A Synthesis Method for Static Anti-Windup Compensators

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A SYNTHESIS METHOD FOR STATIC ANTI-WINDUP COMPENSATORS

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Abstract

Synthesis of static anti-windup compensators is considered. LMI conditions are established for stability and performance analysis of the closed loop system. The performance criterion describes the servo problem for the resulting closed-loop piecewise linear system. The synthesis of the anti-windup compensators will be given by some bilinear matrix inequalities.

1 Introduction

All real world control systems must deal with actuator saturation. This give rise to interesting control challenges. As result of actuator saturation the plant input will be different from the controller output. When this happens the control loop is broken and the controller output does not drive the plant. Thus the states of the controller are updated incorrectly, resulting in serious performance deterioration [2].

A well-known and successful methodology used to cope with this problem is anti-windup compensation or conditioning. This methodology give rise to a compensator that during saturation improves the performance of the closed loop system.

In [4] synthesis of anti-windup compensators is proposed with guaranteed performance. Here the $\mathscr{L}_2$ induced norm has been used to measure performance for the anti-windup compensator.

In [7] the problem of anti-windup compensation has been recognized as being that of returning the system to linear behavior. That is, return of the system output to the one that would have been without saturation. This problem is not directly captured by the work in [4]. In [8] the before mentioned goal is imposed for the synthesis, while the system configuration introduced in [9] is used.

This article presents a simple solution for the latter mentioned problem. In addition to [8] the performance criterion takes into account time varying reference signals to the control system. The methodology used here is based on the results in [6], [5]. There, a general method for characterizing the servo problem for a class of nonlinear systems is presented. It is shown that for piecewise linear systems the servo problem can be described using LMIs.

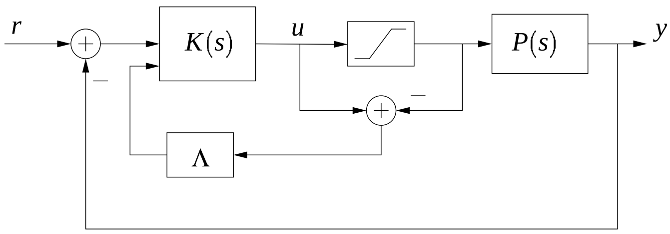

Figure 1: The anti-windup scheme considered in the article.

The outline of the paper is as follows: the next section will pose the anti-windup problem in the framework of piecewise linear systems. Section 3 will summarize the result in [6]. Section 4 presents the main contribution of the article. Section 5 will present a simple example where the method is applied. In Section 6 some concluding remarks are presented.

2 The Anti-windup scheme

In Figure 1 the considered anti-windup scheme is presented. Here the linear controller $K(s)$ is designed to stabilize the plant $P(s)$ without taking the saturation into account.

The problem is to design the static compensator block Λ according to some appropriate performance criterion.

The description of this system as a piecewise linear system presented below is similar to that presented in [3].

The linear plant $P(s)$ has a state-space description given by the matrices A_p, B_p, C_p, D_p. The state-space description of the linear controller $K(s)$ is given by A_c, B_c, C_c, D_c.It is assumed that $P(s)$ is stable and $K(s)$ has been designed such that the closed loop linear system is stable.

The saturation function is defined as

$$sat(u) = \begin{cases} u_m, \, u < u_m \\ u, \, u_m \leq u \leq u_M \\ u_M, \, u > u_M \end{cases}$$

The saturation nonlinearity will give rise to a partitioned state-space for the system, obtaining a piecewise linear system. The

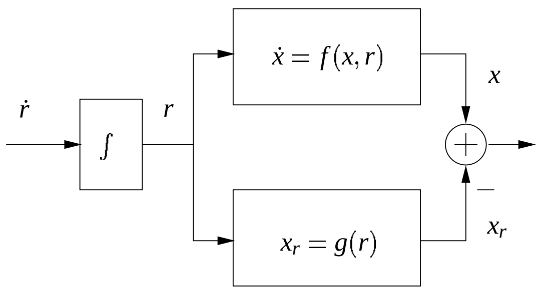

Figure 2: Computable bounds on the map from $\dot{r}$ to $|x - x_r|$ describe the servo problem

three resulting regions will be denoted through this paper as follows X_0 – the linear region, X_1 – the region where $u < u_m$ and X_2 denotes the region where $u > u_M$.

