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Ivan Malloci, Jamal Daafouz, *Member, IEEE,* Claude Iung, *Member, IEEE,* and Rémi Bonidal

***Abstract*— In this article, an alternative LMI solution for the linear quadratic optimal control design is proposed for the discrete-time systems in the singular perturbation form. This approach is particularly adapted for the case of high dimension systems. Moreover, it can be easily extended to the uncertain systems, under some assumptions. An example of practical application to the robust steering control of hot strip mill is presented.**

***Index Terms*— Singular perturbation, LQ control design, LMI, Robust control.**

I. INTRODUCTION

Many industrial systems involve dynamics operating on two or more time scales. In this case, standard control technics can lead to ill-conditioning controllers. In order to avoid such as numerical problems, singular perturbation methods can be used, which consist in decomposing the system into several subsystems, one for each time scale. Then, a different controller is designed for each subsystem. Singular perturbation technics also allow to reduce the controller order. This propriety can be very useful when the system order is too high to implement an effective controller.

In the optimal control framework, first contributions to the singular perturbation theory are given in the continuous-time by [4], [5]. Some extensions to the discrete-time case can be found in [6], [9]. A survey of the most popular optimal control strategies for the singularly perturbed systems is given in [8].

In [10], an alternative LMI solution [1] for the LQ optimal control design is proposed for the continuous-time systems. This approach has been extended to the singularly perturbed systems in [3]. In spite of the authors' knowledge, it does not exist a similar development for the discrete-time case.

In this article, a LMI solution for the LQ control design for the discrete-time singularly perturbed systems is proposed. The advantage associated to the LMI formulation is the existence of several solvers that provide solutions also in the case of high dimension problems. Moreover, we show that the reduced controller can directly be extended to uncertain systems. Experimental results concerning the robust steering control of hot strip mill are given.

The article is organised as follows. In section II, the LQ problem for the linear discrete-time singularly perturbed systems is discussed. In section III, an alternative LMI solution is presented. In section IV, results are extended to uncertain systems. In section V, an example of industrial application, the robust steering control of hot strip mill, is presented.

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I. Malloci, J. Daafouz and C. Iung are with the Centre de Recherche en Automatique de Nancy, UMR 7039 CNRS - Nancy Université, ENSEM, 2, Avenue de la forêt de Haye 54516 Vandoeuvre-lès-Nancy, France. Corresponding author: Jamal.Daafouz@ensem.inpl-nancy.fr.

R. Bonidal is with ArcelorMittal Maizières, R&D Industrial Operations, BP 30320, F-57283 Maizières-lès-Metz Cedex, France.

II. PROBLEM FORMULATION

Consider the linear discrete-time singularly perturbed system in the form

$$\begin{cases} x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k) \\ x_2(k+1) = \varepsilon A_{21}x_1(k) + \\ \qquad\qquad (I_{n_2} + \varepsilon A_{22})x_2(k) + \varepsilon B_2u(k) \\ y(k) = C_1x_1(k) + C_2x_2(k) \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a scalar parameter $\ll 1$, $x_1 \in \mathbb{R}^{n_1}$ is the state vector corresponding to the fast dynamics, $x_2 \in \mathbb{R}^{n_2}$ is the state vector corresponding to the slow dynamics, $u \in \mathbb{R}^r$ is the control signal, $y \in \mathbb{R}^m$ is the output signal and I_n denotes an identity matrix $\in \mathbb{R}^{n\times n}$. The model (1) represents the sampling of a singularly perturbed continuous-time system. Then, results can be extended to the continuous-time systems controlled by digital devices. Let define

$$\begin{aligned} A(\varepsilon) &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon A_{21} & (I_{n_2} + \varepsilon A_{22}) \end{bmatrix}, \\ B(\varepsilon) &= \begin{bmatrix} B_1 \\ \varepsilon B_2 \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \end{aligned} \tag{2}$$

The slow subsystem is defined as:

$$\begin{cases} x_s(k+1) = (I_{n_2} + \varepsilon A_s)x_s(k) + \varepsilon B_s u_s(k) \\ y_s(k) = C_s x_s(k) + D_s u_s(k) \end{cases} \tag{3}$$

with

$$\begin{aligned} A_s &= A_{22} + A_{21}(I_{n_1} - A_{11})^{-1}A_{12} \\ B_s &= B_2 + A_{21}(I_{n_1} - A_{11})^{-1}B_1 \\ C_s &= C_2 + C_1(I_{n_1} - A_{11})^{-1}A_{12} \\ D_s &= C_1(I_{n_1} - A_{11})^{-1}B_1 \end{aligned} \tag{4}$$

and $(I_{n_1} - A_{11})$ invertible; the fast subsystem is defined as:

$$\begin{cases} x_f(k+1) = A_{11}x_f(k) + B_1u_f(k) \\ y_f(k) = C_1x_f(k) \end{cases} . \tag{5}$$

