A small gain condition for interconnections of ISS systems with mixed ISS characterizations

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Abstract

We consider interconnected nonlinear systems with external inputs, where each of the subsystems is assumed to be input-to-state stable (ISS). Sufficient conditions of small gain type are provided guaranteeing that the interconnection is ISS with respect to the external input. To this end we extend recently obtained small gain theorems to a more general type of interconnections. The small gain theorem provided here is applicable to situations where the ISS conditions are formulated differently for each subsystem and are either given in the maximization or the summation sense. Furthermore it is shown that the conditions are compatible in the sense that it is always possible to transform sum formulations to maximum formulations without destroying a given small gain condition. An example shows the advantages of our results in comparison with the known ones.

Index Terms

Control systems, nonlinear systems, large-scale systems, stability criteria, Lyapunov methods.

I. Introduction

Stability of nonlinear systems with inputs can be described in different ways as for example in sense of dissipativity [22], passivity [20], [21], input-to-state stability (ISS) [17] and others. In this paper we consider general interconnections of nonlinear systems and assume that each

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subsystem satisfies an ISS property. The main question of the paper is whether an interconnection of several ISS systems is again ISS. As the ISS property can be defined in several equivalent ways we are interested in finding optimal formulations of the small gain condition that are adapted to a particular formulation. In particular we are interested in a possibly sharp stability condition for the case when the ISS characterization of single systems are different. Moreover we will provide a construction of an ISS Lyapunov function for interconnections of such systems.

Starting with the pioneering works [12], [11] stability of interconnections of ISS systems has been studied by many authors, see for example [15], [1], [3], [10]. In particular it is known that cascades of ISS systems are ISS, while a feedback interconnection of two ISS systems is in general unstable. The first result of the small gain type was proved in [12] for a feedback interconnection of two ISS systems. The Lyapunov version of this result is given in [11]. Here we would like to note the difference between the small gain conditions in these papers. One of them states in [11] that the composition of both gains should be less then identity. The second condition in [12] is similar but it involves the composition of both gains and further functions of the form $(\mathrm{id} + \alpha_i)$. This difference is due to the use of different definitions of ISS in both papers. Both definitions are equivalent but the gains enter as a maximum in the first definition, and a sum of the gains is taken in the second one. The results of [12] and [11] were generalized for an interconnection of $n \geq 2$ systems in [4], [6], [13], [14]. In [4], [6] it was pointed out that a difference in the small gain conditions remains, i.e., if the gains of different inputs enter as a maximum of gains in the ISS definition or a sum of them is taken in the definition. Moreover, it was shown that the auxiliary functions $(\mathrm{id} + \alpha_i)$ are essential in the summation case and cannot be omitted, [4]. In the pure maximization case the small gain condition may also be expressed as a condition on the cycles in the gain matrix, see e.g. [19], [4], [16], [13], [14]. A formulation of ISS in terms of monotone aggregation functions for the case of many inputs was introduced in [16], [5], [7]. For recent results on the small gain conditions for a wider class of interconnections we refer to [13], [8], [14]. In [9] the authors consider necessary and sufficient small gain conditions for interconnections of two ISS systems in dissipative form.

In some applications it may happen that the gains of a part of systems of an interconnection are given in maximization terms while the gains of another part are given in a summation formulation. In this case we speak of mixed ISS formulations. We pose the question whether and where we need the functions $(\mathrm{id} + \alpha_i)$ in the small gain condition to assure stability in this case. In this paper we consider this case and answer this question. Namely we consider n interconnected ISS systems, such that in the ISS definition of some $k \leq n$ systems the gains enter additively. For the remaining systems the definition with maximum is used. Our result contains the known small gain conditions from [4] as a special case $k = 0$ or $k = n$, i.e., if only one type of ISS definition is assumed. An example given in this paper shows the advantages of our results in comparison with the known ones.

This paper is organized as follows. In Section II we present the necessary notation and definitions. Section III discusses properties of gain operators in the case of mixed ISS formulations. In particular we show that the mixed formulation can in principle always be reduced to the maximum formulation. A new small gain condition adapted to the mixed ISS formulation ensuring stability of the considered interconnection is proved in Section IV. Section V provides a construction of ISS Lyapunov functions under mixed small gain conditions. We note some concluding remarks in Section VI.

II. Preliminaries and problem statement

A. Notation

In the following we set $\mathbb{R}_+ := [0, \infty)$ and denote the positive orthant $\mathbb{R}^n_+ := [0, \infty)^n$. The transpose of a vector $x \in \mathbb{R}^n$ is denoted by x^T. On $\mathbb{R}^n$ we use the standard partial order induced by the positive orthant given by

$$x \geq y \iff x_i \geq y_i, \quad i = 1, \dots, n,$$
$$x > y \iff x_i > y_i, \quad i = 1, \dots, n.$$

With this notation $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \ : \ x \geq 0\}$. We write $x \not\geq y \iff \exists\, i \in \{1, \dots, n\} : x_i < y_i$. For a nonempty index set $I \subset \{1, \dots, n\}$ we denote by $|I|$ the number of elements of I. We write y_I for the restriction $y_I := (y_i)_{i \in I}$ of vectors $y \in \mathbb{R}^n_+$. Let R_I be the anti-projection $\mathbb{R}^{|I|}_+ \to \mathbb{R}^n_+$, defined by

$$x \mapsto \sum_{k=1}^{|I|} x_k e_{i_k},$$

where $\{e_k\}_{k=1,\dots,n}$ denotes the standard basis in $\mathbb{R}^n$ and $I = \{i_1, \dots, i_{|I|}\}$.

For a function $v : \mathbb{R}_+ \mapsto \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1, s_2]}(t) = \begin{cases} v(t), & \text{if } t \in [s_1, s_2], \\ 0, & \text{otherwise.} \end{cases}$$

A function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. Note that for any $\alpha \in \mathcal{K}_\infty$ its inverse function α^{-1} always exists and $\alpha^{-1} \in \mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $t \mapsto \beta(s,t)$ is non-increasing and tends to zero for $t \to \infty$. By id we denote the identity map.

Let $|\cdot|$ denote some norm in $\mathbb{R}^n$, and let in particular $|x|_{\max} = \max_i |x_i|$ be the maximum norm. The essential supremum norm of a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}^m$ is denoted by $\|\phi\|_\infty$. L_∞ is the set of measurable functions for which this norm is finite.

B. Problem statement

Consider the system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{1}$$

and assume it is forward complete, i.e., for all initial values $x(0) \in \mathbb{R}^n$ and all essentially bounded measurable inputs u solutions $x(t) = x(t; x(0), u)$ exist for all positive times. Assume also that for any initial value $x(0)$ and input u the solution is unique.

The following notions of stability are used in the remainder of the paper.

Definition 2.1: System (1) is called

(i) *input-to-state stable* (ISS), if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_\infty)\,, \quad \forall x(0) \in \mathbb{R}^n\,, u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)\,, t \geq 0. \tag{2}$$

(ii) *globally stable* (GS), if there exist functions σ, $\hat{\gamma}$ of class $\mathcal{K}$, such that

$$|x(t)| \leq \sigma(|x(0)|) + \hat{\gamma}(\|u\|_\infty)\,, \quad \forall x(0) \in \mathbb{R}^n\,, u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)\,, t \geq 0. \tag{3}$$

(iii) System (1) has the *asymptotic gain* (AG) property, if there exists a function $\overline{\gamma} \in \mathcal{K}$, such that

$$\limsup_{t \to \infty} |x(t)| \leq \overline{\gamma}(\|u\|_\infty)\,, \quad \forall x(0) \in \mathbb{R}^n\,, u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)\,. \tag{4}$$

Remark 2.2: An equivalent definition of ISS is obtained if instead of using summation of terms in (2) the maximum is used as follows:

$$|x(t)| \leq \max\{\tilde{\beta}(|x(0)|, t), \tilde{\gamma}(\|u\|_\infty)\}. \tag{5}$$

Note that for a given system sum and maximum formulations may lead to different comparison functions $\tilde{\beta}$, $\tilde{\gamma}$ in (5) than those in (2). In a similar manner an equivalent definition can be formulated for GS in maximization terms.

Remark 2.3: In [18] it was shown that a system (1) is ISS if and only if it is GS and has the AG property.

We wish to consider criteria for ISS of interconnected systems. Thus consider n interconnected control systems given by

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1,\ldots,x_n,u_1) \\ & \vdots & \\ \dot{x}_n & = & f_n(x_1,\ldots,x_n,u_n) \end{array} \tag{6}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{m_i}$ and the functions $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + m_i} \to \mathbb{R}^{N_i}$ are continuous and for all $r \in \mathbb{R}$ are locally Lipschitz continuous in $x = ({x_1}^T,\ldots,{x_n}^T)^T$ uniformly in u_i for $|u_i| \le r$. This regularity condition for f_i guarantees the existence and uniqueness of solution for the ith subsystem for a given initial condition and input u_i.

