1999 European Control Conference (ECC)
31 August – 3 September 1999, Karlsruhe, Germany

SYNCHRONIZATIONANDOBSERVERSFOR NONLINEARDISCRETETIMESYSTEMS

H.J.C. Huijberts*, **T. Lilge**†, **H. Nijmeijer**‡

* *Faculty of Mathematics and Computing* Science, *Eindhoven University of Technology, P. 0. Box 513, 5600 MB Eindhoven, The Netherlands. Fax: +31-40-2-44-24-89, email: hjch@win.tue.nl.*

† *Institut für Regelungstechnik, Universität Hannover, Appelstrasse 11, D-301 67 Hannover, Germany. Fax: +49-511- 7-62-45-36, email: lilge@irt. uni-hannover. de.*

‡ *Faculty of Mathematical Sciences, University of Twente, P. 0. Box 217, 7500 AE Enschede, The Netherlands, and Faculty of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513., 5600 MB Eindhoven, The Netherlands. Fax: +31-53-4-34-07-33, email: H.Nijmeijer@math.utwente.nl.*

Keywords : Synchronization, observers, nonlinear discrete time systems.

Abstract

A method is described for the synchronization of nonlinear discrete time dynamics. The methodology consists of constructing observer-receiver dynamics that exploit at each time instant the drive signal and buffered past values of the drive signal. In this way, the method can be viewed as a dynamic reconstruction mechanism, in contrast to existing static inversion methods from the theory of dynamical systems.

1 Introduction

Following PECORA and CARROLL [16] a huge interest in the synchronization of two coupled systems has arisen. This research is partly motivated by its possible use in secure communications, cf. [5]. Often, like in [16] a drive/response, or transmitter/receiver, viewpoint is assumed. In a discrete-time context, this typically allows for a description of the transmitter as a n-dimensional dynamical system

$$x_1(k+1) = f_1(x_1(k), x_2(k)) \tag{1}$$
$$x_2(k+1) = f_2(x_1(k), x_2(k)) \tag{2}$$

where $x_1(\cdot)$ and $x_2(\cdot)$ are vectors of dimension m and l, with $m + l = n$ and $x(k) = (x_1(k), x_2(k))$. Given $x_1(\cdot)$ as the drive signal, the receiver dynamics are taken as a copy of (2),

$$\tilde{x}_2(k+1) = f_2(x_1(k), \tilde{x}_2(k)). \tag{3}$$

Synchronization of transmitter and receiver now corresponds to the asymptotic matching of (2) and (3), that is

$$\lim_{k\to\infty} \|x_2(k) - \tilde{x}_2(k)\| = 0. \tag{4}$$

Clearly (4) will not be satisfied in general and, in fact, conditions on f_1 and f_2 that guarantee this condition are only partially known, cf. [15]. For that reason several attempts for achieving synchronization of signals like $x_2(\cdot)$ and $\tilde{x}_2(\cdot)$ have been proposed. In particular we like to recall the (reduced) observer viewpoint advocated in [14] which basically admits the construction of dynamics

$$\tilde{x}_2(k+1) = \tilde{f}_2(x_1(k), \tilde{x}_2(k)). \tag{5}$$

such that (4) holds, whatever initial conditions (1), (2) and (5) have. Although (5) enlarges the idea of using the copy (3) for (2), there are many systems for which (4) will not be met, no matter how f_2 in (5) is chosen.

There is, however, a natural generalization of (5) that consists in exploiting at each time instant k the drive signal $x_1(k)$ and N past values $x_1(k-1), \ldots x_1(k-N)$, as was proposed in e.g. [4],[8],[9]. Thus, as receiver dynamics we use the following system

ISBN 978-3-9524173-5-5

$$\tilde{x}(k+1) = f(\tilde{x}(k), x_1(k), \ldots, x_1(k-N)). \qquad (6)$$

Here, $\tilde{x}(\cdot)$ is n-dimensional, and $\tilde{f}(\cdot,\cdot)$ and N are such that

$$\lim_{k\to\infty} \|x(k) - \tilde{x}(k)\| = 0. \qquad (7)$$

The receiver (6) acts as an 'extended' observer for the system (1,2), in that also past values of the drive signal $x_1(\cdot)$ are used. It turns out that under fairly weak conditions receiver dynamics (6) exist such that the transmitter (1,2) and the receiver (6) synchronize, see Section 3.