Let the LQ performance index

$$J(\varepsilon) = \frac{1}{2}\sum_{k=0}^{\infty}(y(k)'y(k) + u(k)'Ru(k))$$

where $R = R' \succ 0 \in \mathbb{R}^{r\times r}$ is a weighting matrix. Given the optimisation problem

$$(\mathcal{P})\begin{cases} \underset{u}{min}\ J(\varepsilon) \\ under\ (1), \end{cases} \quad (6)$$

if the pair $(A(\varepsilon), B(\varepsilon))$ is stabilisable and the pair $(C, A(\varepsilon))$ is detectable, there exists a stabilising solution $P(\varepsilon) \succeq 0$ for the algebraic Riccati equation:

$$A(\varepsilon)'P(\varepsilon)A(\varepsilon) - A(\varepsilon)'P(\varepsilon)B(\varepsilon)(R + B(\varepsilon)'P(\varepsilon)B(\varepsilon))^{-1}$$
$$B(\varepsilon)'P(\varepsilon)A(\varepsilon) - P(\varepsilon) + C'C = 0.$$

The optimal solution is given by:

$$u(k) = K(\varepsilon)\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad (7)$$

where $K(\varepsilon) = -(R + B(\varepsilon)'P(\varepsilon)B(\varepsilon))^{-1}B(\varepsilon)'P(\varepsilon)A(\varepsilon)$.

When $\varepsilon \to 0$, the system has a two-time scale dynamics. In this case, standard technics lead to ill-conditioning controllers. In order to avoid such as numerical problems, the singular perturbation method may be used : the system is decomposed into two subsystems and a different controller is designed for each of them. Hence, also the optimisation problem (6) must be decomposed into two subproblems :

Slow subproblem: For $\varepsilon = 0$, consider

$$(\mathcal{P}_1)\begin{cases} \underset{u_s}{min}\ J_s^0 = \frac{1}{2}\sum_{k=0}^{\infty}(y_s(k)'y_s(k) + u_s(k)'R_s u_s(k)) \\ under\ (3) \end{cases}$$

with $R_s = R_s' = R + D_s'D_s \succ 0$. If the pair (A_s, B_s) is stabilisable and the pair (C_s, A_s) is detectable, there exists a stabilising solution $P_s \succeq 0$ for the algebraic Riccati equation:

$$(A_s - B_sR_s^{-1}D_s'C_s)'P_s + P_s(A_s - B_sR_s^{-1}D_s'C_s) -$$
$$P_sB_sR_s^{-1}B_s'P_s + C_s'(I_{n_1} - D_sR_s^{-1}D_s')C_s = 0.$$

The optimal solution is given by:

$$u_s(k) = K_s x_s(k) \quad (8)$$

where $K_s = -R_s^{-1}(B_s'P_s + D_s'C_s)$. The state-feedback law (8) guarantees the condition $Re\{\xi(A_s + B_sK_s)\} < 0$, where $\xi(X)$ denotes the spectrum of X. This implies asymptotic stability of the closed-loop system (3) for sufficiently small ε.

Fast subproblem: Consider

$$(\mathcal{P}_2)\begin{cases} \underset{u_f}{min}\ J_f^0 = \frac{1}{2}\sum_{k=0}^{\infty}(y_f(k)'y_f(k) + u_f(k)'Ru_f(k)) \\ under\ \ (5) \end{cases}$$

with $R = R' \succ 0$. If the pair (A_{11}, B_1) is stabilisable and the pair (C_1, A_{11}) is detectable, there exists a stabilising solution $P_f \succeq 0$ for the algebraic Riccati equation:

$$A_{11}'P_fA_{11} - A_{11}'P_fB_1(R + B_1'P_fB_1)^{-1}$$
$$B_1'P_fA_{11} - P_f + C_1'C_1 = 0.$$

The optimal solution is given by:

$$u_f(k) = K_f x_f(k) \quad (9)$$

where $K_f = -(R + B_1'P_fB_1)^{-1}B_1'P_fA_{11}$.

Composite control: From (3) and (5), the composite control law is given by :

$$u_c(k) = u_s(k) + u_f(k) = K^0\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad (10)$$

with $K^0 = \begin{bmatrix} K_f & (K_s - K_f(I_{n_1} - A_{11})^{-1}(A_{12} + B_1K_s)) \end{bmatrix}$.

The control laws (8) and (9) are designed using independent gains K_s and K_f. When $\varepsilon \to 0$, the composite control law (10) is close to the optimal solution (7). An index of performance degradation is given in [9].

III. LMI SOLUTION

The LQ problem (6) may be formulated in a convex form. In this case, the solution can be found solving a LMI problem [10]. This approach has been extended for the continuous-time singularly perturbed systems in [3]. In this section, a similar development is proposed for the discrete-time singularly perturbed systems. Let define the following sets:

$$\mathcal{W} = \left\{ W(\varepsilon) = W(\varepsilon)' = \begin{bmatrix} W_1(\varepsilon) & W_2(\varepsilon) \\ W_2(\varepsilon)' & W_3(\varepsilon) \end{bmatrix} \succ 0 \right\},$$
$$\mathcal{W}_\varepsilon = \left\{ \bar{W}(\varepsilon) = \begin{bmatrix} W(\varepsilon) & S(\varepsilon)' \\ S(\varepsilon) & W_4(\varepsilon) \end{bmatrix} \succeq 0,\ W(\varepsilon) \in \mathcal{W} \right\}, \quad (11)$$

