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# ROBUST NONLINEAR FILTERING OF STOCHASTIC VOLATILITY IN FINANCE

**S.I.Aihara***, **A.Bagchi** †

* The Science University of Tokyo,
Suwa College Toyohira,5000-1
Chino, Nagano, Japan
Tel:+81 266 73 1201 Fax:+81 266 73 9990
E-mail:aihara@ss.suwa.sut.ac.jp

† Faculty of Mathematical Sciences
University of Twente, P.O.Box 217, 7500AE
Enschede, The Netherlands
Tel:+31 53 489 3406 Fax:+31 53 434 0733
E-mail:bagchi@math.utwente.nl

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## Abstract

Volatility of the stock price is the key to the pricing problem of stock related derivatives in finance. Volatility appears in the diffusion term of the usual modeling of stock prices. One popular approach is to take volatility to be stochastic, and assumes that it satisfies a stochastic differential equation. Taking the stock price to be the observation, we may then pose the filtering problem of estimating the volatility on line based on the stock price data. This is an unconventional filtering problem which we solve in this paper. But even more interesting is the fact that this filtering algorithm is inherently not robust. In the rest of the paper we derive the robust form of this filter.

## 1 Introduction

There is a general consensus in the mathematical finance community that the assumption of constant volatility in the Black-Scholes model is not borne out by the actual data. Various alternative models have been proposed, including stochastic ones [1] [2]. One popular model with stochastic volatility is due to [3]. One immediate question that arises is : can one estimate the volatility (preferably online) as one gathers the stock price data ? On a purely theoretical level, the solution to this problem is trivial, as the quadratic variation of the stock price over a time interval should be exactly equal to the integral of the volatility function over the same time interval. In a previous paper [4], we show that this theoretical result has no meaning when applied to real stock price data, and to propose a new formulation of the model leading to a well-defined estimation problem for the case that the noise processes in the model for stock price and volatility are mutually independent. Unfortunately, it seems that we can not apply the method in [4] to the case when these noises are correlated. The purpose of this paper is to reformulate the filtering problem for stochastic volatility whose driving noise has a correlation to the noise in stock price. First we introduce a complementary observation mechanism which does not contain the signal dependent noise. Hence, we can derive the Zakai equation for the newly introduced observation model. After showing that we can convert the introduced observation mechanism to the original one, the Zakai equation for the original model can be derived. At present we don't know the useful transformation to obtain the robust form of the Zakai equation for the noise correlation case and the splitting-up method seems to be not applicable to the noise correlation case in general. However to solve the Zakai equation, we can develop the splitting-up method to fit our filtering problem of stochastic volatility by using the method of a stochastic characteristic curve.

## 2 Problem Formulation

We consider the following model for the price of a stock market:

$$dS(t) = S(t)\{\mu dt + \sqrt{V(t)}dW_1(t)\},\ S(0) = S_o. \quad (2.1)$$

with

$$\begin{aligned} dV(t) &= aV(t)dt + \xi(V(t)dW_2(t) \\ &\quad +\delta\sqrt{V(t)}dW_3(t)), V(0) = V_o, \end{aligned} \quad (2.2)$$

where $S(t)$ denotes the price of stock market at time $t$ , $V(t)$ is stochastically varying volatility and $a, \xi, \delta$ are positive constants. When $\delta$ is zero, we get the usual stochastic volatility model originally proposed by Hull and White [3] . We shall see later that this extra term is essential for deriving the robust form of the filter when the noise terms in the models of the stock and the volatility are correlated.

In this model, we assume that

- (i) The price of stock market is observable. Our first objective is to estimate the stochastic volatility $V(t)$ based



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on the observation of stock prices $S(s); 0 \leq s \leq t$. Our next objective is to solve the pricing problem for options in this framework.

- (ii) $W_1, W_2(t)$ and $W_3$ are Brownian motion processes with

$$
\begin{aligned}
E\{|W_1(t)|^2\} &= E\{|W_2(t)|^2\} \\
&= E\{|W_3(t)|^2\} = t \qquad (2.3) \\
E\{W_1(t)W_2(t)\} &= \rho_c t \qquad (2.4)
\end{aligned}
$$

and $W_3$ is independent of $W_1$ and $W_2$.