Actually, the necessary conditions involved are closely related with *global observability,* cf. [13], or, the Takens-Aeyels-Sauer Reconstruction Theorem, see [18],[1],[2],[17],[15]. However, a crucial difference in our work with the Reconstruction Theorem is that (6) forms a dynamic 'inversion' for the state $x(\cdot)$, whereas in the Reconstruction Theorem one computes the state at some time instant by inverting the observability map, which determines $x_2(k)$ from $x_1(k), ..., x_1(k-N)$. It is interesting to note that an alternative using look-up tables for this procedure was proposed in [12].

The proposed transmitter/receiver synchronization using a receiver of the form (6) can be demonstrated numerically on several examples from the literature, see e.g. [3,11]. In this paper, we will, among others, consider the example from [3]. The present paper is an expanded version of the paper [6].

The organization of this paper is as follows. In the next section we derive conditions under which a discrete time system admits a so called extended observer form. Using this extended observer form, two types of observers are constructed in Section 3. The performance of these observers is illustrated by means of two examples. Section 4 contains some concluding remarks.

2 Existence of extended observer forms

In this section, we focus on a first step in observer design for nonlinear, discrete-time, autonomous, single output systems. This step consists in finding conditions under which a system admits a so called extended observer form. It will be illustrated in the following section that for a system that admits an extended observer form, one can design different types of observers. Thus, consider a system of the form

$$x(k+1) = f(x(k)), \; y(k) = h(x(k)), \qquad (8)$$

for $k = 0, 1, 2, \cdots$, where $x(\cdot)$ is a vector of dimension n and $y(\cdot)$ is a scalar. Assuming that the Jacobian of h is nonzero - which implies that a nontrivial signal from the dynamics is transmitted - we can, at least locally, rewrite (8) in a form like (1,2) with $y(k) = x_1(k)$ being one-dimensional. Within the context of synchronization, it is desired to reconstruct (asymptotically) the (n-1)-dimensional $x_2(\cdot)$ on the basis of the sequence $x_1(k)$ ($k = 1, 2, \cdots$). we will do this using a suitably selected dynamics of the form (6) which basically means that we treat the synchronization problem as an observer problem, cf. [14]. Without loss of generality we can assume that f(0) = 0 and h(0) = 0.

For (8) we define the so-called *observability map* ψ by

$$\psi(x) := \begin{pmatrix} h(x) \\ h \circ f(x) \\ \vdots \\ h \circ f^{n-1}(x) \end{pmatrix} \qquad (9)$$

where $h \circ f(x) := h(f(x)), f^1 := f, f^j := f \circ f^{j-1}$. The system (8) is called *strongly locally observable* around $x = 0$ if the Jacobian $(\partial\psi/\partial x)$ (0) is invertible.

Now consider a strongly locally observable system (8), and define $s_i(x) :=$ ho $f^{i-1}(x) (i = 1, \cdots, n)$. Since (8) is strongly locally observable, $s = \mathrm{col}(s_1, ., s_n)$ forms a new set of coordinates for (8) around x = 0. In what follows, we will assume throughout that s forms a new set of coordinates *globally,* i.e., ψ in (9) is a global diffeomorphism. It is straightforwardly checked that in these new coordinates the system (8) takes the form

$$s(k+1) = \begin{pmatrix} s_2(k) \\ s_3(k) \\ \vdots \\ s_n(k) \\ f_s(s(k)) \end{pmatrix}, \; y(k) = s_1(k) \qquad (10)$$

This form is called the *observable form* of the system.

The system (8) now is said to admit an *extended* observer form *with buffer* N (EOF(N)) if there exist an extended coordinate change parametrized by $y(k-1), \cdots, y(k-N)$ of the form $z(k) = P(s(k), y(k-1), \cdots, y(k-N))$ for (10) andan invertible output transformation $\tilde{y}$ = p(y) such that

$$\begin{aligned} z(k+1) &= Az(k) + \Phi(\tilde{y}(k), \cdot, \tilde{y}(k-N)) \\ \tilde{y}(k) &= Cz(k) \end{aligned} \qquad (11)$$

where the pair (C, A) is observable.