with

$$S(\varepsilon) = \begin{bmatrix} S_1(\varepsilon) & S_2(\varepsilon) \end{bmatrix}, \quad (12)$$

and

$$\mathcal{Q}_\varepsilon = \left\{ \begin{array}{l} \bar{W}(\varepsilon) \in \mathcal{W}_\varepsilon :\ A(\varepsilon)W(\varepsilon)A(\varepsilon)' + \\ A(\varepsilon)S(\varepsilon)'B(\varepsilon)' + B(\varepsilon)S(\varepsilon)A(\varepsilon)' + \\ B(\varepsilon)S(\varepsilon)W(\varepsilon)^{-1}S(\varepsilon)'B(\varepsilon)' - W(\varepsilon) \prec 0 \end{array} \right\} \quad (13)$$

and denote

$$\lim_{\varepsilon\to 0} W(\varepsilon) = \begin{bmatrix} W_1^0 & W_2^0 \\ W_2^{0'} & W_3^0 \end{bmatrix} = W^0,$$
$$\lim_{\varepsilon\to 0} \bar{W}(\varepsilon) = \begin{bmatrix} W^0 & S^{0'} \\ S^0 & W_4^0 \end{bmatrix} = \bar{W}^0,$$
$$\lim_{\varepsilon\to 0} S(\varepsilon) = \begin{bmatrix} S_1^0 & S_2^0 \end{bmatrix} = S^0.$$

An alternative LMI solution to the problem (6) is obtained solving the following optimisation problem [10]:

$$(\mathcal{P}_\varepsilon):\ \underset{\bar{W}(\varepsilon)\in\mathcal{Q}_\varepsilon}{min}\ J(\varepsilon) = Tr\left(\begin{bmatrix} C'C & 0 \\ 0 & R \end{bmatrix}\bar{W}(\varepsilon)\right).$$

Furthermore, if $\bar{W}^*(\varepsilon)$ is optimal, it can be written as:

$$\bar{W}^*(\varepsilon) = \begin{bmatrix} W^*(\varepsilon) & S^*(\varepsilon)' \\ S^*(\varepsilon) & W_4^*(\varepsilon) \end{bmatrix} =$$
$$\begin{bmatrix} W^*(\varepsilon) & W^*(\varepsilon)K(\varepsilon)' \\ K(\varepsilon)W^*(\varepsilon) & K(\varepsilon)W^*(\varepsilon)K(\varepsilon)' \end{bmatrix}$$

where $K(\varepsilon) = S^*(\varepsilon)W^*(\varepsilon)^{-1}$ is the optimal gain and

$$\lim_{\varepsilon\to 0} K(\varepsilon) = K^0 = S^{0*}W^{0*^{-1}}. \quad (14)$$

Hence, $\mathcal{P}_\varepsilon$ may be reformulated as:

$$\min_{W(\varepsilon)\succ 0, S(\varepsilon)} Tr\left(\begin{bmatrix} C'C & 0\\ 0 & R\end{bmatrix}\begin{bmatrix} W(\varepsilon) & S(\varepsilon)'\\ S(\varepsilon) & S(\varepsilon)W(\varepsilon)^{-1}S(\varepsilon)'\end{bmatrix}\right). \tag{15}$$

When ε is small, numerical difficulties to minimise the criterion $J(\varepsilon)$ of $\mathcal{P}_\varepsilon$ arise. This problem is due to the ill-conditioning of the constraint (13). It can be avoided decomposing the original problem $\mathcal{P}_\varepsilon$ into two well-behaved subproblems, as in the LQ case. The criterion can be decomposed as follows:

$$J(\varepsilon) = Tr\left(\begin{matrix} \begin{bmatrix} C_1'\\ C_2'\end{bmatrix}\begin{bmatrix} C_1 & C_2\end{bmatrix}\begin{bmatrix} W_1(\varepsilon) & W_2(\varepsilon)\\ W_2(\varepsilon)' & W_3(\varepsilon)\end{bmatrix} + \\ R\begin{bmatrix} S_1(\varepsilon) & S_2(\varepsilon)\end{bmatrix}\times \\ \begin{bmatrix} W_1(\varepsilon) & W_2(\varepsilon)\\ W_2(\varepsilon)' & W_3(\varepsilon)\end{bmatrix}^{-1}\begin{bmatrix} S_1(\varepsilon)'\\ S_2(\varepsilon)'\end{bmatrix}\end{matrix}\right).$$