The observation process $S(t)$ can be explicitly solved as

$$
\begin{aligned}
S(t) = S_o \exp\{\mu t - \frac{1}{2}\int_0^t V(s)ds \\
+ \int_0^t \sqrt{V(s)}dW_1(s)\} \qquad (2.5)
\end{aligned}
$$

We introduce the process $Z(t)$ defined by

$$
Z(t) = \log\{S(t)/S_o\}. \qquad (2.6)
$$

Then the modified observation process $Z(t)$ is the solution of

$$
dZ(t) = \mu dt - \frac{1}{2}V(t)dt + \sqrt{V(t)}dW_1(t). \qquad (2.7)
$$

Now it seems that we have a usual filtering problem for the signal process $V(t)$ and the observation process $Z(t)$;

$$
\begin{aligned}
dV(t) &= aV(t)dt + \xi(V(t)dW_2(t) \\
&\quad +\delta\sqrt{V(t)}dW_3(t)), V(0) = V_o \qquad (2.8) \\
dZ(t) &= \mu dt - \frac{1}{2}V(t)dt \\
&\quad +\sqrt{V(t)}dW_1(t), Z(0) = 0. \qquad (2.9)
\end{aligned}
$$

However for the above system we can not formulate the usual filtering problem because the observation noise depends on the signal process $V(t)$ (see [5]). Here we shall repeat the argument stated in [4]. In fact, for the theoretical model (2.8) and (2.9), we have the following results:

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{N(n)} |Z(t_{i+1}^{(n)}) - Z(t_i^{(n)})|^2 \\
= \int_0^t V(s)ds \ \ a.s. \qquad (2.10)
\end{aligned}
$$

where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{N(n)}^{(n)} = t_f$ with $\Delta t^{(n)} = \max_{1\leq j\leq N(n)} |t_j^{(n)} - t_{j-1}^{(n)}| \leq \frac{c_1}{n}, \ \ N(n) \leq c_2 n$. More simply, by using the Ito formula, we also have the following filter:

$$
Z^2(t) - 2\int_0^t Z(s)dZ(s) = \int_0^t V(s)ds. \qquad (2.11)
$$

Noting that $V(t)$ is continuous, we can obtain the $V(t)$-process by differentiating $(Z^2(t) - 2\int_0^t Z(s)dZ(s))$ with respect to $t$, i.e.,

$$
X(t,Z) = \frac{d}{dt}(Z^2(t) - 2\int_0^t Z(s)dZ(s)) \qquad (2.12)
$$

is exactly equal to $V(t), \forall t > 0$.

In practice, however, the real data $Z(t)$ is never continuously available in t. We always have discrete observations, however small is the time interval. Therefore, for real data, the formula (2.10) can not be used to estimate the process $V(t)$ for any $t \in (0, T]$.

Our point of departure from the conventional approach in the observation that the real data $Z(t)$ will be very close, but not necessarily identical to the solution of equation (2.7). In practice , $Z(t)$ is only available at discrete time-points, however small the time interval.

# 3 Nonlinear Filtering Problem

## 3.1 Derivation of Zakai Equation

In order to apply the usual non-linear filtering theory to the observation data (2.9), the main difficulty is the dependence of the signal $V(t)$ on the noise. So first we replace this dependence,i.e., we introduce the new observation mechanism:

$$
\tilde{Z}(t) = \int_0^t \frac{\mu - \frac{1}{2}V(s)}{\sqrt{V(s)}}ds + W_1(t). \qquad (3.1)
$$

First we consider the filtering problem for (2.8) with (3.1).

Noting that the initial condition $V(0)$ has an initial probability density $p_o(v)$, we obtain:

**Theorem 3.1** *[6] The unnormalized probability density function $p(t, v|\tilde{\mathcal{Z}}_t)$ is a solution of*

$$
\begin{aligned}
p(t, v|\tilde{\mathcal{Z}}_t) &= p_o(v) + \int_0^t A^* p(s, v|\tilde{\mathcal{Z}}_s)ds \\
&\quad + \int_0^t \tilde{B}^*(\mu, v)p(s, v|\tilde{\mathcal{Z}}_s)d\tilde{Z}(s), \quad (3.2)
\end{aligned}
$$

*where*

$$
\tilde{B}^*(\mu, v)(\cdot) = (\frac{\mu - \frac{1}{2}v}{\sqrt{v}} - \rho_c\xi)(\cdot) - \rho_c\xi v\frac{\partial(\cdot)}{\partial v} \qquad (3.3)
$$

