|T$. Therefore, $N_i((n+1)\triangle) \le N(n\triangle) + \tilde{A}_i(\triangle) + C_2$, where

$$\tilde{A}_{i}(\triangle) = \sum_{j \in \tilde{J}_{i}} \left(A_{j}(m_{j} + 2, m_{j^{+}} + 1) - (\rho_{j} + \delta)(m_{j^{+}} - m_{j}) \right), \tag{9}$$

where we have dropped the dependence of $\tilde{A}_i(\triangle)$ on other parameters for brevity. Generally, if we start with any $I \in U$, we can repeat the above steps to obtain $\sum_{k \in I} N_k((n+1)\triangle) \leq N(n\triangle) + \tilde{A}_I(\triangle) + C_2$, where $\tilde{A}_I(\triangle)$ has the same

form as (9), with \tilde{J}_I defined similarly. Hence, (2) follows with

$$\tilde{A}(\triangle) = \max_{I \subseteq II} \tilde{A}_I(\triangle). \tag{10}$$

Note that for each $I \in U$, we may assume that there exists some $j \in J_I$ for which $m_j = n \triangle - 1$. For if $m_j > n \triangle - 1$ for all $j \in J_I$, then $N_j(m_j + 1) \leq T$ for all $j \in J_I$, which implies

$$\sum_{k \in I} N_k((n+1)\triangle) \le C_1 \triangle + C_2 < N(n\triangle) - \triangle \le N((n+1)\triangle),$$

where in the first inequality we have used the following fact: the total number of arrivals to the network is bounded by $|\mathcal{M}|$ at each time step. Hence, for each $j \in \tilde{J}_I$, $A_j(m_j+2,m_{j+}+1) \leq (m_{j+}-m_j)|\mathcal{M}| \leq |\mathcal{M}|\triangle$. Also, in the third inequality we have used the following fact: $N(\cdot)$ decreases by at most one during each time step. Therefore, $N((n+1)\Delta) \geq N(n\Delta) - \Delta$. The inequality (11) implies that we can exclude I from the maximum in (10). Thus, we may assume such $j \in \tilde{J}_I$ with $m_j = n\Delta - 1$ exists.

Proof of (3)

Given $I\in U$ with the corresponding multiset J_I , let $m_j=n\triangle-1$ for some $j\in J_I$. Hence, there exists a sequence of on-ramps $\{i_1,\ldots,i_K\}\subset \tilde{J}_I$ with $i_1=j$ and $i_K\in I$ such that for k< K, $i_k^+=i_{k+1}$. Therefore, $\sum_{k\in [K]}(m_{i_k^+}-m_{i_k})=\triangle$. Consider the sequence $\{A_{i_k}(m,m+1)\}_{m=n\triangle+1}^\infty,$ where the indices $i_k\in \{i_1,\ldots,i_K\}$ are allowed to depend on time m. For a given $m\in [n\triangle+1,\infty)$, the term $A_{i_k}(m,m+1)$ is independent of the other terms in the sequence, $\rho_{i_k}=\mathbb{E}\left[A_{i_k}(m,m+1)\right]$ is bounded, and $\sigma_{i_k}^2:=\mathbb{E}\left[A_{i_k}^2(m,m+1)-\rho_{i_k}^2\right]$ is (uniformly) bounded for all $i_k\in \{i_1,\ldots,i_K\}$. As a result, $\lim_{\Delta\to\infty}\sum_{m=n\Delta+1}^{(n+1)\Delta}(m-n\Delta)^{-2}\sigma_{i_k}^2$ is also bounded. From Kolmogorov's strong law of large numbers [17, Theorem 10.12], we have, with probability 1,

$$\lim_{\triangle \to \infty} \frac{1}{\triangle} \left(\sum_{m=n\triangle+1}^{(n+1)\triangle} A_{i_k}(m,m+1) - \sum_{m=n\triangle+1}^{(n+1)\triangle} \rho_{i_k} \right) = 0.$$