Given $\Phi(\varepsilon) = W_1(\varepsilon) - W_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)'$, we obtain:

$$\begin{aligned} J(\varepsilon) =& J_s(\varepsilon) + J_f(\varepsilon) = \\ & Tr\left(\begin{matrix}\begin{bmatrix} C_1 & C_2 + C_1W_2(\varepsilon)W_3(\varepsilon)^{-1}\end{bmatrix}\\ \begin{bmatrix}\Phi(\varepsilon) & 0\\ 0 & W_3(\varepsilon)\end{bmatrix}\begin{bmatrix}\star\end{bmatrix}'\end{matrix}\right) + \\ & Tr\left(\begin{matrix} R\begin{bmatrix} S_1(\varepsilon) - S_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)' & S_2(\varepsilon)\end{bmatrix}\times\\ \begin{bmatrix}\Phi(\varepsilon)^{-1} & 0\\ 0 & W_3(\varepsilon)^{-1}\end{bmatrix}\begin{bmatrix}\star\end{bmatrix}'\end{matrix}\right)\end{aligned}$$

with

$$\begin{aligned} J_s(\varepsilon) =& Tr((C_2 + C_1W_2(\varepsilon)'W_3(\varepsilon)^{-1})W_3(\varepsilon)\\ & (C_2 + C_1W_2(\varepsilon)'W_3(\varepsilon)^{-1})' + RS_2(\varepsilon)W_3(\varepsilon)^{-1}S_2(\varepsilon)'),\end{aligned}$$

$$\begin{aligned} J_f(\varepsilon) =& Tr((C_1(W_1(\varepsilon) - W_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)')C_1' + \\ & R(S_1(\varepsilon) - S_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)')\\ & (W_1(\varepsilon) - W_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)')^{-1}\\ & (S_1(\varepsilon) - S_2(\varepsilon)W_3(\varepsilon)^{-1}W_2(\varepsilon)')').\end{aligned}$$

Let define

$$W_2^0 = (I_{n_1} - A_{11})^{-1}(A_{12}W_3^0 + B_1S_2^0), \tag{16}$$

$$W_s = W_3^0,\ S_s = S_2^0, \tag{17}$$

$$\begin{aligned} W_f &= W_1^0 - W_2^0(W_3^0)^{-1}W_2^{0'},\\ S_f &= S_1^0 - S_2^0(W_3^0)^{-1}W_2^{0'}.\end{aligned} \tag{18}$$

Then:

$$\begin{aligned} \lim_{\varepsilon\to 0} C_2 + C_1W_2(\varepsilon)W_3(\varepsilon)^{-1} &= C_2 + C_1(I_{n_1} - A_{11})^{-1}A_{12} + \\ & C_1(I_{n_1} - A_{11})^{-1}B_1S_sW_s^{-1} = C_s + D_sS_sW_s^{-1},\end{aligned}$$

$$\begin{aligned} \lim_{\varepsilon\to 0} J_s(\varepsilon) = J_s^0 =& Tr(C_sW_sC_s' + C_sS_s'D_s' + D_sS_sC_s'\\ & D_sS_sW_s^{-1}S_s'D_s' + RS_sW_s^{-1}S_s'),\end{aligned}$$

$$\lim_{\varepsilon\to 0} J_f(\varepsilon) = J_f^0 = Tr(C_1W_fC_1' + RS_fW_f^{-1}S_f').$$

The last two equations can be written in the form:

$$J_s^0 = Tr\left(\begin{bmatrix} C_s'C_s & C_s'D_s\\ D_s'C_s & D_s'D_s + R\end{bmatrix}\begin{bmatrix} W_s & S_s\\ S_s' & S_sW_s^{-1}S_s'\end{bmatrix}\right), \tag{19}$$

$$J_f^0 = Tr\left(\begin{bmatrix} C_1'C_1 & 0\\ 0 & R\end{bmatrix}\begin{bmatrix} W_f & S_f\\ S_f' & S_fW_f^{-1}S_f'\end{bmatrix}\right), \tag{20}$$

with

$$J^0 = J_s^0 + J_f^0. \tag{21}$$

In terms of variables, J_s^0 depends on W_s and S_s whereas J_f^0 depends on W_f and S_f. Hence, two independent optimisation subproblems can be defined:

Slow subproblem:

$$(\mathcal{P}_s): \min_{\bar{W}_s\in\mathcal{Q}_s} Tr\left(\begin{bmatrix} C_s'C_s & C_s'D_s\\ D_s'C_s & D_s'D_s + R\end{bmatrix}\bar{W}_s\right),$$

with

$$\mathcal{W}_s = \left\{\bar{W}_s = \begin{bmatrix} W_s & S_s'\\ S_s & V_s\end{bmatrix}\succ 0\right\}$$

and

$$\mathcal{Q}_s = \left\{\bar{W}_s\in\mathcal{W}_s : A_sW_s + W_sA_s' + B_sS_s + S_s'B_s' \prec 0\right\}.$$

Fast subproblem:

$$(\mathcal{P}_f): \min_{\bar{W}_f\in\mathcal{Q}_f} Tr\left(\begin{bmatrix} C_1'C_1 & 0\\ 0 & R\end{bmatrix}\bar{W}_f\right),$$

with

$$\mathcal{W}_f = \left\{\bar{W}_f = \begin{bmatrix} W_f & S_f'\\ S_f & V_f\end{bmatrix}\succ 0\right\}$$

and

$$\mathcal{Q}_f = \left\{\begin{matrix}\bar{W}_f\in\mathcal{W}_f : A_fW_fA_f' + A_fS_f'B_f' + \\ B_fS_fA_f' + B_fS_fW_f^{-1}S_f'B_f' - W_f \prec 0\end{matrix}\right\}$$

that, using the Schur complement, becomes

$$\mathcal{Q}_f = \left\{\bar{W}_f\in\mathcal{W}_f : \begin{bmatrix} W_f & (\star)'\\ A_fW_f + B_fS_f & W_f\end{bmatrix}\succ 0\right\}.$$

The next theorem gives a solution for the problem $\mathcal{P}_s$ - $\mathcal{P}_f$.