*and*

$$
\begin{aligned}
A^*(\cdot) &= \frac{\partial}{\partial v}(\frac{\xi^2(v^2 + \delta^2 v)}{2}\frac{\partial(\cdot)}{\partial}) \\
&\quad + \frac{\partial}{\partial v}((\xi^2(v + \frac{\delta^2}{2}) - av)(\cdot)). \qquad (3.4)
\end{aligned}
$$

Now we need to reconstruct the observation data $Z(t)$ from $\tilde{Z}(t)$. Noting that

$$
dZ(t) = \sqrt{V(t)}d\tilde{Z}(t),
$$

i.e., in the Zakai-equation the signal $V(t)$ is fixed as $v$, we obtain the following theorem:

**Theorem 3.2** *The unnormalized probability density function $p(t,v|\mathcal{Z}_t)$ for the original observation data $Z(t)$ is a solution of*

$$\begin{aligned} p(t,v|\mathcal{Z}_t) &= p_o(v) + \int_0^t A^* p(s,v|\mathcal{Z}_s)ds \\ &\quad + \int_0^t B^*(\mu,v)p(s,v|\mathcal{Z}_s)dZ(s), \end{aligned} \tag{3.5}$$

*where*

$$B^*(\mu,v)(\cdot) = (\frac{\mu}{v} - \frac{1}{2} - \frac{\rho_c\xi}{\sqrt{v}})(\cdot) - \rho_c\xi\sqrt{v}\frac{\partial(\cdot)}{\partial v}. \tag{3.6}$$

### 3.2 Realization of Zakai-equation

If the observation noise is independent of the system noise, it is possible to transform the Zakai equation to the robust form. Hence we can use the real observation data . However in our situation at present we don't know the robust form. So a possible way to solve the derived Zakai equation by using the observation data is to introduce the time discretized approximation of the Zakai equation as used by Pardoux [7]. To do this we need to extend the splitting up method in [8] to the noise dependent case . Let

$$\Delta = \frac{T}{n+1}$$

We split $[0,T]$ in steps $0, \Delta, \cdots, (n+1)\Delta$. Consider an interval $[i\Delta, (i+1)\Delta[, i = 0,1,\cdots,n$. We set

$$\frac{dp_{1i}^{(n)}(t)}{dt} - (A^* - \frac{1}{2}B^*B^* - \frac{\eta}{2})p_{1i}^{(n)}(t) = 0, \quad t \in [i\Delta, (i+1)\Delta[ \tag{3.7}$$

$$dP_{2i}^{(n)}(t) + \frac{\eta}{2}p_{2i}^{(n)}(t)dt = B^* p_{2i}^{(n)}(t) \circ dZ(t), \quad t \in [i\Delta, (i+1)\Delta[ \tag{3.8}$$

with

$$p_{1i}^{(n)}(i\Delta) = p_i^{(n)} \tag{3.9}$$

$$p_{2i}^{(n)}(i\Delta) = p_{i+1/2}^{(n)} \tag{3.10}$$

and the sequence $p_i^{(n)}$ , and $p_{i+1/2}^{(n)}$ are defined as follows

$$\begin{cases} p_{i+1/2}^{(n)} = p_{1i}^{(n)}((i+1)\Delta - 0) \\ p_{i+1}^{(n)} = p_{2i}^{(n)}((i+1)\Delta - 0), \;\; p_0^{(n)} = p_o \end{cases} , \tag{3.11}$$

where $\eta$ is a sufficiently large constant and $\circ$ denotes the Stratonovich integral.

By using the method of a stochastic characteristic curve, we get

**Proposition 3.1** *The explicit solution of (3.8) is given by*

$$\begin{aligned} p_{2i}^{(n)}(t,v) = \exp\{&\frac{2\mu}{\rho_c\xi}\{\frac{1}{\sqrt{v}+\frac{1}{2}\rho_c\xi(Z(t)-Z(i\Delta))} - \frac{1}{\sqrt{v}}) \\ &-\frac{1}{2}(Z(t)-Z(i\Delta)) + \frac{\eta}{2}(t-i\Delta)\} \\ &\times(\frac{v}{(\sqrt{v}+\frac{1}{2}\rho_c\xi(Z(t)-Z(i\Delta)))^2}) \\ \times p_{1i}^{(n)}((i+1)\Delta - 0, &((\sqrt{v}+\frac{1}{2}\rho_c\xi(Z(t)-Z(i\Delta)))\vee 0)^2) \\ &\text{for } t \in [i\Delta, (i+1)\Delta[ \end{aligned}$$