The interconnection (6) can be written as (1) with $x := (x_1^T,\ldots,x_n^T)^T$, $u := (u_1^T,\ldots,u_n^T)^T$ and

$$f(x,u) = \left(f_1(x_1,\ldots,x_n,u_1)^T,\ldots,f_n(x_1,\ldots,x_n,u_n)^T\right)^T.$$

If we consider the individual subsystems, we treat the state $x_j, j \ne i$ as an independent input for the ith subsystem.

We now intend to formulate ISS conditions for the subsystems of (6), where some conditions are in the sum formulation as in (2) while other are given in the maximum form as in (5). Consider the index set $I := \{1,\ldots,n\}$ partitioned into two subsets I_Σ, $I_{\max}$ such that $I_{\max} = I \setminus I_\Sigma$.

The ith subsystem of (6) is ISS, if there exist functions β_i of class $\mathcal{KL}$, γ_{ij}, $\gamma_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values $x_i(0)$ and inputs $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ there exists a unique solution $x_i(\cdot)$ satisfying for all $t \ge 0$

$$|x_i(t)| \le \beta_i(|x_i(0)|,t) + \sum_{j=1}^n \gamma_{ij}(\|x_{j[0,t]}\|_\infty) + \gamma_i(\|u\|_\infty)\,, \quad \text{if} \quad i \in I_\Sigma\,, \tag{7}$$

and

$$|x_i(t)| \le \max\{\beta_i(|x_i(0)|,t), \max_j\{\gamma_{ij}(\|x_{j[0,t]}\|_\infty)\}, \gamma_i(\|u\|_\infty)\}\,, \quad \text{if} \quad i \in I_{\max}\,. \tag{8}$$

Remark 2.4: Note that without loss of generality we can assume that $I_\Sigma = \{1,\dots,k\}$ and $I_{\max} = \{k+1,\dots,n\}$ where $k := |I_\Sigma|$. This can be always achieved by a permutation of the subsystems in (6).

Since ISS implies GS and the AG property, there exist functions $\sigma_i, \hat\gamma_{ij}, \hat\gamma_i \in \mathcal{K}\cup\{0\}$, such that for any initial value $x_i(0)$ and input $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ there exists a unique solution $x_i(t)$ and for all $t \geq 0$

$$|x_i(t)| \leq \sigma_i(|x_i(0)|) + \sum_{j=1}^n \hat\gamma_{ij}(\|x_{j[0,t]}\|_\infty) + \hat\gamma_i(\|u\|_\infty)\,, \qquad \text{if} \quad i \in I_\Sigma\,, \tag{9}$$

$$|x_i(t)| \leq \max\{\sigma_i(|x_i(0)|), \max_j\{\hat\gamma_{ij}(\|x_{j[0,t]}\|_\infty)\}, \hat\gamma_i(\|u\|_\infty)\}\,, \qquad \text{if} \quad i \in I_{\max}\,, \tag{10}$$

which are the defining inequalities for the GS property of the i-th subsystem.

The AG property is defined in the same spirit by assuming that there exist functions $\overline\gamma_{ij}$, $\widetilde\gamma_i \in \mathcal{K}\cup\{0\}$, such that for any initial value $x_i(0)$ and inputs $x_j \in L_\infty(\mathbb{R}_+,\mathbb{R}^{N_j})$, $i \neq j$, $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ there exists a unique solution $x_i(t)$ and

$$\limsup_{t\to\infty} |x_i(t)| \leq \sum_{j=1}^n \overline\gamma_{ij}(\|x_j\|_\infty) + \overline\gamma_i(\|u\|_\infty)\,, \qquad \text{if} \quad i \in I_\Sigma\,, \tag{11}$$

$$\limsup_{t\to\infty} |x_i(t)| \leq \max\{\max_j\{\overline\gamma_{ij}(\|x_j\|_\infty)\}, \overline\gamma_i(\|u\|_\infty)\}\,, \qquad \text{if} \quad i \in I_{\max}\,. \tag{12}$$

We collect the gains $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ of the ISS conditions (7), (8) in a matrix $\Gamma = (\gamma_{ij})_{n\times n}$, with the convention $\gamma_{ii} \equiv 0,\ i = 1,\dots,n$. The operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is then defined by

$$\Gamma(s) := (\Gamma_1(s),\dots,\Gamma_n(s))^T\ , \tag{13}$$

where the functions $\Gamma_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ are given by $\Gamma_i(s) := \gamma_{i1}(s_1) + \dots + \gamma_{in}(s_n)$ for $i \in I_\Sigma$ and $\Gamma_i(s) := \max\{\gamma_{i1}(s_1),\dots,\gamma_{in}(s_n)\}$ for $i \in I_{\max}$. In particular, if $I_\Sigma = \{1,\dots,k\}$ and $I_{\max} = \{k+1,\dots,n\}$ we have

$$\Gamma(s) = \begin{pmatrix} \gamma_{12}(s_2) + \dots + \gamma_{1n}(s_n) \\ \vdots \\ \gamma_{k1}(s_1) + \dots + \gamma_{kn}(s_n) \\ \max\{\gamma_{k+1,1}(s_1),\dots,\gamma_{k+1,n}(s_n)\} \\ \vdots \\ \max\{\gamma_{n1}(s_1),\dots,\gamma_{n,n-1}(s_{n-1})\} \end{pmatrix}. \tag{14}$$

In [4] small gain conditions were considered for the case $I_\Sigma = I = \{1,\dots,n\}$, respectively $I_{\max} = I$. In [16], [7] more general formulations of ISS are considered, which encompass the case studied in this paper. In this paper we exploit the special structure to obtain more specific results than available before.

Our main question is whether the interconnection (6) is ISS from u to x. To motivate the approach we briefly recall the small gain conditions for the cases $I_\Sigma = I$, resp. $I_{\max} = I$, which imply ISS of the interconnection, [4]. If $I_\Sigma = I$, we need to assume that there exists a $D := \mathrm{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$ such that

$$\Gamma \circ D(s) \not\geq s, \, \forall s \in \mathbb{R}^n_+ \backslash \{0\} \, , \tag{15}$$

and if $I_{\max} = I$, then the small gain condition

$$\Gamma(s) \not\geq s, \, \forall s \in \mathbb{R}^n_+ \backslash \{0\} \tag{16}$$

is sufficient. In case that both I_Σ and $I_{\max}$ are not empty we can use

$$\max_{i=1,\ldots,n} \{x_i\} \leq \sum_{i=1}^n x_i \leq n \max_{i=1,\ldots,n} \{x_i\} \tag{17}$$

to pass to the situation with $I_\Sigma = \emptyset$ or $I_{\max} = \emptyset$. But this leads to more conservative gains. To avoid this conservativeness we are going to obtain a new small gain condition for the case $I_\Sigma \neq I \neq I_{\max}$. As we will see there are two essentially equivalent approaches to do this. We may use the weak triangle inequality

$$a + b \leq \max\{(\mathrm{id} + \eta) \circ a, (\mathrm{id} + \eta^{-1}) \circ b\} \, , \tag{18}$$

which is valid for all functions $a, b, \eta \in \mathcal{K}_\infty$ as discussed in Section III-A to pass to a pure maximum formulation of ISS. However, this method involves the right choice of a large number of weights in the weak triangular inequality which can be a nontrivial problem. Alternatively tailor-made small gain conditions can be derived. The expressions in (15), (16) prompt us to consider the following small gain condition. For a given $\alpha \in \mathcal{K}_\infty$ let the diagonal operator $D_\alpha : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by

$$D_\alpha(s) := (D_1(s_1), \ldots, D_n(s_n))^T \, , \quad s \in \mathbb{R}^n_+ \, , \tag{19}$$

where $D_i(s_i) := (\mathrm{id} + \alpha)(s_i)$ for $i \in I_\Sigma$ and $D_i(s_i) := s_i$ for $i \in I_{\max}$. The small gain condition on the operator Γ corresponding to a partition $I = I_\Sigma \cup I_{\max}$ is then

$$\exists \; \alpha \in \mathcal{K}_\infty \quad : \quad \Gamma \circ D_\alpha(s) \not\geq s, \, \forall s \in \mathbb{R}^n_+ \backslash \{0\}. \tag{20}$$

We will abbreviate this condition as $\Gamma \circ D_\alpha \not\geq \mathrm{id}$. In this paper we will prove that this small gain condition guarantees the ISS property of the interconnection (6) and show how an ISS-Lyapunov function can be constructed if this condition is satisfied in the case of a Lyapunov formulation of ISS.

Before developing the theory we discuss an example to highlight the advantage of the new small gain condition (20), cf. Theorem 4.4. In order not to cloud the issue we keep the example as simple as possible.