Theorem 1: Assume that problems $\mathcal{P}_s$ and $\mathcal{P}_f$ admit, respectively, solutions

$$\bar{W}_s = \begin{bmatrix} W_s & S_s'\\ S_s & V_s\end{bmatrix},\ \bar{W}_f = \begin{bmatrix} W_f & S_f'\\ S_f & V_f\end{bmatrix}.$$

Then:

i) The solution $\bar{W}(\varepsilon)$ of the problem $\mathcal{P}_\varepsilon$ satisfies

$$\lim_{\varepsilon\to 0}\bar{W}(\varepsilon) = \begin{bmatrix} W^0 & S^{0'}\\ S^0 & W_4^0\end{bmatrix} = \bar{W}^0,$$

$$\begin{aligned}\lim_{\varepsilon\to 0} J(\varepsilon) = J^0 = J_s^0 + J_f^0 = \\ Tr\left(\begin{bmatrix} C'C & 0\\ 0 & R\end{bmatrix}\begin{bmatrix} W^0 & S^{0'}\\ S^0 & W_4^0\end{bmatrix}\right)\end{aligned}$$

with

$$W^0 = \begin{bmatrix} W_f + W_2^0W_s^{-1}W_2^{0'} & W_2^0\\ W_2^{0'} & W_s\end{bmatrix}, \tag{22}$$

$$S^0 = \begin{bmatrix} S_f + S_sW_s^{-1}W_2^{0'} & S_s\end{bmatrix}, \tag{23}$$

$$W_4^0 = S^0W^{0^{-1}}S^{0'}. \tag{24}$$

ii) There exists $\varepsilon_0 > 0$ such that the problem $\mathcal{P}_\varepsilon$ admits the approximate solution

$$\bar{W}^0 = \begin{bmatrix} W^0 & S^{0'} \\ S^0 & W_4^0 \end{bmatrix}$$

$\forall\, \varepsilon \in (0, \varepsilon_0]$.

iii) For $\varepsilon \in (0, \varepsilon_0]$, the controller gain (14) is given by $K^0 = S^0 W^{0^{-1}}$.

Proof:

i) From (11), denote

$$W(\varepsilon)^{-1} = \begin{bmatrix} \Sigma(\varepsilon) & \Omega(\varepsilon) \\ \Omega(\varepsilon)' & \Upsilon(\varepsilon) \end{bmatrix} \tag{25}$$

with

$$\begin{aligned} \Sigma(\varepsilon) &= (W_1(\varepsilon) - W_2(\varepsilon) W_3(\varepsilon)^{-1} W_2(\varepsilon)')^{-1} \\ \Omega(\varepsilon) &= -\Sigma(\varepsilon) W_2(\varepsilon) W_3(\varepsilon)^{-1} \\ \Upsilon(\varepsilon) &= W_3(\varepsilon)^{-1} + \\ &\quad W_3(\varepsilon)^{-1} W_2(\varepsilon)' \Sigma(\varepsilon) W_2(\varepsilon) W_3(\varepsilon)^{-1}. \end{aligned} \tag{26}$$

Then, substituting (2), (11) and (25) in (13), we obtain:

$$\begin{bmatrix} H_1(\varepsilon) & H_2(\varepsilon) \\ H_2(\varepsilon)' & \varepsilon H_3(\varepsilon) \end{bmatrix} \prec 0 \tag{27}$$

with $H_1(\varepsilon)$, $H_2(\varepsilon)$ and $H_3(\varepsilon)$ defined in equations (28)-(30). When $\varepsilon \to 0$, using (26) we obtain (31)-(33).

Equation (16) verifies (32). Furthermore, substituting (16) and (18) in (31), we obtain (34) and, substituting (4), (16) and (17) in (33), we obtain (35).

Finally, (34) and (35) represent the constraints of the problems $\mathcal{P}_f$ and $\mathcal{P}_s$, respectively. Equations (28)-(35) are defined in the next page.