*where $p_{1i}^{(n)}(t,v)$ is a solution of the deterministic equation given by (3.7).*

To solve the deterministic $p_{1i}^{(n)}(t,v)$-equation, the transformation $v = e^x$ is useful for avoiding the singularity at $v=0$,

**Proposition 3.2** *Denoting*

$$\tilde{p}_{1i}^{(n)}(t,x) = p_{1i}^{(n)}(t,e^x),$$

*we have*

$$\begin{aligned} &\frac{\partial\tilde{p}_{1i}^{(n)}(t,x)}{\partial t} - \frac{\partial}{\partial x}\{\frac{\xi^2}{2}(1+(\delta^2-\rho_c^2)e^{-x})\frac{\partial\tilde{p}_{1i}^{(n)}(t,x)}{\partial x}\} \\ &+\{\frac{\xi^2}{2} - a + \rho_c\xi(\frac{5}{4}\rho_c\xi - \eta e^{-\frac{x}{2}} + \frac{1}{2}e^{\frac{x}{2}})\}e^{-x}\frac{\partial\tilde{p}_{1i}^{(n)}(t,x)}{\partial x} \\ &\quad +\{\frac{1}{4}(\rho_c\xi e^{-\frac{1}{2}x} - \eta e^{-x} + 1)^2 + \frac{1}{2}\eta^2 e^{-2x} \\ &\quad\quad -\frac{1}{8} + \frac{\eta}{2}\}\tilde{p}_{1i}^{(n)}(t,x) = 0, \\ &\quad\quad \tilde{p}_{1i}^{(n)}(t,x) = p_{2i}^{(n)}((i+1)\Delta - 0, e^x). \end{aligned}$$

From above propositions, we can realize the Zakai-equation. The remaining problem is to show the mathematical property to support the convergence of the proposed algorithm

We assume that the volatility process $V(t)$ in $(\epsilon,\infty)$ a.s. $\forall t \geq 0$. So we introduce the Hilbert spaces for $\epsilon > 0$

$$H = L^2(\epsilon,\infty) \subset V = H^1(\epsilon,\infty) \cap \{\phi(\epsilon) = 0\}$$

and identify $H$ with its dual. We denote by $V'$ the dual of $V$ and norms on $H$ and $V$ by $|\cdot|$ and $||\cdot||$, respectively.

The duality between $V$ and $V'$is referred to as $<\cdot,\cdot>$.

**Proposition 3.3** *We assume that*

$$\delta > \rho_c. \tag{3.12}$$

*Hence the system (3.7), (3.8) define in a unique way $p_{1i}^{(n)}, p_{2i}^{(n)}$ in $L^2_{\mathcal{F}}(T;V), L^2_{\mathcal{F}}(T;H)$, respectively.*

*Proof.* Noting that $\forall\phi \in V$

$$\begin{aligned} &-<A^*\phi,\phi> + \frac{1}{2}<B^*B^*\phi,\phi> \\ &= \frac{\xi^2}{2}\{|v\frac{\partial\phi}{\partial v}|^2 + (\delta^2-\rho_c^2)|\sqrt{v}\frac{\partial\phi}{\partial v}|^2\} \\ &\quad +(\{-\frac{1}{2}(\xi^2-a) + \frac{1}{2}(\frac{\mu}{v}-\frac{1}{2})^2 \\ &\quad -\frac{3}{4}(\frac{\mu}{v}-\frac{1}{2})\frac{\rho_c\xi}{\sqrt{v}} + \frac{1}{4}\frac{\rho_c^2\xi^2}{v}\}\phi,\phi). \end{aligned}$$

Hence form (3.12) we have

$$-<A^*\phi,\phi>+\frac{1}{2}<B^*B^*\phi,\phi>\geq\beta||\phi||^2+\lambda|\phi|^2 \quad (3.13)$$

for some $\beta > 0$ and $\lambda > 0$. It follows from (3.7) and (3.13) that

$$|p_{1i}^{(n)}(t)|^2+\beta\int_{i\Delta}^t ||p_{1i}^{(n)}(s)||^2ds + (\frac{\eta}{2}-\lambda)\int_{i\Delta}^t |p_{1i}^{(n)}(s)|^2ds \leq |p_i^{(n)}|^2. \quad (3.14)$$