Example 2.5: We consider an interconnection of $n = 3$ systems given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + \gamma_{13}(|x_3|) + \gamma_1(u) \\ \dot{x}_2 &= -x_2 + \max\{\gamma_{21}(|x_1|), \gamma_{23}(|x_3|)\} \\ \dot{x}_3 &= -x_3 + \max\{\gamma_{32}(|x_2|), \gamma_3(u)\} \end{aligned} \tag{21}$$

where the γ_{ij} are given $\mathcal{K}_\infty$ functions. Using the variation of constants method and the weak triangle inequality (18) we see that the trajectories can be estimated by:

$$\begin{aligned} |x_1(t)| &\leq \beta_1(|x(0)|, t) + \gamma_{13}(||x_{3[0,t]}||_\infty) + \gamma_1(\|u\|_\infty) \\ |x_2(t)| &\leq \max\{\beta_2(|x(0)|, t), (\mathrm{id} + \eta) \circ \gamma_{21}(||x_{1[0,t]}||_\infty), (\mathrm{id} + \eta) \circ \gamma_{23}(||x_{3[0,t]}||_\infty)\} \\ |x_3(t)| &\leq \max\{\beta_3(|x(0)|, t), (\mathrm{id} + \eta) \circ \gamma_{32}(||x_{2[0,t]}||_\infty), (\mathrm{id} + \eta) \circ \gamma_3(\|u\|_\infty)\}\,, \end{aligned} \tag{22}$$

where the β_i are appropriate $\mathcal{KL}$ functions and $\eta \in \mathcal{K}_\infty$ is arbitrary.

This shows that each subsystem is ISS. In this case we have

$$\Gamma = \begin{pmatrix} 0 & 0 & \gamma_{13} \\ (\mathrm{id} + \eta) \circ \gamma_{21} & 0 & (\mathrm{id} + \eta) \circ \gamma_{23} \\ 0 & (\mathrm{id} + \eta) \circ \gamma_{32} & 0 \end{pmatrix}.$$

Then the small gain condition (20) requires that there exists an $\alpha \in \mathcal{K}_\infty$ such that

$$\begin{pmatrix} \gamma_{13}(s_3) \\ \max\{(\mathrm{id} + \eta) \circ \gamma_{21} \circ (\mathrm{id} + \alpha)(s_1), (\mathrm{id} + \eta) \circ \gamma_{23}(s_3)\} \\ (\mathrm{id} + \eta) \circ \gamma_{32}(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{23}$$

for all $s \in \mathbb{R}^3_+ \setminus \{0\}$. If (23) holds then considering $s^T(r) := (\gamma_{13} \circ (\mathrm{id} + \eta) \circ \gamma_{32}(r), r, (\mathrm{id} + \eta) \circ \gamma_{32}(r))^T$, $r > 0$ we obtain that the following two inequalities are satisfied

$$(\mathrm{id} + \alpha) \circ \gamma_{13} \circ (\mathrm{id} + \eta) \circ \gamma_{32} \circ (\mathrm{id} + \eta) \circ \gamma_{21}(r) < r, \tag{24}$$

$$(\mathrm{id} + \eta) \circ \gamma_{23} \circ (\mathrm{id} + \eta) \circ \gamma_{32}(r) < r. \tag{25}$$

It can be shown by contradiction that (24) and (25) imply (23).

To give a simple example assume the that the gains are linear and given by $\gamma_{13} := \gamma_{21} := \gamma_{23} := \gamma_{32}(r) = 0.9\, r$, $r \geq 0$. Choosing $\alpha = \eta = 1/10$ we see that the inequalities (24) and (25)) are satisfied. So by Theorem 4.4 we conclude that system (1) is ISS. In this simple example we also see that a transformation to the pure maximum case would have been equally simple. An application of the weak triangle inequality for the first row with $\eta = \alpha$ would have led to the pure maximization case. In this case the small gain condition may be expressed as a cycle condition [19], [4], [16], [13], [14], which just yields the conditions (24) and (25).
We would like to note that application of the small gain condition from [4] will not help us to prove stability for this example, as can be seen from the following example.

Example 2.6: In order to apply results from [4] we could (e.g. by using (17)) obtain estimates of the form

$$\begin{aligned} |x_1(t)| &\leq \beta_1(|x(0)|, t) + \gamma_{13}(||x_{3[0,t]}||_\infty) + \gamma_1(\|u\|_\infty) \\ |x_2(t)| &\leq \beta_2(|x(0)|, t) + \gamma_{21}(||x_{1[0,t]}||_\infty) + \gamma_{23}(||x_{3[0,t]}||_\infty) \\ |x_3(t)| &\leq \beta_3(|x(0)|, t) + \gamma_{32}(||x_{2[0,t]}||_\infty) + \gamma_3(\|u\|_\infty)\,. \end{aligned} \tag{26}$$

With the gains from the previous example the corresponding gain matrix is

$$\Gamma = \begin{pmatrix} 0 & 0 & 0.9 \\ 0.9 & 0 & 0.9 \\ 0 & 0.9 & 0 \end{pmatrix},$$

and in the summation case with linear gains the small gain condition is $r(\Gamma) < 1$, [4]. In our example $r(\Gamma) > 1.19$, so that using this criterion we cannot conclude ISS of the interconnection.

The previous examples motivate the use of the refined small gain condition developed in this paper for the case of different ISS characterizations. In the next section we study properties of the gain operators and show that mixed ISS formulations can in theory always be transformed to a maximum formulation without losing information on the small gain condition.

III. Gain Operators

In this section we prove some auxiliary results for the operators satisfying small gain condition (20). In particular, it will be shown that a mixed (or pure sum) ISS condition can always be reformulated as a maximum condition in such a way that the small gain property is preserved.[1]

The following lemma recalls a fact, that was already noted in [4].

Lemma 3.1: For any $\alpha \in \mathcal{K}_\infty$ the small gain condition $D_\alpha \circ \Gamma \not\geq \mathrm{id}$ is equivalent to $\Gamma \circ D_\alpha \not\geq \mathrm{id}$.

Proof: Note that D_α is a homeomorphism with inverse $v \mapsto D_\alpha^{-1}(v) := \big(D_1^{-1}(v_1), \dots, D_n^{-1}(v_n)\big)^T$. By monotonicity of D_α and D_α^{-1} we have $D_\alpha \circ \Gamma(v) \not\geq v$ if and only if $\Gamma(v) \not\geq D_\alpha^{-1}(v)$. For any $w \in \mathbb{R}_+^n$ define $v = D_\alpha(w)$. Then $\Gamma \circ D_\alpha(w) \not\geq w$. This proves the equivalence. ∎

For convenience let us introduce $\mu : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\mu(w, v) := (\mu_1(w_1, v_1), \dots, \mu_n(w_n, v_n))^T, w \in \mathbb{R}_+^n,\ v \in \mathbb{R}_+^n, \tag{27}$$

where $\mu_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ is such that $\mu_i(w_i, v_i) := w_i + v_i$ for $i \in I_\Sigma$ and $\mu_i(w_i, v_i) := \max\{w_i, v_i\}$ for $i \in I_{\max}$. The following counterpart of Lemma 13 in [4] provides the main technical step in the proof of the main results.

Lemma 3.2: Assume that there exists an $\alpha \in \mathcal{K}_\infty$ such that the operator Γ as defined in (13) satisfies $\Gamma \circ D_\alpha \not\geq \mathrm{id}$ for a diagonal operator D_α as defined in (19). Then there exists a $\phi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}_+^n$,

$$w \leq \mu(\Gamma(w), v) \tag{28}$$

implies $\|w\| \leq \phi(\|v\|)$.

Proof: Without loss of generality we assume $I_\Sigma = \{1, \dots, k\}$ and $I_{\max} = I \setminus I_\Sigma$, see Remark 2.4, and hence Γ is as in (14). Fix any $v \in \mathbb{R}_+^n$. Note that for $v = 0$ there is nothing to show, as then $w \neq 0$ yields an immediate contradiction to the small gain condition. So assume $v \neq 0$.

We first show, that for those $w \in \mathbb{R}_+^n$ satisfying (28) at least some components of w have to be bounded. To this end let $\widetilde{D} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be defined by

$$\widetilde{D}(s) := \big(s_1 + \alpha^{-1}(s_1), \dots, s_k + \alpha^{-1}(s_k), s_{k+1}, \dots, s_n\big)^T,\ s \in \mathbb{R}_+^n$$

and let $s^* := \widetilde{D}(v)$. Assume there exists $w = (w_1, \dots, w_n)^T$ satisfying (28) and such that $w_i > s_i^*$, $i = 1, \dots, n$. In particular, for $i \in I_\Sigma$ we have

$$s_i^* < w_i \leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i \tag{29}$$

[1]We would like to thank one of the anonymous reviewers for posing the question whether this is possible.

and hence from the definition of s^* it follows that

$$s_i^* = v_i + \alpha^{-1}(v_i) < \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i.$$

And so $v_i < \alpha(\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n))$. From (29) it follows

$$w_i \leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i < (id + \alpha) \circ (\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n)). \tag{30}$$

Similarly, by the construction of w and the definition of s^* we have for $i \in I_{\max}$

$$v_i = s_i^* < w_i \leq \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n), v_i\}\,, \tag{31}$$

and hence

$$w_i \leq \max\{\gamma_{i1}(w_1), \ldots, \gamma_{in}(w_n)\}. \tag{32}$$

From (30), (32) we get $w \leq D_\alpha \circ \Gamma(w)$. By Lemma 3.1 this contradicts the assumption $\Gamma \circ D_\alpha \not\geq$ id. Hence some components of w are bounded by the respective components of $s^1 := s^*$. Iteratively we will prove that all components of w are bounded.