ii) Replacing in (27) the unknown values of $W_1(\varepsilon)$, $W_2(\varepsilon)$, $W_3(\varepsilon)$, $S_1(\varepsilon)$, $S_2(\varepsilon)$ with W_1^0, W_2^0, W_3^0, S_1^0, S_2^0, we obtain

$$\begin{bmatrix} H_1^0 & \varepsilon G \\ \varepsilon G' & \varepsilon(H_3^0 + \varepsilon L) \end{bmatrix} \prec 0, \tag{36}$$

with H_1^0 and H_3^0 defined in (34) and (35),

$$\begin{aligned} G = &A_{11} W_1^0 A_{21}' + A_{12} W_2^{0'} A_{21}' + A_{11} W_2^0 A_{22}' + \\ &A_{12} W_3^0 A_{22}' + A_{11} S_1^{0'} B_2' + A_{12} S_2^{0'} B_2' + \\ &B_1 S_1^0 A_{21}' + B_1 S_2^0 A_{22}' + B_1 (S_1^0 \Sigma^0 S_1^{0'} + \\ &S_2^0 \Omega^{0'} S_1^{0'} + S_1^0 \Omega^0 S_2^{0'} + S_2^0 \Upsilon^0 S_2^{0'}) B_2' \end{aligned}$$

and

$$\begin{aligned} L = &A_{21} W_1^0 A_{21}' + A_{21} W_2^{0'} A_{22}' + A_{22} W_2^0 A_{21}' + \\ &A_{22} W_3^0 A_{22}' + A_{21} S_1^{0'} B_2' + A_{22} S_2^{0'} B_2' + \\ &B_2 S_2^0 A_{21}' + B_2 S_2^0 A_{22}' + B_2 (S_1^0 \Sigma^0 S_1^{0'} + \\ &S_2^0 \Omega^{0'} S_1^{0'} + S_1^0 \Omega^0 S_2^{0'} + S_2^0 \Upsilon^0 S_2^{0'}) B_2'. \end{aligned}$$

Σ^0, Ω^0 and Υ^0 are obtained replacing $W_1(\varepsilon)$, $W_2(\varepsilon)$, $W_3(\varepsilon)$, $S_1(\varepsilon)$, $S_2(\varepsilon)$ in (26). The conditions $H_1^0 \prec 0$ and $H_3^0 \prec 0$ imply that there exists a scalar $\varepsilon_0 > 0$ such that

$$H_1^0 - \varepsilon G (H_3^0 + \varepsilon L)^{-1} G' \prec 0,$$

is verified $\forall\, \varepsilon \in (0, \varepsilon_0]$. Then, using the Schur complement, also (36) is verified $\forall\, \varepsilon \in (0, \varepsilon_0]$.

iii) Consider

$$u_s(k) = K_s x_s(k) = S_s W_s^{-1} x_s(k)$$

and

$$u_f(k) = K_f x_f(k) = S_f W_f^{-1} x_f(k).$$

The composite controller is given by

$$u_c(k) = u_s(k) + u_f(k) = K_s x_s(k) + K_f x_f(k).$$

To derive the slow model, we assumed that $x_s(k) = x_2(k)$ when $x_1(k+1) \simeq x_1(k)$, i.e. when the transient behaviour is finished. Moreover, to derive the fast model we assumed that $x_f(k) = x_1(k) - (I_{n_1} - A_{11})^{-1}(A_{12} x_s(k) + B_1 u_s(k)) = x_1(k) - (I_{n_1} - A_{11})^{-1}(A_{12} + B_1 K_s) x_s(k)$ when $x_2(k+1) \simeq x_2(k)$, i.e. during the fast transient). Then, we have

$$\begin{aligned} u_c(k) = &S_f W_f^{-1} x_1(k) + S_s W_s^{-1} x_2(k) - \\ &S_f W_f^{-1} (I_{n_1} - A_{11})^{-1} \times \\ &(A_{12} + B_1 S_s W_s^{-1}) x_2(k) = \\ &K^0 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \end{aligned}$$

which corresponds to (10). In order to prove that $K^0 = S^0 W^{0^{-1}}$, the formula of the inverse of partitioned matrix can be applied to (22). We find

$$W_0^{-1} = \begin{bmatrix} W_f^{-1} & -W_f^{-1} W_2^0 W_s^{-1} \\ -W_s^{-1} W_2^{0'} W_f^{-1} & W_t \end{bmatrix},$$

with $W_t = W_s^{-1} + W_s^{-1} W_2^{0'} W_f^{-1} W_2^0 W_s^{-1}$. Then

$$K^0 = \begin{bmatrix} S_f W_f^{-1} & S_s W_s^{-1} - S_f W_f^{-1} (I_{n_1} - A_{11})^{-1} \times \\ & (A_{12} + B_1 S_s W_s^{-1}) \end{bmatrix}$$

which concludes the proof.

■

Remark 1: Let $K_f = S_f W_f^{-1} = 0$. We obtain the reduced control law

$$u_r(k) = \begin{bmatrix} 0 & K_s \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{37}$$

where $K_s = S_s W_s^{-1}$ is the optimal controller gain of the slow subsystem. If A_{11} is Schur, the closed-loop system is still asymptotically stable $\forall\, \varepsilon \in (0, \varepsilon_0]$. An upper bound of the performance degradation is given in [9].