For $p_{2i}^{(n)}$-process, we convert (3.7) to the Ito form:

$$dp_{2i}^{(n)}(t)+\frac{\eta}{2}p_{2i}^{(n)}(t)dt=\frac{1}{2}B^*B^*p_{2i}^{(n)}(t)dt + B^*p_{2i}^{(n)}(t)dW(t). \quad (3.15)$$

It follows that $\forall\phi\in V$

$$\begin{aligned} |B^*(t)\phi|^2+&<B^*(t)B^*(t)\phi,\phi> \\ =\quad & (\{2(\frac{\mu}{v}-\frac{1}{2}\frac{\rho_c\xi}{\sqrt{v}})^2-\frac{1}{2}(\frac{\mu}{v}-\frac{1}{2}\frac{\rho_c\xi}{\sqrt{v}})\frac{\rho_c\xi}{\sqrt{v}}\}\phi,\phi) \\ & -\frac{\rho_c\xi}{4}|v^{-1/4}\phi|^2-\frac{\mu\rho_c\xi}{2}|v^{-3/4}\phi|^2 \\ \leq\quad & C_1|\phi|^2 \end{aligned} \quad (3.16)$$

and by using the integrating by parts formula we get

$$(B^*(t)\phi,\phi)=(\{\frac{\mu}{v}-\frac{1}{2}-\frac{3\rho_c\xi}{4\sqrt{v}}\}\phi,\phi) \leq C_2|\phi|^2 \quad (3.17)$$

By using the results given by Pardoux [7], we have

$$\begin{aligned} |p_{2i}^{(n)}(t)|^2+\eta\int_{i\Delta}^t|p_{2i}^{(n)}(s)|^2ds &= |p\ (n)_{2i}(i\Delta)|^2 \\ &+\int_{i\Delta}^t(B_bp_{2i}^{(n)}(s),p_{2i}^{(n)}(s))ds \\ &+2\int_{i\Delta}^t(B^*p_{2i}^{(n)}(s),p_{2i}^{(n)}(s))dW(s), \end{aligned} \quad (3.18)$$

where

$$(B_b\phi,\phi)=<B^*B^*\phi,\phi>+|B^*\phi|^2. \quad (3.19)$$

Taking a mathematical expectation to (3.18), we obtain

$$E\{|p_{2i}^{(n)}(t)|^2\}+(\eta-c_1)E\{\int_{i\Delta}^t|p_{2i}^{(n)}(s)|^2ds\} \leq E\{|p_{2i}^{(n)}(i\Delta)|^2\} \quad (3.20)$$

### 3.3 A priori estimates

We begin by establishing *a priori* estimates.

**Proposition 3.4** *The processes $p_{1i}^{(n)}, p_{2i}^{(n)}$ satisfy*

$$E\int_0^T||p_{1i}^{(n)}(t)||^2dt\leq C, \quad (3.21)$$

$$E\int_0^T|p_{2i}^{(n)}(t)|^2dt\leq C \quad (3.22)$$

$$E\{|p_{1i}^{(n)}(t)|^4\}\leq C, \quad (3.23)$$

$$E\{|p_{2i}^{(n)}(t)|^4\}\leq C, \forall t\in[0,T] \quad (3.24)$$

*where $C$ does not depend on $T$ or $i$ for a convenient choice of $\eta$.*

*Proof.* It follows from (3.14) and (3.18) that

$$\begin{aligned} E\{|p_{1i}^{(n)}((i+1)\Delta-0)|^2+|p_{2i}^{(n)}((i+1)\Delta-0)|^2\} \\ -E\{|p_{1i}^{(n)}(i\Delta)|^2+|p_{2i}^{(n)}(i\Delta)|^2\} \\ +C(\eta)E\{\int_{i\Delta}^{(i+1)\Delta}(||p_{1i}^{(n)}(s)||^2+|p_{2i}^{(n)}(s)|^2)ds\}\leq 0 \end{aligned} \quad (3.25)$$

for a convenient choice of $\eta$. Hence using (3.11) we get

$$\begin{aligned} E\{|p_{i+1}^{(n)}|^2\}-E\{|p_i^{(n)}|^2\} \\ +C(\eta)E\{\int_i^{(i+1)\Delta}(||p_{1i}^{(n)}(s)||^2+|p_{2i}^{(n)}(s)|^2)ds\}\leq 0 \end{aligned} \quad (3.26)$$