Fix a w satisfying (28). Then $w \not\leq s^1$ and so there exists an index set $I_1 \subset I$, possibly depending on w, such that $w_i > s_i^1$, $i \in I_1$ and $w_i \leq s_i^1$, for $i \in I_1^c = I \setminus I_1$. Note that by the first step I_1^c is nonempty. We now renumber the coordinates so that

$$w_i > s_i^1 \text{ and } \quad w_i \leq \sum_{j=1}^n \gamma_{ij}(w_j) + v_i\,, \qquad i = 1, \ldots, k_1, \tag{33}$$

$$w_i > s_i^1 \text{ and } \quad w_i \leq \max\{\max_j \gamma_{ij}(w_j), v_i\}\,, \quad i = k_1 + 1, \ldots, n_1, \tag{34}$$

$$w_i \leq s_i^1 \text{ and } \quad w_i \leq \sum_{j=1}^n \gamma_{ij}(w_j) + v_i,\,, \qquad i = n_1 + 1, \ldots, n_1 + k_2 \tag{35}$$

$$w_i \leq s_i^1 \text{ and } \quad w_i \leq \max\{\max_j \gamma_{ij}(w_j), v_i\}\,, \quad i = n_1 + k_2 + 1, \ldots, n\,, \tag{36}$$

where $n_1 = |I_1|$, $k_1 + k_2 = k$. Using (35), (36) in (33), (34) we get

$$w_i \leq \quad \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i, \qquad i = 1, \ldots, k_1, \tag{37}$$

$$w_i \leq \max\{\max_{j=1,\ldots,n_1} \gamma_{ij}(w_j), \max_{j=n_1+1,\ldots,n} \gamma_{ij}(s_j^1), v_i\}, \quad i = k_1 + 1, \ldots, n_1\,. \tag{38}$$

Define $v^1 \in \mathbb{R}_+^{n_1}$ by

$$\begin{aligned} v_i^1 &:= \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i\,, \qquad i = 1, \ldots, k_1\,, \\ v_i^1 &:= \max\{\max_{j=n_1+1,\ldots,n} \gamma_{ij}(s_j^1), v_i\}\,, \quad i = k_1 + 1, \ldots, n_1. \end{aligned}$$

Now (37), (38) take the form:

$$w_i \leq \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + v_i^1\,, \qquad i = 1, \ldots, k_1, \tag{39}$$

$$w_i \leq \max\{\max_{j=1,\ldots,n_1} \gamma_{ij}(w_j), v_i^1\}\,, \quad i = k_1+1, \ldots, n_1. \tag{40}$$

Let us represent $\Gamma = \begin{pmatrix} \Gamma_{I_1 I_1} & \Gamma_{I_1 I_1^c} \\ \Gamma_{I_1^c I_1} & \Gamma_{I_1^c I_1^c} \end{pmatrix}$ and define the maps $\Gamma_{I_1 I_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1 I_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1^c I_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n-n_1}$ and $\Gamma_{I_1^c I_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n-n_1}$ analogous to Γ. Let

$$D_{I_1}(s) := ((id+\alpha)(s_1), \ldots, (id+\alpha)(s_{k_1}), s_{k_1+1}, \ldots, s_{n_1})^T\ .$$

From $\Gamma \circ D_\alpha(s) \not\geq s$ for all $s \neq 0$, $s \in \mathbb{R}_+^n$ it follows by considering $s = (z^T, 0)^T$ that $\Gamma_{I_1 I_1} \circ D_{I_1}(z) \not\geq z$ for all $z \neq 0$, $z \in \mathbb{R}_+^{n_1}$. Using the same approach as for $w \in \mathbb{R}_+^n$ it can be proved that some components of $w^1 = (w_1, \ldots, w_{n_1})^T$ are bounded by the respective components of $s^2 := \widetilde{D}_{I_1}(v^1)$.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s_i^{j+1}\}, \tag{41}$$

with $I_{j+1}^c := I \setminus I_{j+1}$ and

$$s^{j+1} := \widetilde{D}_{I_j} \circ (\mu^j(\Gamma_{I_j I_j^c}(s^j_{I_j^c}), v_{I_j})), \tag{42}$$

where $\widetilde{D}_{I_j}$ is defined analogously to $\widetilde{D}$, the map $\Gamma_{I_j I_j^c} : \mathbb{R}_+^{n-n_j} \to \mathbb{R}_+^{n_j}$ acts analogously to Γ on vectors of the corresponding dimension, $s^j_{I_j^c} = (s_i^j)_{i \in I_j^c}$ is the restriction defined in the preliminaries and μ^j is appropriately defined similar to the definition of μ.

The nesting (41), (42) will end after at most $n-1$ steps: there exists a maximal $l \leq n$, such that

$$I \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_l \neq \emptyset$$

and all components of w_{I_l} are bounded by the corresponding components of s^{l+1}. Let

$$s_\varsigma := \max\{s^*, R_{I_1}(s^2), \ldots, R_{I_l}(s^{l+1})\} := \begin{pmatrix} \max\{(s^*)_1, (R_{I_1}(s^2))_1, \ldots, (R_{I_l}(s^{l+1}))_1\} \\ \vdots \\ \max\{(s^*)_n, (R_{I_1}(s^2))_n, \ldots, (R_{I_l}(s^{l+1}))_n\} \end{pmatrix}$$

where R_{I_j} denotes the anti-projection $\mathbb{R}_+^{|I_j|} \to \mathbb{R}_+^n$ defined above.

By the definition of μ for all $v \in \mathbb{R}_+^n$ it holds

$$0 \leq v \leq \mu(\Gamma, \mathrm{id})(v) := \mu(\Gamma(v), v).$$

Let the n-fold composition of a map $M : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ of the form $M \circ \ldots \circ M$ be denoted by $[M]^n$. Applying $\widetilde{D}$ we have

$$0 \leq v \leq \widetilde{D}(v) \leq \widetilde{D}\circ(\mu(\Gamma, \mathrm{id}))(v) \leq \cdots \leq [\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n(v). \tag{43}$$

From (42) and (43) for w satisfying (28) we have $w \leq s_\varsigma \leq [\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n(v)$. The term on the right-hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (28) also satisfies $w \leq [\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n(|v|_{\max}, \ldots, |v|_{\max})^T$ and taking the maximum-norm on both sides yields $|w|_{\max} \leq \phi(|v|_{\max})$ for some function ϕ of class $\mathcal{K}_\infty$. For example, ϕ can be chosen as

$$\phi(r) := \max\{([\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n(r, \ldots, r))_1, \ldots, ([\widetilde{D} \circ \mu(\Gamma, \mathrm{id})]^n(r, \ldots, r))_n\}.$$

This completes the proof of the lemma. ■

We also introduce the important notion of Ω-paths [7]. This concept is useful in the construction of Lyapunov functions and will also be instrumental in obtaining a better understanding of the relation between max and sum small gain conditions.

Definition 3.3: A continuous path $\sigma \in \mathcal{K}^n_\infty$ is called an Ω-path with respect to Γ if

(i) for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;

(ii) for every compact set $P \subset (0, \infty)$ there are finite constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \ldots, n$ we have

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C, \quad \forall r \in P \tag{44}$$

(iii) for all $r > 0$ it holds that $\Gamma(\sigma(r)) < \sigma(r)$.

By [7, Theorem 8.11] the existence of an Ω-path σ follows from the small gain condition (16) provided an irreducibility condition is satisfied. To define this notion we consider the directed graph $G(\mathcal{V}, \mathcal{E})$ corresponding to Γ with nodes $\mathcal{V} = \{1, \ldots, n\}$. A pair $(i, j) \in \mathcal{V} \times \mathcal{V}$ is an edge in the graph if $\gamma_{ij} \neq 0$. Then Γ is called irreducible if the graph is strongly connected, see e.g. the appendix in [4] for further discussions on this topic.

We note that if Γ is reducible, then it may be brought into upper block triangular form by a permutation of the indices

$$\Gamma = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \dots & \Upsilon_{1d} \\ 0 & \Upsilon_{22} & \dots & \Upsilon_{2d} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \Upsilon_{dd} \end{pmatrix} \tag{45}$$

where each block $\Upsilon_{jj} \in (\mathcal{K}_\infty \cup \{0\})^{d_j \times d_j}$, $j = 1, \dots, d$, is either irreducible or 0.

The following is an immediate corollary to [7, Theorem 8.11], where the result is only implicitly contained.