The following result gives conditions under which a system admits an EOF(N).

Theorem 2.1 Consider a *system (IO) in observable form, and let* $N \in \{0, 1, \cdots, n-1\}$ *be given. Then (10)* admits an *extended observer* form *with buffer N* ***if*** *and only* ***if*** *there exist an invertible mapping* $\tilde{p} : \mathbb{R} \to \mathbb{R}$ *and mappings* $\tilde{\phi}_1, \cdot\cdot, \tilde{\phi}_{n-N} : \mathbb{R}^{N+1} \to \mathbb{R}$ *such that*

$$f_s(s) = \tilde{p}\left(\sum_{j=1}^{n-N} \tilde{\phi}_j \left(s_{n-j+1}, \cdots, s_{n-N-j+1}\right)\right) \qquad (12)$$

Proof *(necessity)* Assume that (10) admits an EOF(N). Since the pair (C, A) in (11) is observable, we may assume without loss of generality that (C, A) is in dual Brunovsky canonical form. Comparing (10) and (11), we then see that

$$z_1(k) = \tilde{y}(k) = p(s_1(k))$$

which implies on its turn that

$$z_2(k) = z_1(k+1) - \phi_1(p(y(k)), \cdots, p(y(k-N))) = p(s_2(k)) - \phi_1(p(y(k)), \cdots, p(y(k-N)))$$

Continuing in this way, this gives that

$$z_i(k) = p(s_i(k)) - \sum_{j=0}^{i-2} \phi_j(p(y(k+j)), \cdots, p(y(k-N+j))) \tag{13}$$

For $i = n$, this gives together with (10), (11)

$$p(f_s(s(k))) = \sum_{j=0}^{n-1} \phi_{n-j}(p(y(k+j)), \cdots, p(y(k-N+j))) = \sum_{j=0}^{n-1} \phi_{n-j}(p(s_{j+1}(k)), \cdots, p(s_1(k)), p(y(k-1)), \cdots, p(y(k-N+j)))$$

Since the left hand side of the above equality does not depend on past values of y, and the past values of y occur in a triangular way in the right hand side, we conclude that in fact the right hand side of the above equality does not depend on the past values of y. This gives that we have

$$p(f_s(s)) = \sum_{j=1}^{n-N} \phi_j(p(s_{n-j+1}), \cdots, p(s_{n-N-j+1})) + \sum_{j=n-N+1}^{n} \phi_j(p(s_{n-j+1}), \cdots, p(s_1))$$

From this equality, our claim is established by defining $\tilde{p} := p^{-1}$,

$$\tilde{\phi}_j(s_{n-j+1}, \cdots, s_{n-N-j+1}) := \phi_j(p(s_{n-j+1}), \cdots, p(s_{n-N-j+1}))$$

for $j = 1, \cdots, n$ -N $-$ 1, and

$$\phi_{n-N}(s_{N+1}, \cdots, s_1) := \phi_{n-N}(p(s_{N+1}), \ . \ , p(s_1)) + \sum_{j=n-N+1}^{n} \phi_j(p(s_{n-j+1}), \cdots, p(s_1))$$

_ *(sufficiency)* Assume that there exist functions p, $\phi_1, \cdot, \tilde{\phi}_{n-N}$ such that (12) holds. Define p := $\tilde{p}^{-1}$, $\phi_j := \tilde{\phi}_j \circ p_e (j = 1, \cdots, n-N)$, where $p_e : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is given by $p_e = \mathrm{col}(p, \cdots, p)$. Further, define new extended coordinates z according to (13). It is then straightforwardly checked that the system in the coordinates z is in extended observer form with buffer N. ■

From Theorem 2.1 it follows that a system (10) always admits an EOF(n- 1). For the other extreme case, N = 0, it follows from Theorem 2.1 that (10) admits an EOF(0) if and only if there exist an invertible function $\tilde{p} : \mathbb{R} \to \mathbb{R}$ and functions $\phi_1, \cdots, \phi_n : \mathbb{R} \to \mathbb{R}$ such that

$$f_s(s) = \tilde{p}(\phi_1(s_n) + \cdots + \phi_n(s_1))$$

If one requires $\tilde{p}$ to be the identity, necessary and sufficient conditions for existence of $\phi_1, \cdots, \phi_n$ are that $(\partial^2 f_s / \partial s_i \partial s_j) = 0$ for $i \neq$ j. This was proved in [10]. To generalize this result to the case where $\tilde{p}$ is not required to be the identity, we define the one-forms

$$\omega_i := \left(\frac{\partial f_s}{\partial s_i}\right) ds_i \ (i = 1, \cdots, n) \tag{14}$$