$$\begin{aligned} H_1(\varepsilon) =&A_{11}W_1(\varepsilon)A_{11}' + A_{12}W_2(\varepsilon)'A_{11}' + A_{11}W_2(\varepsilon)A_{12}' + A_{12}W_3(\varepsilon)A_{12}' + A_{11}S_1(\varepsilon)'B_1' + A_{12}S_2(\varepsilon)'B_1' + \\ &B_1S_1(\varepsilon)A_{11}' + B_1S_2(\varepsilon)A_{12}' + B_1(S_1(\varepsilon)\Sigma(\varepsilon)S_1(\varepsilon)' + S_2(\varepsilon)\Omega(\varepsilon)'S_1(\varepsilon)' + \ S_1(\varepsilon)\Omega(\varepsilon)S_2(\varepsilon)' + \\ &S_2(\varepsilon)\Upsilon(\varepsilon)S_2(\varepsilon)')B_1' - W_1(\varepsilon), \end{aligned} \tag{28}$$

$$\begin{aligned} H_2(\varepsilon) =&A_{11}W_1(\varepsilon)A_{21}'\varepsilon + A_{12}W_2(\varepsilon)'A_{21}'\varepsilon + A_{11}W_2(\varepsilon)(I_{n_2} + \varepsilon A_{22})' + A_{12}W_3(\varepsilon)(I_{n_2} + \varepsilon A_{22})' + \\ &A_{11}S_1(\varepsilon)'B_2'\varepsilon + A_{12}S_2(\varepsilon)'B_2'\varepsilon + B_1S_1(\varepsilon)\varepsilon A_{21}' + B_1S_2(\varepsilon)(I_{n_2} + \varepsilon A_{22})' + B_1(S_1(\varepsilon)\Sigma(\varepsilon)S_1(\varepsilon)' + \\ &S_2(\varepsilon)\Omega(\varepsilon)'S_1(\varepsilon)' + S_1(\varepsilon)\Omega(\varepsilon)S_2(\varepsilon)' + S_2(\varepsilon)\Upsilon(\varepsilon)S_2(\varepsilon)')B_2'\varepsilon - W_2(\varepsilon), \end{aligned} \tag{29}$$

$$\begin{aligned} H_3(\varepsilon) =&\frac{1}{\varepsilon}(\varepsilon A_{21}W_1(\varepsilon)A_{21}'\varepsilon + \varepsilon A_{21}W_2(\varepsilon)(I_{n_2} + \varepsilon A_{22})' + (I_{n_2} + \varepsilon A_{22})W_2'(\varepsilon)A_{21}'\varepsilon + (I_{n_2} + \varepsilon A_{22})W_3(\varepsilon) \\ &(I_{n_2} + \varepsilon A_{22})' + \varepsilon A_{21}S_1(\varepsilon)'B_2'\varepsilon + (I_{n_2} + \varepsilon A_{22})S_2(\varepsilon)'B_2'\varepsilon + \varepsilon B_2S_1(\varepsilon)A_{21}'\varepsilon + \varepsilon B_2S_2(\varepsilon)(I_{n_2} + \varepsilon A_{22})' + \\ &\varepsilon B_2(S_1(\varepsilon)\Sigma(\varepsilon)S_1(\varepsilon)' + S_2(\varepsilon)\Omega(\varepsilon)'S_1(\varepsilon)' + S_1(\varepsilon)\Omega(\varepsilon)S_2(\varepsilon)' + S_2(\varepsilon)\Upsilon(\varepsilon)S_2(\varepsilon)')B_2'\varepsilon - W_3(\varepsilon)). \end{aligned} \tag{30}$$

$$\begin{aligned} H_1^0 =&A_{11}W_1^0A_{11}' + A_{12}W_2^{0'}A_{11}' + A_{11}W_2^0A_{12}' + A_{12}W_3^0A_{12}' + A_{11}S_1^{0'}B_1' + A_{12}S_2^{0'}B_1' + B_1S_1^0A_{11}' + B_1S_2^0A_{12}' + \\ &B_1S_1^0(W_1^0 - W_2^0W_3^{0^{-1}}W_2^{0'})^{-1}S_1^{0'}B_1' - B_1S_2^0W_3^{0^{-1}}W_2^{0'}(W_1^0 - W_2^0W_3^{0^{-1}}W_2^{0'})^{-1}S_1^{0'}B_1' - \\ &B_1S_1^0(W_1^0 - W_2^0W_3^{0^{-1}}W_2^{0'})^{-1}W_2^0W_3^{0^{-1}}S_2^{0'}B_1' + B_1S_2^0W_3^{0^{-1}}S_2^{0'}B_1' + B_1S_2^0W_3^{0^{-1}}W_2^{0'} \\ &(W_1^0 - W_2^0W_3^{0^{-1}}W_2^{0'})^{-1}W_2^0W_3^{0^{-1}}S_2^{0'}B_1' - W_1^0 \prec 0. \end{aligned} \tag{31}$$

$$H_2^0 = A_{11}W_2^0 + A_{12}W_3^0 + B_1S_2^0 - W_2^0 = 0, \tag{32}$$

$$H_3^0 =W_2^{0'}A_{21}' + A_{21}W_2^0 + A_{22}W_3^0 + W_3^0A_{22}' + S_2^{0'}B_2' + B_2S_2^0 \prec 0. \tag{33}$$