Corollary 3.4: Assume that Γ defined in (13) is irreducible. Then Γ satisfies the small gain condition if and only if an Ω-path σ exists for $D \circ \Gamma$.

Proof: The hard part is the implication that the small gain condition guarantees the existence of an Ω-path, see [7]. For the converse direction assume that an Ω-path exists for $D \circ \Gamma$ and that for a certain $s \in \mathbb{R}^n_+, s \neq 0$ we have $D \circ \Gamma(s) \geq s$. By continuity and unboundedness of σ we may choose a $\tau > 0$ such that $\sigma(\tau) \geq s$ but $\sigma(\tau) \not\geq s$. Then $s \leq D \circ \Gamma(s) \leq D \circ \Gamma(\sigma(\tau)) < \sigma(\tau)$. This contradiction proves the statement. ■

A. From Summation to Maximization

We now use the previous consideration to show that an alternative approach is possible for the treatment of the mixed ISS formulation, which consists of transforming the complete formulation in a maximum formulation. Using the weak triangle inequality (18) iteratively the conditions in (7) may be transformed into conditions of the form (8) with

$$|x_i(t)| \;\leq\; \beta_i(|x_i(0)|, t) + \sum_{j=1}^{n} \gamma_{ij}(\|x_{j[0,t]}\|_\infty) + \gamma_i(\|u\|_\infty) \tag{46}$$

$$\leq\; \max\{\tilde{\beta}_i(|x_i(0)|, t), \max_j\{\tilde{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty)\}, \tilde{\gamma}_i(\|u\|_\infty)\} \tag{47}$$

for $i \in I_\Sigma$. To get a general formulation we let $j_1, \dots, j_{k_i}$ denote the indices j for which $\gamma_{ij} \neq 0$. Choose auxiliary functions $\eta_{i0}, \dots, \eta_{ik_i} \in \mathcal{K}_\infty$ and define $\chi_{i0} := (\mathrm{id} + \eta_{i0})$ and $\chi_{il} = (\mathrm{id} + \eta_{i0}^{-1}) \circ \ldots \circ (\mathrm{id} + \eta_{i(l-1)}^{-1}) \circ (\mathrm{id} + \eta_{il})$, $l = 1, \dots, k_i$, and $\chi_{i(k_i+1)} = (\mathrm{id} + \eta_{i0}^{-1}) \circ \cdots \circ (\mathrm{id} + \eta_{ik_i}^{-1})$. Choose a permutation $\pi_i : \{0, 1, \dots, k_i + 1\} \to \{0, 1, \dots, k_i + 1\}$ and define

$$\tilde{\beta}_i := \chi_{i\pi_i(0)} \circ \beta_i \,, \quad \tilde{\gamma}_{ij_l} := \chi_{i\pi_i(l)} \circ \gamma_{ij_l} \,, \; l = 1, \dots, k_i \,, \quad \tilde{\gamma}_i := \chi_{i\pi_i(k_i+1)} \circ \gamma_i \,, \tag{48}$$

and of course $\tilde{\gamma}_{ij} \equiv 0$, $j \notin \{j_1, \dots, j_{k_1}\}$. In this manner the inequalities (46) are valid and a maximum ISS formulation is obtained. Performing this for every $i \in I_\Sigma$ we obtain an operator $\tilde{\Gamma} : \mathbb{R}_n^+ \to \mathbb{R}_n^+$ defined by

$$\left(\tilde{\Gamma}_1(s), \dots, \tilde{\Gamma}_n(s)\right)^T, \tag{49}$$

where the functions $\tilde{\Gamma}_i : \mathbb{R}_+^n \to \mathbb{R}_+$ are given by $\tilde{\Gamma}_i(s) := \max\{\tilde{\gamma}_{i1}(s_1), \dots, \tilde{\gamma}_{in}(s_n)\}$ for $i \in I_\Sigma$ and $\tilde{\Gamma}_i(s) := \max\{\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n)\}$ for $i \in I_{\max}$. Here the $\tilde{\gamma}_{ij}$'s are given by (48), whereas the γ_{ij}'s are the original gains.

As it turns out the permutation is not really necessary and it is sufficient to peel off the summands one after the other. We will now show that given a gain operator Γ with a mixed or pure sum formulation which satisfies the small gain condition $D \circ \Gamma \not\geq \mathrm{id}$, it is always possible to switch to a maximum formulation which also satisfies the corresponding small gain condition $\tilde{\Gamma} \not\geq \mathrm{id}$. In the following statement k_i is to be understood as defined just after (47).

Proposition 3.5: Consider a gain operator Γ of the form (13). Then the following two statements are equivalent

(i) the small gain condition (20) is satisfied,

(ii) for each $i \in I_\Sigma$ there exist $\eta_{i,0}, \dots, \eta_{i,(k_i+1)} \in \mathcal{K}_\infty$, such that the corresponding small gain operator $\tilde{\Gamma}$ satisfies the small gain condition (16).

Remark 3.6: We note that in the case that a system (6) satisfies a mixed ISS condition with operator Γ, then the construction in (46) shows that the ISS condition is also satisfied in the maximum sense with the operator $\tilde{\Gamma}$. On the other hand the construction in the proof does not guarantee that if the ISS condition is satisfied for the operator $\tilde{\Gamma}$ then it will also be satisfied for the original Γ.

Proof: "$\Rightarrow$": We will show the statement under the condition that Γ is irreducible. In the reducible case we may assume that Γ is in upper block triangular form (45). In each of the diagonal blocks we can perform the transformation described below and the gains in the off-diagonal blocks are of no importance for the small gain condition.

In the irreducible case we may apply Corollary 3.4 to obtain a continuous map $\sigma : [0, \infty) \to \mathbb{R}_+^n$, where $\sigma_i \in \mathcal{K}_\infty$ for every component function of σ and so that

$$D \circ \Gamma \circ \sigma(\tau) < \sigma(\tau), \quad \text{for all} \quad \tau > 0. \tag{50}$$

Define the homeomorphism $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $T : s \mapsto (\sigma_1(s_1), \dots, \sigma_n(s_n))$. Then $T^{-1} \circ D \circ \Gamma \circ T \not\geq \mathrm{id}$ and we have by (50) for $e = \sum_{i=1}^n e_i$, that

$$T(\tau e) = \sigma(\tau) > D \circ \Gamma \circ \sigma(\tau) = D \circ \Gamma \circ T(\tau e)\,,$$

so that for all $\tau > 0$

$$T^{-1} \circ D \circ \Gamma \circ T(\tau e) < \tau e\,. \tag{51}$$

We will show that $T^{-1} \circ \tilde{\Gamma} \circ T(\tau e) < \tau e$ for an appropriate choice of the functions η_{ij}. By the converse direction of Corollary 3.4 this shows that $T^{-1} \circ \tilde{\Gamma} \circ T \not\geq \mathrm{id}$ and hence $\tilde{\Gamma} \not\geq \mathrm{id}$ as desired.

Consider now a row corresponding to $i \in I_\Sigma$ and let $j_1, \dots, j_{k_i}$ be the indices for which $\gamma_{ij} \neq 0$. For this row (51) implies

$$\sigma_i^{-1} \circ (\mathrm{id} + \alpha) \circ \left(\sum_{j \neq i} \gamma_{ij}(\sigma_j(r)) \right) < r\,, \quad \forall r > 0\,, \tag{52}$$

or equivalently

$$(\mathrm{id} + \alpha) \circ \left(\sum_{j \neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1} \right) \circ \sigma_i(r) < \sigma_i(r)\,, \quad \forall r > 0\,. \tag{53}$$

This shows that

$$(\mathrm{id} + \alpha) \circ \left(\sum_{j \neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1} \right) < \mathrm{id}\,, \quad \text{on } (0, \infty)\,. \tag{54}$$

Note that this implies that $\left(\mathrm{id} - \sum_{j \neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}\right) \in \mathcal{K}_\infty$ because $\alpha \in \mathcal{K}_\infty$. We may therefore choose $\hat{\gamma}_{ij} > \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}, j = j_1, \dots, j_{k_i}$ in such a manner that

$$\mathrm{id} - \sum_{l=1}^{k_i} \hat{\gamma}_{ij_l} \quad \in \mathcal{K}_\infty\,.$$

Now define for $l = 1, \dots, k_i$

$$\eta_{il} := \left(\mathrm{id} - \sum_{k \leq l} \hat{\gamma}_{ij_k} \right) \circ \hat{\gamma}_{ij_l}^{-1} \quad \in \mathcal{K}_\infty\,.$$

It is straightforward to check that

$$(\mathrm{id} + \eta_{il}) = \left(\mathrm{id} - \sum_{k<l} \hat{\gamma}_{ij_k} \right) \circ \hat{\gamma}_{ij_l}^{-1}\,, \quad (\mathrm{id} + \eta_{il}^{-1}) = \left(\mathrm{id} - \sum_{k<l} \hat{\gamma}_{ij_k} \right) \circ \left(\mathrm{id} - \sum_{k \leq l} \hat{\gamma}_{ij_k} \right)^{-1}\,.$$

With $\chi_{il} := (\mathrm{id} + \eta_{i1}^{-1}) \circ \dots \circ (\mathrm{id} + \eta_{i(l-1)}^{-1}) \circ (\mathrm{id} + \eta_{il})$ it follows that

$$\chi_{il} \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} = (\mathrm{id} + \eta_{i1}^{-1}) \circ \dots \circ (\mathrm{id} + \eta_{i,l-1}^{-1}) \circ (\mathrm{id} + \eta_{il}) \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} = \hat{\gamma}_{ij_l}^{-1} \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} < \mathrm{id}\,.$$

This shows that it is possible to choose $\eta_{ij}, i \in I_\Sigma$ such that all the entries in $T^{-1} \circ \tilde{\Gamma} \circ T$ are smaller than the identity. This shows the assertion.