Hence, with probability one, there exists $\triangle_1, \varepsilon > 0$ such that $\varepsilon < \delta$ and for $\triangle > \triangle_1$ we have

$$\sum_{m=n\triangle+1}^{(n+1)\triangle} A_{i_k}(m,m+1) - \sum_{m=n\triangle+1}^{(n+1)\triangle} \rho_{i_k} < \varepsilon \triangle.$$

This implies that, with probability one, for $\triangle > \triangle_1$ we have

$$\sum_{k \in [K]} \left(A_{i_k} (m_{i_k} + 2, m_{i_k^+} + 1) - (\rho_k + \delta) (m_{i_k^+} - m_{i_k}) \right)$$

$$< (\varepsilon - \delta) \triangle.$$

Furthermore, for any other $j \in \tilde{J}_I \setminus \{i_1,\ldots,i_K\}$, with probability one, there exists $M_j > 0$ such that $m_{j^+} - m_j > M_j$ implies $A_j(m_j + 2, m_{j^+} + 1) - (\rho_j + \delta)(m_{j^+} - m_j) < (\varepsilon - \delta)(m_{j^+} - m_j)$, and if $m_{j^+} - m_j \leq M_j$, then $A_j(m_j + 2, m_{j^+} + 1) - (\rho_j + \delta)(m_{j^+} - m_j) \leq (|\mathcal{M}| - \rho_j - \delta)M_j$,

since the total number of arrivals to the network is bounded by $|\mathcal{M}|$ at each time step. Therefore, with probability one, for $\triangle > \max\{\triangle_1, \triangle_2 := \frac{1}{\delta - \varepsilon} \sum_j (|\mathcal{M}| - \rho_j - \delta) M_j\}$ we have

$$\frac{1}{\triangle} \sum_{j \in \tilde{J}_I} \left(A_j(m_j + 2, m_{j+} + 1) - (\rho_j + \delta)(m_{j+} - m_j) \right)$$

$$< (\varepsilon - \delta) + \frac{1}{\triangle} \sum_j (|\mathcal{M}| - \rho_j - \delta) M_j < 0$$
(12)

which in turn implies that, with probability one, $\limsup_{\Delta \to \infty} \tilde{A}_I(\Delta)/\Delta < 0$. It follows that, with probability one, $\limsup_{\Delta \to \infty} \tilde{A}(\Delta)/\Delta < 0$.

Finally,

$$\left| \frac{\tilde{A}_I(\triangle)}{\triangle} \right| \le \frac{1}{\triangle} \sum_{j \in \tilde{J}_I} (|\mathcal{M}| + \rho_j + \delta) (m_{j^+} - m_j) \le C_2, \tag{13}$$

where we have used $A_j(m_j+2,m_{j^+}+1) \leq (m_{j^+}-m_j)|\mathcal{M}|$ in the first inequality using the fact that the total number of arrivals to the network is bounded by $|\mathcal{M}|$ at each time step. Therefore, the sequences $\left\{\tilde{A}(\triangle)/\Delta\right\}_{\Delta=1}^{\infty}$ and $\left\{\left(\tilde{A}(\triangle)/\Delta\right)^2\right\}_{\Delta=1}^{\infty}$ are upper bounded by integrable functions. Fatou's lemma [17] then implies

$$\begin{split} \limsup_{\Delta \to \infty} \mathbb{E} \left[\frac{\tilde{A}(\Delta)}{\Delta} \right] &\leq \mathbb{E} \left[\limsup_{\Delta \to \infty} \frac{\tilde{A}(\Delta)}{\Delta} \right] < 0, \\ \limsup_{\Delta \to \infty} \mathbb{E} \left[\left(\frac{\tilde{A}(\Delta)}{\Delta} \right)^2 \right] &\leq \mathbb{E} \left[\limsup_{\Delta \to \infty} \left(\frac{\tilde{A}(\Delta)}{\Delta} \right)^2 \right] < C_3, \end{split}$$

where $C_3 > C_2^2$ is some constant. This in turn gives (3).