Summing up the relation (3.26) from 0 to $n$, we deduce

$$E\int_0^T||p_{1i}^{(n)}(t)||^2dt\leq C, E\int_0^T|p_{2i}^{(n)}(t)|^2dt\leq C, \quad E\{|p_i^{(n)}|^2\}\leq C, \quad (3.27)$$

where $C$ is independent of $T$ or $i$ but depends on $p_o$ and $\mu$. It is also easy to obtain form (3.14) and (3.18)

$$E\{|p_{1i}^{(n)}(t)|^2\}\leq C, E\{|p_{2i}^{(n)}(t)|^2\}\leq C. \quad (3.28)$$

From (3.14) we deduce

$$E\{|p_{1l}^{(n)}(t)|^4\}\leq E\{|p_{1i}^{(n)}(i\Delta)|^4\}. \quad (3.29)$$

By using the Ito formula, from (3.18) we obtain

$$\begin{aligned} E\{|p_{2i}^{(n)}(t)|^4\}&+2\eta E\{\int_{i\Delta}^t|p_{2i}^{(n)}(s)|^4ds\} \\ &=E\{|p_{2i}^{(n)}(i\Delta)|^4\} \\ &+2E\{\int_{i\Delta}^t(B_bp_{2i}^{(n)}(s),p_{2i}^{(n)}(s))|p_{2i}^{(n)}(s)|^2ds\} \\ &+E\{\int_{i\Delta}^t|(p_{2i}^{(n)}(s),B^*p_{2i}^{(n)}(s))|^2ds\}. \end{aligned}$$

It follows from (3.16) and (3.17) that

$$E\{|p_{2i}^{(n)}(t)|^4\}+(2\eta-2C_1-C_2)E\{\int_{i\Delta}^t|p_{2i}^{(n)}(s)|^4ds\} \leq E\{|p_{2i}^{(n)}(i\Delta)^4\}$$

Choosing

$$2\eta \geq 2C_1 + C_2,$$

we get

$$E\{|p_{2i}^{(n)}(t)|^4\} \leq E\{|p_{2i}^{(n)}(i\Delta)|^4\}.$$

Hence we also get the estimate (3.24).

### 3.4 Convergence

From Proposition 3.3, we can extract subsequence, still denoted $p_{1i}^{(n)}, p_{2i}^{(n)}$ such that

$$\begin{aligned} p_{1i}^{(n)} &\rightarrow \ p_1 \text{ in } L^2_{\mathcal{F}}(T, V) \text{ weakly} \\ p_{2i}^{(n)} &\rightarrow \ p_1 \text{ in } L^2_{\mathcal{F}}(T, H) \text{ weakly} \end{aligned}$$

and

$$p_{1i}^{(n)}, p_{2i}^{(n)} \rightarrow p_1, p_2 \text{ in } L^\infty(T; L^2(\Omega; H)) \text{ weakly star.}$$

We can present the following two lemmas from [8].

**Lemma 3.1** *The functions $p_1$ and $p_2$ are equal to a common function $\xi$.*

*Proof.* It follows from (3.7) that

$$\begin{aligned} & p_{i+1/2}^{(n)} - p_{1i}^{(n)}(t) \\ & + \int_t^{(i+1)\Delta} (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})p_{1i}^{(n)}(s)ds = 0 \end{aligned}$$

and from (3.15)

$$\begin{aligned} & p_{2i}^{(n)}(t) - p_{i+1/2}^{(n)} + \int_{i\Delta}^t \frac{\mu}{2} p_{2i}^{(n)}(s)ds \\ = & \int_{i\Delta}^t \frac{1}{2}B^*B^*p_{2i}^{(n)}(s)ds + \int_{i\Delta}^t B^*p_{2i}^{(n)}(s)dW(s). \end{aligned}$$

Summing up , we get

$$\begin{aligned} & (p_{2i}^{(n)}(t) - p_{1i}^{(n)}(t)) \\ & + \int_t^{(i+1)\Delta} (-A^* + \frac{1}{2}B^*B^* + \frac{\mu}{2})p_{1i}^{(n)}(s)ds \\ & + \int_{i\Delta}^t \frac{\mu}{2}p_{2i}^{(n)}(s)ds = \int_{i\Delta}^t \frac{1}{2}B^*B^*p_{2i}^{(n)}(s)ds \\ & + \int_{i\Delta}^t B^*p_{2i}^{(n)}(s)dW(s). \end{aligned} \tag{3.30}$$