"$\Leftarrow$": To show the converse direction let the small gain condition (16) be satisfied for the operator $\tilde{\Gamma}$. Consider $i \in I_\Sigma$.

We consider the following two cases for the permutation π used in (48). Define $p := \min\{\pi(0), \pi(k_i+1)\}$. In the first case $\{\pi(0), \pi(k_i+1)\} = \{k_i, k_i+1\}$, i.e., $\pi(l) < p, \forall\ l \in \{1, \dots, k_i\}$. Alternatively, the second case is $\exists l \in \{1, \dots, k_i\} : \pi(l) > p$.

We define $\alpha_i \in \mathcal{K}_\infty$ by

$$\alpha_i := \begin{cases} \eta_{ip}^{-1} \circ \sum\limits_{\pi(l)>p} \gamma_{ij_l} \circ \left(\sum\limits_j \gamma_{ij}\right)^{-1}, & \text{if } \exists j \in \{1, \dots, k_i\} : \pi(j) > p\,, \\ \eta_{i,p-1} \circ \gamma_{i,j_{\pi^{-1}(p-1)}} \circ \left(\sum\limits_j \gamma_{ij}\right)^{-1}, & \text{if } \forall j \in \{1, \dots, k_i\} \quad \pi(j) < p\,. \end{cases} \tag{55}$$

Consider the ith row of $D \circ \Gamma$ and the case $\exists j \in \{1, \dots, k_i\} : \pi(j) > p$. (Note that for no $l \in \{1, \dots, k_i\}$ we have $\pi(l) = p$).

$$\begin{aligned} (\mathrm{id} + \alpha_i) \circ \sum_j \gamma_{ij} &= \sum_j \gamma_{ij} + \alpha_i \circ \sum_j \gamma_{ij} \\ &= \sum_j \gamma_{ij} + \eta_{ip}^{-1} \circ \sum_{\pi(l)>p} \gamma_{ij_l} \circ \left(\sum_j \gamma_{ij}\right)^{-1} \circ \sum_j \gamma_{ij} \\ &= \sum_j \gamma_{ij} + \eta_{ip}^{-1} \circ \sum_{\pi(l)>p} \gamma_{ij_l} \\ &= \sum_{\pi(l)<p} \gamma_{ij_l} + (\mathrm{id} + \eta_{ip}^{-1}) \circ \sum_{\pi(l)>p} \gamma_{ij_l}\,. \end{aligned} \tag{56}$$

Applying the weak triangle inequality (18) first to the rightmost sum in the last line of (56) and then to the remaining sum we obtain

$$\begin{aligned} & \sum_{\pi(l)<p} \gamma_{ij_l} + (\mathrm{id} + \eta_{ip}^{-1}) \circ \sum_{\pi(l)>p} \gamma_{ij_l} \\ \leq\ & \sum_{\pi(l)<p-1} \gamma_{ij_l} + \max\{(\mathrm{id} + \eta_{i,p-1}) \circ \gamma_{i,\pi^{-1}(p-1)}, \\ & (\mathrm{id} + \eta_{i,p-1}^{-1}) \circ (\mathrm{id} + \eta_{ip}^{-1}) \circ \max_{\pi(l)>p}\{(\mathrm{id} + \eta_{i,p+1}^{-1}) \circ \ldots \circ (\mathrm{id} + \eta_{i,\pi(l)-1}^{-1}) \circ (\mathrm{id} + \eta_{i\pi(l)}) \circ \gamma_{ij_l}\}\} \\ \leq\ & \ldots \quad \leq \max_l \{\chi_{i\pi(l)} \circ \gamma_{ij_l}\}\,. \end{aligned} \tag{57}$$

The last expression is the defining equation for $\widetilde{\Gamma}_i(s_1, \dots, s_n) = \max\limits_{l=1,\dots,k_i} \{\chi_{i\pi(l)} \circ \gamma_{ij_l}(s_{j_l})\}$. Thus from (56), (57) we obtain $\widetilde{\Gamma}_i \geq (D \circ \Gamma)_i$.

Consider now the case $\forall l \in \{1,\dots,k_i\} \quad \pi(l) < p$. A similar approach shows that $\widetilde{\Gamma}_i \geq (D \circ \Gamma)_i$. Following the same steps as in the first case we obtain

$$\begin{aligned}(\mathrm{id}+\alpha_i)\circ\sum_j \gamma_{ij} &= \sum_j \gamma_{ij} + \eta_{i,p-1}\circ\gamma_{i,j_{\pi^{-1}(p-1)}} \\ &= \sum_{\pi(l)<p-1}\gamma_{ij_l} + (\mathrm{id}+\eta_{i,p-1})\circ\gamma_{i,j_{\pi^{-1}(p-1)}} \\ &\leq \sum_{\pi(l)<p-2}\gamma_{ij_l} + \max\{(\mathrm{id}+\eta_{i,p-2})\circ\gamma_{ij_{\pi^{-1}(p-2)}}, \\ &\qquad (\mathrm{id}+\eta_{i,p-2}^{-1})\circ(\mathrm{id}+\eta_{i,(p-1)})\circ\gamma_{i,j_{\pi^{-1}(p-1)}}\} \\ &\leq \dots \quad \leq \max_l\{\chi_{i\pi(l)}\circ\gamma_{ij_l}\}\,. \end{aligned} \tag{58}$$

Again from (58) $\widetilde{\Gamma}_i \geq (D\circ\Gamma)_i$.

Taking $\alpha = \min \alpha_i \in \mathcal{K}_\infty$ it holds that $\widetilde{\Gamma} \geq D\circ\Gamma$. Thus if $\widetilde{\Gamma} \not\geq \mathrm{id}$, then $D\circ\Gamma \not\geq \mathrm{id}$. ■

IV. Small gain theorem

We now turn back to the question of stability. In order to prove ISS of (6) we use the same approach as in [4]. The main idea is to prove that the interconnection is GS and AG and then to use the result of [18] by which AG and GS systems are ISS.

So, let us first prove small gain theorems for GS and AG.

Theorem 4.1: Assume that each subsystem of (6) is GS and a gain matrix is given by $\Gamma = (\hat{\gamma}_{ij})_{n\times n}$. If there exists D as in (19) such that $\Gamma\circ D(s) \not\geq s$ for all $s \neq 0, s \geq 0$, then the system (1) is GS.

Proof: Let us take the supremum over $\tau \in [0,t]$ on both sides of (9), (10). For $i \in I_\Sigma$ we have

$$\|x_{i[0,t]}\|_\infty \leq \sigma_i(|x_i(0)|) + \sum_{j=1}^{n}\hat{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty) + \hat{\gamma}_i(\|u\|_\infty) \tag{59}$$

and for $i \in I_{\max}$ it follows

$$\|x_{i[0,t]}\|_\infty \leq \max\{\sigma_i(|x_i(0)|), \max_j\{\hat{\gamma}_{ij}(\|x_{j[0,t]}\|_\infty)\}, \hat{\gamma}_i(\|u\|_\infty)\}. \tag{60}$$

Let us denote $w = \left(\|x_{1[0,t]}\|_\infty, \dots, \|x_{n[0,t]}\|_\infty\right)^T$,

$$v \;=\; \begin{pmatrix} \mu_1(\sigma_1(|x_1(0)|), \hat{\gamma}_1(\|u\|_\infty)) \\ \vdots \\ \mu_n(\sigma_n(|x_n(0)|), \hat{\gamma}_n(\|u\|_\infty)) \end{pmatrix} = \mu(\sigma(|x(0)|), \hat{\gamma}(\|u\|_\infty)),$$

where we use notation μ and μ_i defined in (27). From (59), (60) we obtain $w \leq \mu(\Gamma(w), v)$. Then by Lemma 3.2 there exists $\phi \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \|x_{[0,t]}\|_\infty &\leq \phi(\|\mu(\sigma(|x(0)|), \hat{\gamma}(\|u\|_\infty))\|) \\ &\leq \phi(\|\sigma(|x(0)|) + \hat{\gamma}(\|u\|_\infty)\|) \\ &\leq \phi(2\|\sigma(|x(0)|)\|) + \phi(2\|\hat{\gamma}(\|u\|_\infty)\|) \end{aligned} \tag{61}$$

for all $t > 0$. Hence for every initial condition and essentially bounded input u the solution of the system (1) exists for all $t \geq 0$ and is uniformly bounded, since the right-hand side of (61) does not depend on t. The estimate for GS is then given by (61). ■

Theorem 4.2: Assume that each subsystem of (6) has the AG property and that solutions of system (1) exist for all positive times and are uniformly bounded. Let a gain matrix Γ be given by $\Gamma = (\overline{\gamma}_{ij})_{n\times n}$. If there exists a D as in (19) such that $\Gamma \circ D(s) \not\geq s$ for all $s \neq 0, s \geq 0$, then system (1) satisfies the AG property.