Assuming without loss of generality that $\omega_i \neq 0 \ (i = 1, \cdots, n)$, we let $\tau_1, \cdots, \tau_n$ be vector fields satisfying $\tau_i \rfloor \omega_j = \delta_{ij}$. Then the following result may be proved (see [7], where also conditions for existence of an EOF(k) with $0 < k <$ n $- 1$ are presented).

Theorem 2.2 *The system (8) admits* an *EOF*(0) *if and only if for* $i, j = 1, \cdots, n$ we have

$$d\omega_i \wedge \omega_j + d\omega_j \wedge \omega_i = 0 \tag{15}$$

and

$$\mathcal{L}_{\tau_i}([\tau_i, \tau_j] \rfloor \omega_i) = \mathcal{L}_{\tau_j}([\tau_j, \tau_i] \rfloor \omega_j) \tag{16}$$

■

In [7]), also conditions for existence of an EOF(k) with $0 < k <$ n $- 1$ presented.

3 Observer design for N = $n - 1$

From Theorem 2.1 it follows that (8) always admits an EOF(n $-$ 1). Following the sufficiency part of the proof of Theorem 2.1, this EOF(n $-$ 1) takes the form

$$\begin{aligned} z(k+1) &= Az(k) + \Phi(y(k-n+1), \ldots, y(k)) \\ y(k) &= z_1(k) \end{aligned} \tag{17}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdot & \cdot & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

and

$$\Phi(y(k-n+1),, y(k)) = \begin{pmatrix} f_s(y(k-n+1),\cdots,y(k)) \\ 0 \\ \\ \\ 0 \end{pmatrix}$$

From (17), two types of observers may be derived. An observer of **type 1** has the form

$$\begin{aligned} \hat{z}(k+1) &= A\hat{z}(k) + \Phi(y(k-n+1), \ldots, y(k)) + \\ &\quad q(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= \hat{z}_1(k), \quad k \geq n-1 \end{aligned} \tag{18}$$

where $q \in I\!R^n$ is chosen in such a way that the polynomial $\sum_{i=1}^{n} q_i \lambda^{i-1} + \lambda^n$ has its poles within the unit circle. It is then straightforwardly checked that indeed (18) is an observer for (17).

The derivation of an *observer of type 2* starts from the observation that the solutions of (17) satisfy $z_i(k) = 0$ ($i = 2, \cdots, n$; $k \geq n-1$). This suggests to consider an observer of the form

$$\begin{aligned} \hat{z}(k+1) &= \Phi(y(k-n+1), \cdots, y(k)) + \\ &\quad \Psi(y(k), \hat{y}(k), \hat{z}_2(k), \cdots, \hat{z}_n(k)) \\ \hat{y}(k) &= \hat{z}_1(k) \end{aligned} \tag{19}$$

where

$$\Psi(y(k), \hat{y}(k), \hat{z}_2(k), \cdots, \hat{z}_n(k)) = \begin{pmatrix} \lambda_1(\hat{y}(k) - y(k)) \\ \lambda_2 \hat{z}_2(k) \\ \vdots \\ \lambda_n \hat{z}_n(k) \end{pmatrix}$$

Defining the error signal e $:= \hat{z} - z$, we obtain the error dynamics

$$e(k+1) = \Lambda e(k), \quad k \geq n-1$$

where A = $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. From these error dynamics it follows that the convergence rate of the i-th component can now be assigned by λ_i, without affecting the other components.

Comparing both observers, we see that the convergence rate of each of the components of observer type 2 can be assigned independently, while this is not the case for observer type 1. Thus, observer type 2 will give a better transient behavior than observer type 1. On the other hand, however, observer type 1 with properly chosen q is in general more robust to (measurement) noise than observer type 2 (cf. [8],[9]).

We conclude this section with two examples.