We also regard the operator $A^*$ as

$$A^* \in \mathcal{L}(H; (V \cap H^2)')$$

Hence

$$||A^*\phi||_{(V\cap H^2)'} \leq C_1|\phi|^2.$$

For the operator $B^*B^*$ we find that

$$B^*B^* \in \mathcal{L}(H; (V \cap H^2)')$$

with $||B^*B^*\phi||_{(V\cap H^2)')} \leq C_2|\phi|^2$. Hence

$$\begin{aligned} & E\{||p_{2i}^{(n)}(t) - p_{1i}^{(n)}(t)||^2_{(V\cap H^2)'}\} \\ \leq \ & 4E(\int_t^{(i+1)\Delta} ||(-A^* + \frac{1}{2}B^*B^* \\ & + \frac{\eta}{2})p_{1i}^{(n)}(s)||_{(V\cap H^2)'}ds)^2 \\ & +4E(\int_{i\Delta}^t \frac{\eta}{2}||p_{2i}^{(n)}(s)||_{(V\cap H^2)'}ds)^2 \\ & +4E(\int_{i\Delta}^t \frac{1}{2}||B^*B^*p_{2i}^{(n)}(s)||_{(V\cap H^2)'}ds)^2 \\ & +4E\{||\int_{i\Delta}^t B^*p_{2i}^{(n)}(s)dW(s)||^2_{(V\cap H^2)'}\} \\ \leq \ & \text{Const.}\{\Delta E\{\int_{i\Delta}^{(i+1)\Delta} (|p_{1i}^{(n)}(t)|^2 + |p_{2i}^{(n)}(t)|^2)dt\} \\ & +E\{\int_{i\Delta}^{(i+1)\Delta} |p_{2i}^{(n)}(t)|^2 dt\} \\ \leq \ & C\sqrt{\Delta}\{\sqrt{\Delta}E\{\int_{i\Delta}^{(i+1)\Delta} (|p_{1i}^{(n)}(t)|^2 + |p_{2i}^{(n)}(t)|^2)dt\}\} \\ & +(E\{\int_{i\Delta}^{(i+1)\Delta} |p_{2i}^{(n)}(t)|^4 dt\})^{1/2}\} \\ \leq \ & \text{Const.}\sqrt{\Delta} \end{aligned}$$

Since $p_{2i}^{(n)} - p_{1i}^{(n)} \rightarrow p_2 - p_1$, weakly in $L^2_{\mathcal{F}}(T; (V \cap H^2)')$, we have

$$\int_0^T E\{||p_2 - p_1||_{(V\cap H^2)'}\}dt = 0$$

hence $p_1 = p_2 = \xi$.

**Lemma 3.2** $\xi = p$

*Proof.* See [8] .

**Theorem 3.3** *We have*

$$\begin{aligned} p_{1i}^{(n)} &\rightarrow \ p \text{ in } L^2_{\mathcal{F}}(T; V) \text{ strongly} \\ p_{2i}^{(n)} &\rightarrow \ p \text{ in } L^2_{\mathcal{F}}(T; H) \text{ strongly} \\ p_{1i}^{(n)}(t), p_{2i}^{(n)}(t) &\rightarrow \ p(t) \text{ in } L^2(\Omega; H), \text{ strongly} \\ & \qquad \forall t \in [0, T[ \end{aligned}$$

*Proof.* It is easy to show that

$$\begin{aligned} & E\{|p_{2i}^{(n)}((i+1)\Delta)|^2\} - E\{|p_{2i}^{(n)}(i\Delta)|^2\} \\ & +E\{\int_{i\Delta}^{(i+1)\Delta} (\eta|p_{2i}^{(n)}(s)|^2 - (B_b p_{2i}^{(n)}(s), p_{2i}^{(n)}(s))ds\} = 0. \end{aligned}$$

From (3.7) we also have

$$\begin{aligned} & E\{|p_{1i}^{(n)}(t)|^2\} - E\{|p_{1i}^{(n)}(i\Delta)|^2\} \\ & +2E\{\int_{i\Delta}^t < (A^* + \frac{1}{2}B^*B^* + \frac{\mu}{2})p_{1i}^{(n)}(s), p_{1i}^{(n)}(s) > ds\} = 0 \end{aligned}$$