Remark 4.3: The existence of solutions for all times is essential, otherwise the assertion is not true. See Example 14 in [4].

Proof: Let τ be an arbitrary initial time. From the definition of the AG property we have for $i \in I_\Sigma$

$$\limsup_{t\to\infty} |x_i(t)| \leq \sum_{j=1}^{n} \overline{\gamma}_{ij}(\|x_{j[\tau,\infty]}\|_\infty) + \overline{\gamma}_i(\|u\|_\infty) \tag{62}$$

and for $i \in I_{\max}$

$$\limsup_{t\to\infty} |x_i(t)| \leq \max\{\max_j\{\overline{\gamma}_{ij}(\|x_{j[\tau,\infty]}\|_\infty)\}, \overline{\gamma}_i(\|u\|_\infty)\}. \tag{63}$$

Since all solutions of (6) are bounded we obtain by [4, Lemma 7] that

$$\limsup_{t\to\infty} |x_i(t)| = \limsup_{\tau\to\infty}(\|x_{i[\tau,\infty]}\|_\infty) =: l_i(x_i), i = 1, \dots, n.$$

By this property from (62), (63) and [18, Lemma II.1] it follows that

$$l_i(x_i) \leq \sum_{j=1}^{n} \gamma_{ij}(l_j(x_j)) + \overline{\gamma}_i(\|u\|_\infty)$$

for $i \in I_\Sigma$ and

$$l_i(x_i) \leq \max\{\max_j\{\gamma_{ij}(l_j(x_j))\}, \overline{\gamma}_i(\|u\|_\infty)\}$$

for $i \in I_{\max}$. Using Lemma 3.2 we conclude

$$\limsup_{t\to\infty} \|x(t)\| \leq \phi(\|u\|_\infty) \tag{64}$$

for some ϕ of class $\mathcal{K}$, which is the desired AG property. ■

Theorem 4.4: Assume that each subsystem of (6) is ISS and let Γ be defined by (13). If there exists a D as in (19) such that $\Gamma \circ D(s) \not\geq s$ for all $s \neq 0, s \geq 0$, then system (1) is ISS.

Proof: Since each subsystem is ISS it follows in particular that it is GS with gains $\hat{\gamma}_{ij} \leq \gamma_{ij}$. By Theorem 4.1 the whole interconnection (1) is then GS. This implies that solutions of (1) exists for all times.

Another consequence of ISS property of each subsystem is that each of them has the AG property with gains $\overline{\gamma}_{ij} \leq \gamma_{ij}$. Applying Theorem 4.2 the whole system (1) has the AG property.

This implies that (1) is ISS by Theorem 1 in [18]. ■

Remark 4.5: Note that applying Theorem 1 in [18] we lose information about the gains. As we will see in the second main result in Section V gains can be constructed in the framework of Lyapunov theory.

Remark 4.6: A more general formulation of ISS conditions for interconnected systems can be given in terms of so-called *monotone aggregation functions* (MAFs, introduced in [16], [7]). In this general setting small gain conditions also involve a scaling operator D. Since our construction relies on Lemma 3.2 a generalization of the results in this paper could be obtained if sums are replaced by general MAFs and maximization is retained. We expect that the assertion of the Theorem 4.4 remains valid in the more general case, at least if the MAFs are subadditive.

The following section gives a Lyapunov type counterpart of the small gain theorem obtained in this section and shows an explicit construction of an ISS Lyapunov function for interconnections of ISS systems.

V. Construction of ISS Lyapunov functions

Again we consider an interconnection of n subsystems in form of (6) where each subsystem is assumed to be ISS and hence there is a smooth ISS Lyapunov function for each subsystem. We will impose a small gain condition on the Lyapunov gains to prove the ISS property of the whole system (1) and we will look for an explicit construction of an ISS Lyapunov function for it. For our purpose it is sufficient to work with not necessarily smooth Lyapunov functions defined as follows.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, where $\alpha(r) = 0$ if and only if $r = 0$, is called positive definite.

A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *proper and positive definite* if there are $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)\,, \quad \forall x \in \mathbb{R}^n.$$

Definition 5.1: A continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an *ISS Lyapunov function for the system* (1) if

1) it is proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^n \backslash \{0\}$

2) there exists $\gamma \in \mathcal{K}$, and a positive definite function α such that in all points of differentiability of V we have

$$V(x) \geq \gamma(\|u\|) \Rightarrow \nabla V(x) f(x,u) \leq -\alpha(\|x\|). \tag{65}$$

Note that we do not require an ISS Lyapunov function to be smooth. However as a locally Lipschitz continuous function it is differentiable almost everywhere.

Remark 5.2: In Theorem 2.3 in [7] it was proved that the system (1) is ISS if and only if it admits an (not necessarily smooth) ISS Lyapunov function.

ISS Lyapunov function for subsystems can be defined in the following way.

Definition 5.3: A continuous function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called an *ISS Lyapunov function for the subsystem* i in (6) if

1) it is proper and positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_i} \backslash \{0\}$

2) there exist $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$, $i \neq j$, $\gamma_i \in \mathcal{K}$ and a positive definite function α_i such that in all points of differentiability of V_i we have

for $i \in I_\Sigma$

$$V_i(x_i) \geq \gamma_{i1}(V_1(x_1)) + \ldots + \gamma_{in}(V_n(x_n)) + \gamma_i(\|u\|) \Rightarrow$$
$$\nabla V_i(x_i) f_i(x,u) \leq -\alpha_i(\|x_i\|) \tag{66}$$

and for $i \in I_{\max}$

$$V_i(x_i) \geq \max\{\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{in}(V_n(x_n)), \gamma_i(\|u\|)\} \Rightarrow$$

$$\nabla V_i(x_i) f_i(x,u) \leq -\alpha_i(\|x_i\|). \tag{67}$$

Let the matrix $\overline{\Gamma}$ be obtained from matrix Γ by adding external gains γ_i as the last column and let the map $\overline{\Gamma}: \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n$ be defined by:

$$\overline{\Gamma}(s,r) := \{\overline{\Gamma}_1(s,r), \dots, \overline{\Gamma}_n(s,r)\} \tag{68}$$

for $s \in \mathbb{R}_+^n$ and $r \in \mathbb{R}_+$, where $\overline{\Gamma}_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is given by $\overline{\Gamma}_i(s,r) := \gamma_{i1}(s_1) + \cdots + \gamma_{in}(s_n) + \gamma_i(r)$ for $i \in I_\Sigma$ and by $\overline{\Gamma}_i(s,r) := \max\{\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), \gamma_i(r)\}$ for $i \in I_\Sigma$.

Before we proceed to the main result of this section let us recall a related result from [7] adapted to our situation:

Theorem 5.4: Consider the interconnection given by (6) where each subsystem i has an ISS Lyapunov function V_i with the corresponding Lyapunov gains γ_{ij}, γ_i, $i,j = 1,\dots,n$ as in (66) and (67). Let $\overline{\Gamma}$ be defined as in (68). Assume that there is an Ω-path σ with respect to Γ and a function $\phi \in \mathcal{K}_\infty$ such that

$$\overline{\Gamma}(\sigma(r), \phi(r)) < \sigma(r), \quad \forall r > 0. \tag{69}$$

Then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)).$$

We note that this theorem is a special case of [7, Theorem 5.3] that was stated for a more general $\overline{\Gamma}$ than here. Moreover it was shown that an Ω-path needed for the above construction always exists if Γ is irreducible and $\Gamma \not\geq \mathrm{id}$ in $\mathbb{R}_+^n$. The pure cases $I_\Sigma = I$ and $I_{\max} = I$ are already treated in [7], where the existence of ϕ that makes Theorem 5.4 applicable was shown under the condition $D \circ \Gamma \not\geq \mathrm{id}$ for the case $I_\Sigma = I$ and $\Gamma \not\geq \mathrm{id}$ for the case $I_{\max} = I$.

The next result gives a counterpart of [7, Corollaries 5.5 and 5.6] specified for the situation where both I_Σ and $I_{\max}$ can be nonempty.