As a first example, consider the transmitter system

$$\begin{aligned} x_1(k+1) &= \mu(1-\epsilon)x_1(k)(1-x_1(k)) + \epsilon\, x_2(k) \\ x_2(k+1) &= \mu(1-\epsilon)x_2(k)(1-x_2(k)) + \epsilon\, x_1(k) \end{aligned} \tag{20}$$

presented by Badola et al. in [3]. Taking $x_1(k)$ as the drive signal (m = $l = 1$), Badola et al. investigated the synchronization of $x_2(k)$ and the receiver signal $x_3(k)$ of which the dynamics were taken as

$$x_3(k+1) = \mu(1-\epsilon)x_3(k)(1-x_3(k)) + \epsilon\, x_1(k). \tag{21}$$

Our aim is to apply an observer presented in the previous section as receiver dynamics for transmitter (20). With $y(k) = x_1(k)$, it is possible to design observers as in the previous section in order to get the estimates $\hat{x}_1(k)$, $\hat{x}_2(k)$ for the signals $x_1(k)$, $x_2(k)$. The resulting observer equations are omitted for reasons of space. For the subsequent simulations, the initial conditions $x_1(0)$ = 0.2, $x_2(0)$ = 0.4, $\hat{x}_1(0) = \hat{x}_2(0)$ = 0.7 and parameters μ = 3.7, ϵ = 0.09 were used. Note that (20) represents two weakly coupled logistic equations, each of them exhibiting chaotic behavior. Following [3], $x_2(k)$ and x_3 (k) do not synchronize for these parameters and $x_3(0) = \hat{x}_2(0)$ = 0.7 while the observers obtained here show satisfactory behavior. Exemplary simulations of the observer errors applying observer type 1 and 2 can be seen in figure 1 for $\lambda_1 = \lambda_2$ = 0.5 (for the observer type 1, this corresponds to the choice q_0 = **0.25**, q_1 = -1). Both observers provide very good estimations after 20 iterations with a maximum absolute observer error less than 0.002. As already mentioned in the previous section, observer type 2 shows smaller observer errors during transient time than observer type 1.

As a second example, we want to extend system (20) to the third order transmitter system

$$\begin{aligned} x_1(k+1) &= \mu(1-\epsilon)x_1(k)(1-x_1(k)) + \epsilon\, x_2(k) \\ x_2(k+1) &= \mu(1-\epsilon)x_2(k)(1-x_2(k)) + \epsilon\, x_3(k) \\ x_3(k+1) &= \mu(1-\epsilon)x_3(k)(1-x_3(k)) + \epsilon\, x_1(k) \end{aligned} \tag{22}$$

with the drive signal $y(k) = x_1(k) (m = 1, l = 2)$. In this case, observing the unknown signals $x_2(k)$ and $x_3(k)$ is more difficult because $x_3(k)$ does not directly influence the measured drive signal $x_1(k)$ but only via x_2 (k). For this reason, the coupling parameter ϵ was increased up to 0.35 while the second parameter μ = 3.7 was not changed. For $x_1(0) = 0.2$, $x_2(0) = 0.4$, $x_3(0) = 0.6$, $\hat{x}_i(0) = 0.7$, $i = 1, 2, 3$, and eigenvalues of the observer error dynamics λ_i = 0.5, $i = 1, 2, 3$ (for the observer type 1, this corresponds to the choice $q_0 = -0.125, q_1 = 0.75, q_2 = -1.5$), the observer errors applying observer types 1 and 2 are shown in figure 2.

It can be seen that $|e_3(k)|$ reaches very high values (up to 7500 with observer type 1) during transient time. Nevertheless, after 20 iterations the maximum absolute observer error is less than 0.007.

The examples show the efficiency of observers taken as receiver dynamics in synchronization problems, especially when taking into consideration that synchronization of transmitter system and observer is guaranteed if