Summing up these equations and using (3.11), we obtain

$$\begin{aligned}&E\{|p_{1i}^{(n)}(t)|^2\} - |p_o|^2\\&+2E\{\int_{i\Delta}^{t} < (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})p_{1i}^{(n)}(s), p_{1i}^{(n)}(s) > ds\}\\&+E\{\int_0^{\Delta[t/\Delta]} (\mu|p_{2i}^{(n)}(s)|^2 - (B_b p_{2i}^{(n)}(s), p_{2i}^{(n)}(s))ds\} = 0\end{aligned}$$

and

$$\begin{aligned}&E\{|p_{2i}^{(n)}(t)|^2\} - E\{|p_{1/2}^{(n)}|^2\}\\&+E\{\int_0^t (\eta|p_{2i}^{(n)}(s)|^2 - (B_b p_{2i}^{(n)}(s), p_{2i}^{(n)}(s))ds\}\\&+2E\{\int_{\Delta}^{\Delta[t/\Delta]+\Delta} < (A^* + \frac{1}{2}B^*B^* + \frac{\mu}{2})\\&\qquad \times p_{1i}^{(n)}(s), p_{1i}^{(n)}(s) > ds\} = 0.\end{aligned}$$

Here we shall show the strong convergence of $p_{1i}^{(n)}$-process. Introduce

$$\begin{aligned}\chi^{(n)}(t) &= E\{|p(t) - p_{1i}^{(n)}(t)|^2\}\\&+2E\{\int_0^t < (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})\\&\quad \times (p(s) - p_{1i}^{(n)}(s)), p(s) - p_{1i}^{(n)}(s) > ds\}\\&+E\{\int_0^{\Delta[t/\Delta]} (\mu|p(s) - p_{2i}^{(n)}(s)|^2\\&\quad -(B_b(p(s) - p_{2i}^{(n)}(s)), p(s) - p_{2i}^{(n)}(s)))ds\}\\&= \chi_1^{(n)}(t) + \chi_2^{(n)}(t) + \chi_3^{(n)}(t)\end{aligned}$$

where

$$\begin{aligned}\chi_1^{(n)}(t) &= E\{|p(t)|^2\}\\&+2E\{\int_0^t < (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})p(s), p(s) > ds\}\\&+E\{\int_0^{\Delta[t/\Delta]} (\mu|p(s)|^2 - (B_b p(s), p(s)))ds\},\\\chi_2^{(n)}(t) &= -2E\{(p(t), p_{1i}^{(n)}(t))\}\\&-2E\{\int_0^t (< (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})p(s), p_{1i}^{(n)}(s) >\\&+ < (-A^* + \frac{1}{2}B^*B^* + \frac{\eta}{2})p_{1i}^{(n)}(s), p(s) >)ds\}\\&-2\mu E\{\int_0^{\Delta[t/\Delta]} (p(s), p_{2i}^{(n)}(s))ds\}\\&+2E\{\int_0^{\Delta[t/\Delta]} (B_b p(s), p_{2i}^{(n)}(s))ds\}\end{aligned}$$

and

$$\chi_3^{(n)}(t) = |p_o|^2.$$

Noting that

$$< B^*B^*\phi, \phi > -(B_b\phi, \phi) = -|B^*\phi|^2, \qquad (3.31)$$

we have

$$\begin{aligned}\lim_{n\to\infty} \chi_1^{(n)}(t) &= E\{|p(t)|^2\} + 2E\{\int_0^t (< -A^*p(s), p(s) >\\&\quad -\frac{1}{2}|B^*p(s)|^2 + \eta|p(s)|^2)ds\} = |p_o|^2\end{aligned}$$

From the results of weak convergence of $p_{1i}^{(n)}$- process, we have

$$\begin{aligned}\lim_{n\to\infty} \chi_2^{(n)}(t) &= -2E\{|p(t)|^2\} - 4E\{\int_0^t (< -A^*p, p >\\&\quad -\frac{1}{2}|B^*p|^2 + \mu|p|^2)ds\} = -2|p_o|^2.\end{aligned}$$

From (3.31), choosing $\eta > \lambda$, we have $0 \le \chi^{(n)}(t)$. Hence

$$\begin{aligned}p_{1i}^{(n)} &\to p \text{ in } L^2_{\mathcal{F}}(T; V) \text{ strongly}\\p_{1i}^{(n)}(t) &\to p(t) \text{ in } L^2(\Omega; H)\ \ \forall t \ge 0 \text{ strongly.}\end{aligned}$$

Similarly we can check the convergence of $p_{2i}^{(n)}$-process.

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