Theorem 5.5: Assume that each subsystem of (6) has an ISS Lyapunov function V_i and the corresponding gain matrix is given by (68). If Γ is irreducible and if there exists D_α as in (19) such that $\Gamma \circ D_\alpha(s) \not\geq s$ for all $s \neq 0, s \geq 0$ is satisfied, then the system (1) is ISS and an ISS Lyapunov function is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)), \tag{70}$$

where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to $D \circ \Gamma$.

Proof: From the structure of D_α it follows that

$$\begin{aligned} \sigma_i &> (\mathrm{id} + \alpha) \circ \Gamma_i(\sigma), \qquad && i \in I_\Sigma, \\ \sigma_i &> \Gamma_i(\sigma), && i \in I_{\max}. \end{aligned}$$

The irreducibility of Γ ensures that $\Gamma(\sigma)$ is unbounded in all components. Let $\phi \in \mathcal{K}_\infty$ be such that for all $r \geq 0$ the inequality $\alpha(\Gamma_i(\sigma(r))) \geq \max_{i=1,\dots,n} \gamma_i(\phi(r))$ holds for $i \in I_\Sigma$ and $\Gamma_i(\sigma(r)) \geq \max_{i=1,\dots,n} \gamma_i(\phi(r))$ for $i \in I_{\max}$. Note that such a ϕ always exists and can be chosen as follows. For any $\gamma_i \in \mathcal{K}$ we choose $\tilde{\gamma}_i \in \mathcal{K}_\infty$ such that $\tilde{\gamma}_i \geq \gamma_i$. Then ϕ can be taken as $\phi(r) := \frac{1}{2} \min\{\min_{i \in I_\Sigma, j \in I} \tilde{\gamma}_j^{-1}(\alpha(\Gamma_i(\sigma(r)))), \min_{i \in I_{\max}, j \in I} \tilde{\gamma}_j^{-1}(\Gamma_i(\sigma(r)))\}$. Note that ϕ is a $\mathcal{K}_\infty$ function since the minimum over $\mathcal{K}_\infty$ functions is again of class $\mathcal{K}_\infty$. Then we have for all $r > 0, i \in I_\Sigma$ that

$$\sigma_i(r) > D_i \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) + \alpha(\Gamma_i(\sigma(r))) \geq \Gamma_i(\sigma(r)) + \gamma_i(\phi(r)) = \overline{\Gamma}_i(\sigma(r), \phi(r))$$

and for all $r > 0, i \in I_{\max}$

$$\sigma_i(r) > D_i \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) \geq \max\{\Gamma_i(\sigma(r)), \gamma_i(\phi(r))\} = \overline{\Gamma}_i(\sigma(r), \phi(r)).$$

Thus $\sigma(r) > \overline{\Gamma}(\sigma(r), \phi(r))$ and the assertion follows from Theorem 5.4. ■

The irreducibility assumption on Γ means in particular that the graph representing the interconnection structure of the whole system is strongly connected. To treat the reducible case we consider an approach using the irreducible components of Γ. If a matrix is reducible it can be transformed to an upper block triangular form via a permutation of the indices, [2].

The following result is based on [7, Corollaries 6.3 and 6.4].

Theorem 5.6: Assume that each subsystem of (6) has an ISS Lyapunov function V_i and the corresponding gain matrix is given by (68). If there exists D_α as in (19) such that $\Gamma \circ D_\alpha(s) \not\geq s$ for all $s \neq 0, s \geq 0$ is satisfied, then the system (1) is ISS, moreover there exists an Ω-path σ and $\phi \in \mathcal{K}_\infty$ satisfying $\overline{\Gamma}(\sigma(r), \phi(r)) < \sigma(r), \forall\ r > 0$ and an ISS Lyapunov function for the whole system (1) is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)).$$

Proof: After a renumbering of subsystems we can assume that $\overline{\Gamma}$ is of the form (45). Let D be the corresponding diagonal operator that contains id or $\mathrm{id} + \alpha$ on the diagonal depending on the new enumeration of the subsystems. Let the state x be partitioned into $z_i \in \mathbb{R}^{d_i}$ where d_i is the size of the ith diagonal block Υ_{ii}, $i = 1, \dots, d$. And consider the subsystems Σ_j of the whole system (1) with these states

$$z_j := (x_{q_j+1}^T, x_{q_j+2}^T, \dots, x_{q_{j+1}}^T)^T,$$

where $q_j = \sum_{l=1}^{j-1} d_l$, with the convention that $q_1 = 0$. So the subsystems Σ_j correspond exactly to the strongly connected components of the interconnection graph. Note that each $\Upsilon_{jj}, j = 1, \dots, d$ satisfies a small gain condition of the form $\Upsilon_{jj} \circ D_j \not\geq \mathrm{id}$ where $D_j : \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$ is the corresponding part of D_α.

For each Σ_j with the gain operator $\Upsilon_{jj}, j = 1, \dots, d$ and external inputs $z_{j+1}, \dots, z_d, u$ Theorem 5.5 implies that there is an ISS Lyapunov function $W_j = \max_{i=q_j+1,\dots,q_{j+1}} \widehat{\sigma}_i^{-1}(V_i(x_i))$ for Σ_j, where $(\widehat{\sigma}_{q_j+1}, \dots, \widehat{\sigma}_{q_{j+1}})^T$ is an arbitrary Ω-path with respect to $\Upsilon_{jj} \circ D_j$. We will show by induction over the number of blocks that an ISS Lyapunov function for the whole system (1) of the form $V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i))$ exists, for an appropriate σ.

For one irreducible bock there is nothing to show. Assume that for the system corresponding to the first $k-1$ blocks an ISS Lyapunov function exists and is given by $\widetilde{V}_{k-1} = \max_{i=1,\dots,q_k} \sigma_i^{-1}(V_i(x_i))$. Consider now the first k blocks with state $(\widetilde{z}_{k-1}, z_k)$, where $\widetilde{z}_{k-1} := (z_1, \dots, z_{k-1})^T$. Then we have the implication

$$\begin{aligned} \widetilde{V}_{k-1}(\widetilde{z}_{k-1}) \;&\geq\; \widetilde{\gamma}_{k-1,k}(W_k(z_k)) + \widetilde{\gamma}_{k-1,u}(\|u\|) \quad \Rightarrow \\ \nabla \widetilde{V}_{k-1}(\widetilde{z}_{k-1}) \widetilde{f}_{k-1}(\widetilde{z}_{k-1}, z_k, u) \;&\leq\; -\widetilde{\alpha}_{k-1}(\|\widetilde{z}_{k-1}\|)\,, \end{aligned}$$

where $\widetilde{\gamma}_{k-1,k}, \widetilde{\gamma}_{k-1,u}$ are the corresponding gains, $\widetilde{f}_{k-1}$, $\widetilde{\alpha}_{k-1}$ are the right hand side and dissipation rate of the first $k-1$ blocks.

The gain matrix corresponding to the block k then has the form

$$\overline{\Gamma}_k = \begin{pmatrix} 0 & \widetilde{\gamma}_{k-1,k} & \widetilde{\gamma}_{k-1,u} \\ 0 & 0 & \gamma_{k,u} \end{pmatrix}.$$

For $\overline{\Gamma}_k$ by [7, Lemma 6.1] there exist an Ω-path $\widetilde{\sigma}^k = (\widetilde{\sigma}_1^k, \widetilde{\sigma}_2^k)^T \in \mathcal{K}_\infty^2$ and $\phi \in \mathcal{K}_\infty$ such that $\overline{\Gamma}_k(\widetilde{\sigma}^k, \phi) < \widetilde{\sigma}^k$ holds. Applying Theorem 5.4 an ISS Lyapunov function for the whole system exists and is given by

$$\widetilde{V}_k \;=\; \max\{(\widetilde{\sigma}_1^k)^{-1}(\widetilde{V}_{k-1}), (\widetilde{\sigma}_2^k)^{-1}(W_k)\}$$

A simple inductive argument shows that the final Lyapunov function is of the form $V(x) = \max_{k=1,\dots,d}(\sigma_k^{-1}(W_k(z_k))$, where for $k = 1, \dots, d-1$ we have (setting $\sigma_2^0 = \mathrm{id}$)

$$\sigma_k^{-1} = \left(\tilde{\sigma}_1^{d-1}\right)^{-1} \circ \cdots \circ \left(\tilde{\sigma}_1^{k}\right)^{-1} \circ \left(\tilde{\sigma}_2^{k-1}\right)^{-1}$$

and $\sigma_d = \tilde{\sigma}_2^{d-1}$. This completes the proof. ■

VI. Conclusion

We have considered large-scale interconnections of ISS systems. The mutual influence of the subsystems on each other may either be expressed in terms of summation or maximization of the corresponding gains. We have shown that such a formulation may always be reduced to a pure maximization formulation, however the presented procedure requires the knowledge of an Ω-path of the gain matrix, which amounts to having solved the problem. Also an equivalent small gain condition has been derived which is adapted to the particular problem. A simple example shows the effectiveness and advantage of this condition in comparison to known results. Furthermore, the Lyapunov version of the small gain theorem provides an explicit construction of ISS Lyapunov function for the interconnection.

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