The anti-windup compensation block Λ enters the controller as follows:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c e + \Lambda_1 (u - sat(u)) \\ y_c &= C_c x_c + D_c e + \Lambda_2 (u - sat(u)) \end{aligned}$$

where $\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}$. Thus, in the three partitions the dynamics will be given by:

$$\begin{cases} \dot{\bar{x}} = A_1 \bar{x} + a_1 + B_1 r \\ y = C_1 \bar{x} + D_1 r \end{cases}, \bar{x} \in X_1$$

$$\begin{cases} \dot{\bar{x}} = A_2 \bar{x} + B_2 r \\ y = C_2 \bar{x} + D_2 r \end{cases}, \bar{x} \in X_2$$

$$\begin{cases} \dot{\bar{x}} = A_3 \bar{x} + a_3 + B_3 r \\ y = C_3 \bar{x} + D_3 r \end{cases}, \bar{x} \in X_3$$

Here the matrices A_i, B_i depend linearly on the parameter $\Lambda_1 (I + \Lambda_2)^{-1}$. For details about the matrices see [3].

3 The Servo Problem

The servo problem for a general nonlinear systems can be analyzed in a framework presented in Figure 2. The problem is to obtain information about the difference between the system trajectory (x) and a predetermined trajectory x_r in presence of an input signal r. The exogenous input considered in this framework will be the time derivative of r. Choosing $\mathscr{L}_2$ norm as measure for the signals, it is natural a choice of the $\mathscr{L}_2$ gain to characterize the systems behavior. Thus by computing the $\mathscr{L}_2$ gain from the input signal's derivative ($\dot{r}$) to the "distance" between system trajectories (x) and reference trajectories (x_r), one obtains information relating the convergence of the studied system trajectories.

The following theorem presents the result from [6]. This gives an upper bound on the $\mathscr{L}_2$ gain from the input signal derivative to the "distance" from the system state to a defined trajectory x_r.

Theorem 1 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz. For every $r \in \mathscr{R} \subset \mathbb{R}^m$ let $x_r \in \mathbb{R}^n$ be a unique solution to $0 = f(x_r, r)$.*

If there exists $\gamma > 0$ and a non-negative $\mathscr{C}^1$ function V, with $V(x_r, r) = 0$ and

$$\begin{bmatrix} \frac{\partial V}{\partial x} f(x,r) + |x - x_r|^2 & \frac{1}{2}\frac{\partial V}{\partial r} \\ \frac{1}{2}\left(\frac{\partial V}{\partial r}\right)^T & -\gamma^2 I \end{bmatrix} < 0 \tag{1}$$

for all $(x,r) \in \mathscr{S}$, then for each solution to

$$\dot{x} = f(x,r),\ x(0) = x_{r_0},\ r(0) = r_0 \tag{2}$$

such that $r(t) \in \mathscr{R}$ and $(x(t), r(t)) \in \mathscr{S}$ for all t, it holds that

$$\int_0^T |x - x_r|^2 dt \le \gamma^2 \int_0^T |\dot{r}|^2 dt \tag{3}$$

Proof:
Multiplying (1) from left and right with $\begin{bmatrix} \mathbf{1} & \dot{r}^T \end{bmatrix}$ one obtains:

$$\frac{\partial V}{\partial x} f(x,r) + |x - x_r|^2 + \frac{\partial V}{\partial r}\dot{r} - \gamma^2 |\dot{r}|^2 < 0$$

that is

$$\frac{dV}{dt} + |x - x_r|^2 - \gamma^2 |\dot{r}|^2 < 0$$

which in turns by integration on $[0, T]$ gives

$$V(x(T), r(T)) + \int_0^T |x - x_r|^2 dt - \gamma^2 \int_0^T |\dot{r}|^2 dt < 0$$

and inequality (3) results since $V(x,r) \ge 0$.

Obviously, for a generic nonlinear system as considered in (2) it might be difficult to find a $V(x,r)$ such that (1) is fulfilled. In case of piecewise linear systems convex optimization can be used to compute the mentioned upper bound.

Consider now a particular kind of nonlinear systems, a piecewise linear system, of the form:

$$\dot{x} = A_i x + B_i r, \quad x(t) \in X_i \tag{4}$$

with $\{X_i\}_{i \in I} \subseteq \mathbb{R}^n$ a partition of the state space into a number of convex polyhedral cells with disjoint interior. Suppose that for any constant $r \in \mathscr{R}$ the piecewise linear system has a unique equilibrium point.