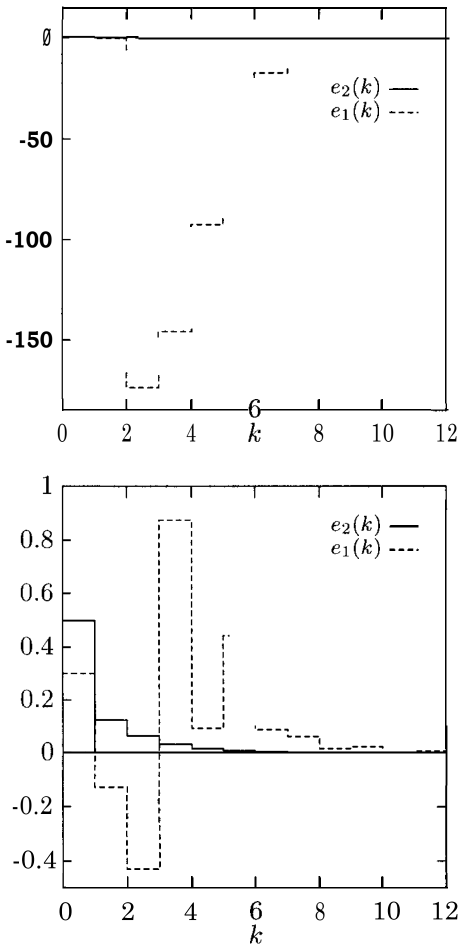

Fig. 1: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)\,(i = 1, 2)$ for system (20) and respectively observer type 1 (18) and observer type 2 (19)

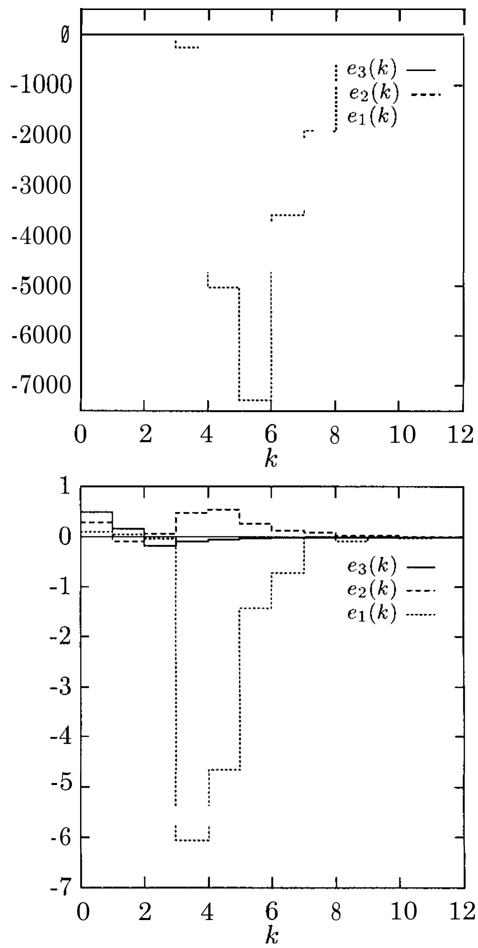

Fig. 2: Observer errors $e_i(k) = \hat{x}_i(k) - x_i(k)\,(i = 1, 2, 3)$ for system (22) and respectively observer type 1 (18) and observer type 2 (19)

the system is globally observable. Moreover, the eigenvalues of the observer error dynamics and consequently the convergence rate are selectable. For synchronization as presented in [3], one is neither able to guarantee synchronization nor able to influence the number of steps until synchronizations occurs.

4 Concluding remarks

We have presented a control perspective on synchronization of discrete time transmitter systems. The methodology of designing an observer as the receiver system enables the exponential synchronization of transmitter and receiver, and does not require any condition on conditional Lyapunov exponents as is often the case when identical transmitter and receiver systems are used (cf. [15],[16]). Essentially, the observer scheme that is used in this paper exploits at each time instant k the last $n-1$ measurements of the drive signal $y(k), y(k-1), \ldots\ldots y(k-n+1)$, with n being the dimension of the transmitter dynamics, and can be viewed as a dynamic mechanism for the (Takens-Aeyels-Sauer) Reconstruction Theorem, provided the system satisfies a global observability condition. Contrary to [3], our results are valid no matter how the initial conditions are chosen.

The observer viewpoint on the synchronization problem has also been advocated for continuous time systems, see [14], but the scheme as we used here in discrete time has no direct analogue in continuous time. An obvious way to proceed in continuous time therefore could exist in (fast) sampling of the continuous time transmitter and then design a discrete time observer as receiver. In that case the synchronization error becomes small -depending on the sampling time - but not identically zero. However, in many applications this will not be a big problem.

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