Furthermore, consider symmetric matrices S_{ij} that satisfy the inequality:

$$\begin{bmatrix} x - x_r \\ r \end{bmatrix}^T S_{ij} \begin{bmatrix} x - x_r \\ r \end{bmatrix} > 0,\ x \in X_i,\ r \in \mathscr{R}_j \tag{5}$$

Define

$$\overline{B}_j \triangleq \begin{bmatrix} A_j^{-1} B_j \\ 1 \end{bmatrix}, \qquad \overline{I} \triangleq \begin{bmatrix} I_n & 0 \\ 0 & 0_m \end{bmatrix} \tag{6}$$

$$\overline{A}_{ij} \triangleq \begin{bmatrix} A_i & -A_iA_j^{-1}B_j + B_i \\ 0 & 0 \end{bmatrix} \quad (7)$$

The following proposition is useful for application of Theorem 1.

Proposition 1 *Let $f(x,r) = A_ix + B_ir$, $x_r = -A_j^{-1}B_jr$ with $\underline{x}(0) = x_r(0)$, $r(0) = r_0$. If there exist $\gamma > 0$, $P > 0$ such that $\overline{P} = diag\{P, 0\}$ satisfies*

$$\begin{bmatrix} \overline{A}_{ij}^T\overline{P} + \overline{PA}_{ij} + S_{ij} + \overline{I} & \overline{PB}_j \\ \overline{B}_j^T\overline{P} & -\gamma^2 I \end{bmatrix} < 0, i \neq j \quad (8)$$

$$\begin{bmatrix} A_j^TP + PA_j + I & PA_j^{-1}B_j \\ (A_j^{-1}B_j)^TP & -\gamma^2 I \end{bmatrix} < 0 \quad (9)$$

then $V(x,r) = (x - x_r)^TP(x - x_r)$ satisfies (1) for all $x \in X_i$, $r(t) \in \mathscr{R}_j$.

Remark 3.1 *In particular, in the case when $\dot{r}(t) = 0$, for $t > T$, by finding a finite $\gamma > 0$ it is shown that all trajectories of the nonlinear system (4) will converge to x_r.*

Remark 3.2 *When the local linear systems contain affine terms the argument vector of the Lyapunov function will be extended to $(x\ r\ 1)$. Then the definitions in (6), (7) become:*

$$\overline{B}_j \triangleq \begin{bmatrix} A_j^{-1}B_j \\ 1 \\ 0 \end{bmatrix}, \quad (10)$$

$$\overline{A}_{ij} \triangleq \begin{bmatrix} A_i & -A_iA_j^{-1}B_j + B_i & a_i \\ 0 & 0 & 0 \end{bmatrix} \quad (11)$$

The conservatism of the theorems can be reduced by considering piecewise quadratic Lyapunov function. In this case the Lyapunov function will be piecewise $\mathscr{C}^1$ instead of $\mathscr{C}^1$. Imposing that is non-increasing at the points of discontinuity, the results hold (see [1]).

4 Synthesis of static anti-windup compensators

The anti-windup problem can naturally be posed as a servo problem for the nonlinear system (i.e. the closed-loop piecewise linear system). The goal is to return to the behavior of the linear system as fast as possible. In this context, x_r introduced in the previous section can be used to define a trajectory that describes the linear behavior of the system. That is, define

$$x_r = -A_2^{-1}B_2r \quad (12)$$

Computing the $\mathscr{L}_2$ gain from the derivative of the input signal to $x - x_r$, gives a measure on the behavior of the system trajectories with respect to x_r. Notice that the input signal is smoothly time varying.

It is reasonable to assume that the reference signals have such a magnitude that they can be achieved by the system output without violating the saturation constraints in stationarity. Using this assumption it is enough to use only the x_r defined by (12) in the synthesis.

Thus, a solution to the considered anti-windup problem is given by Proposition 1, with $i = 1,3$ and $j = 2$ using the definition in (10) and (11). Unfortunately, if one is searching also for the parameters Λ_1, Λ_2 in the same time as solving for P, the matrix inequality becomes a BMI (bilinear matrix inequality). Iterative approaches can be used to solve this kind of problems, however no formal proof for convergence exists.

Regarding the S-procedure terms (S_{ij}) describing the partitions, a suitable description of the state space partition can be obtained by constructing a polyhedral cell bounding using the matrix:

$$\begin{bmatrix} H_{x_i} & H_{r_i} + H_{x_i}(-A_j^{-1}B_j) & H_{e_i} \end{bmatrix}$$

where the state space partition for the nonlinear system is given by the hyperplanes:

$$\begin{bmatrix} H_{x_i} & H_{r_i} & H_{e_i} \end{bmatrix} \begin{bmatrix} x \\ r \\ 1 \end{bmatrix} = 0$$

with $i = 1,3$ and $j = 2$. Notice that also the size of S_{ij} will be increased due to presence of affine dynamics and partition.

5 Example

To demonstrate the method, a simple SISO example with a PI controller will be used. This example has been studied also in [3], [8]. The plant and controller are:

$$P(s) = \frac{0.5s^2 + 0.5s + 1}{s^2 + 0.2s + 0.2}$$

$$K(s) = 2\left(1 + \frac{1}{s}\right)$$

The saturation on the control signal is set to ± 0.5. The output in case there is no saturation acting on the plant is shown in Figure 3, while in the case of saturation acting on the control output the performance deteriorates considerably (see Figure 4). The reference signal in both cases is a step filtered by a first order linear system with a time constant of 0.01 seconds.

Due to the integrator in the controller the LMIs are not strictly feasible. For this reason a leakage is introduced in the integrator by moving its pole to -0.01. For practical applications it is reasonable to consider a “forgetting factor” in the integrator.

Applying the algorithm, a static compensator of the form $\Lambda_1 = -0.45$, $\Lambda_2 = 0$ is found. Piecewise quadratic Lyapunov functions are used in the algorithm. The best upper bound found on the $\mathscr{L}_2$ gain from $\dot{r}$ to $x - x_r$ is 5.0856. For a lower bound on this $\mathscr{L}_2$ gain, a local analysis in the linear region can be carried out. This way, a lower bound of 3.8639 is obtained. The

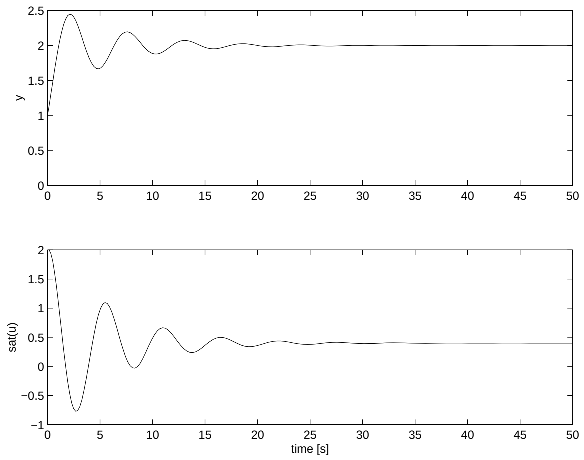

Figure 3: Output and control signal when no saturation is present.

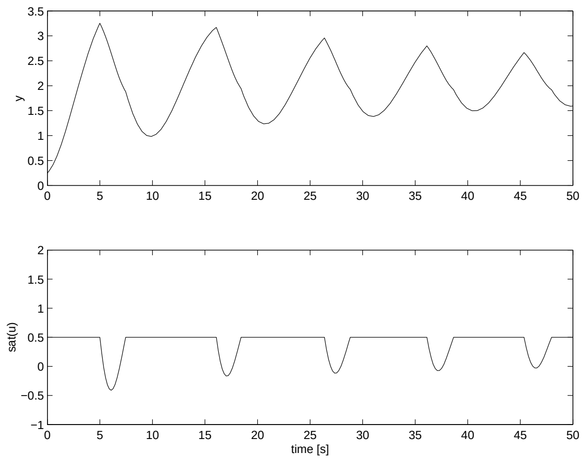

Figure 4: Output and control signal when saturation is acting on the control signal.

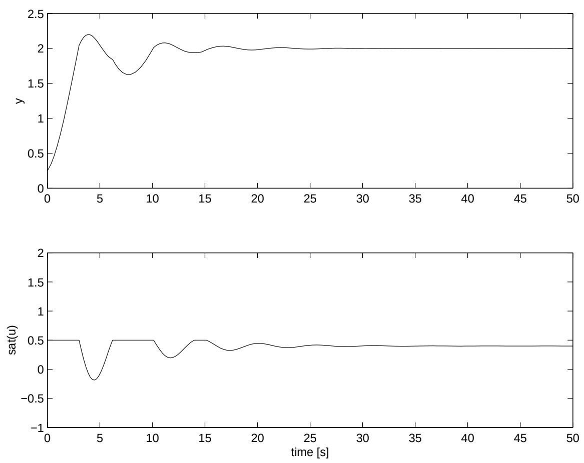

Figure 5: Output and control signal when saturation is acting on the control signal. Anti-windup compensation is applied on the control system.

output of the compensated system is shown in Figure 5. Notice the significant improvement in the performance of the control system.

6 Conclusions

A synthesis method for static anti-windup compensators has been presented. The $\mathscr{L}_2$ gain from $\dot{r}$ to $x - x_r$ is used as a performance measure for the compensated system. A simple example has been shown to demonstrate the method.

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