$$\begin{aligned} H_1^0 =&A_{11}W_fA_{11}' + A_{11}S_f'B_1' + B_1S_fA_{11}' + B_1S_fW_f^{-1}S_f'B_1' - W_f + A_{11}W_2^0W_3^{-1}(W_2^0A_{11}' + W_3A_{12}' + S_2^{0'}B_1') + \\ &B_1S_2^0W_3^{-1}(W_2^0A_{11}' + W_3A_{12}' + S_2^{0'}B_1') + A_{12}(W_2^0A_{11}' + W_3A_{12}' + S_2^{0'}B_1') - W_2^0W_3^{-1}W_2^{0'} = \\ &A_{11}W_fA_{11}' + A_{11}S_f'B_1' + B_1S_fA_{11}' + B_1S_fW_f^{-1}S_f'B_1' - W_f \prec 0. \end{aligned} \tag{34}$$

$$H_3^0 = A_sW_s + W_sA_s' + B_sS_s + S_s'B_s' \prec 0. \tag{35}$$

IV. ROBUST REDUCED CONTROLLER

Consider the uncertain discrete-time linear system

$$\begin{cases} x(k+1) = \mathcal{A}(\varepsilon)x(k) + \mathcal{B}(\varepsilon)u(k) \\ y(k) = Cx(k) \end{cases} \tag{38}$$

where $\mathcal{A}(\varepsilon)$ and $\mathcal{B}(\varepsilon)$ are the polytopic domains $\mathcal{A}(\varepsilon) = \sum_{i=1}^{N}\lambda_i^A A^i(\varepsilon)$ and $\mathcal{B}(\varepsilon) = \sum_{i=1}^{N}\lambda_i^B B^i(\varepsilon)$. λ_i^A and λ_i^B denote the uncertainty and belong to the unit simplex $\Lambda = \{\sum_{i=1}^{N}\lambda_i = 1, \lambda_i \geq 0\}$ and $i \in \Gamma = \{1, ..., N\}$, where N is the number of uncertain parameters. As in the linear case, we can separate the fast and slow manifold:

$$\begin{cases} x_1(k+1) = A_{11}^i x_1(k) + A_{12}^i x_2(k) + B_1^i u(k) \\ x_2(k+1) = \varepsilon A_{21}^i x_1(k) + \\ \qquad\qquad (I_{n_2} + \varepsilon A_{22}^i)x_2(k) + \varepsilon B_2^i u(k) \\ y(k) = C_1x_1(k) + C_2x_2(k). \end{cases} \tag{39}$$

The slow subsystem is:

$$\begin{cases} x_s(k+1) = (I_{n_2} + \varepsilon A_s^i)x_s(k) + \varepsilon B_s^i u_s(k) \\ y_s(k) = C_s^i x_s(k) + D_s^i u_s(k). \end{cases}$$

with

$$\begin{aligned} A_s^i &= A_{22}^i + A_{21}^i(I_{n_1} - A_{11}^i)^{-1}A_{12}^i \\ B_s^i &= B_2^i + A_{21}^i(I_{n_1} - A_{11}^i)^{-1}B_1^i \\ C_s^i &= C_2 + C_1(I_{n_1} - A_{11}^i)^{-1}A_{12}^i \\ D_s^i &= C_1(I_{n_1} - A_{11}^i)^{-1}B_1^i. \end{aligned}$$

Let choose the weighting matrix R such that $R = R' = T'T \succ 0$. The next theorem designs a reduced controller able to stabilise the uncertain system (38) and minimise the performance index

$$J_s^{0,i} = Tr\left(\begin{bmatrix} C_s^{i'}C_s^i & C_s^{i'}D_s^i \\ D_s^{i'}C_s^i & D_s^{i'}D_s^i + R \end{bmatrix}\begin{bmatrix} W_s & S_s \\ S_s' & S_sW_s^{-1}S_s' \end{bmatrix}\right),$$

which is defined in (19) for the linear case.

Theorem 2: If A_{11}^i is Schur and there exist matrices $X_s = X_s' \succ 0$, $W_s = W_s' \succ 0$ and S_s of appropriate dimension such that the LMI optimisation problem

$$\min_{X_s, S_s, W_s} Tr\left(X_s\right) \tag{40}$$

under

$$\begin{bmatrix} X_s & C_s^i W_s + D_s^i S_s & T S_s \\ (\star)' & W_s & 0 \\ (\star)' & (\star)' & W_s \end{bmatrix} \succeq 0 \tag{41}$$

and

$$A_s^i W_s + W_s A_s^{i\,\prime} + B_s^i S_s + S_s' B_s^{i\,\prime} \prec 0 \tag{42}$$

has a solution, then the reduced controller $K_r = [0 \;\; S_s W_s^{-1}]$ guarantees the asymptotical stability of the closed-loop system (38), $\forall\, i \in \Gamma$.

Proof: Using the Schur complement, (41) can be written as

$$\begin{aligned} X_s \succeq & C_s^i W_s C_s^{i\,\prime} + C_s^i S_s{}' D_s^{i\,\prime} + D_s^i S_s C_s^{i\,\prime} + \\ & D_s^i S_s W_s^{-1} S_s{}' D_s^{i\,\prime} + T S_s W_s^{-1} S_s{}' T'. \end{aligned}$$

Then

$$\begin{aligned} Tr(X_s) \succeq & Tr(C_s^i W_s C_s^{i\,\prime} + C_s^i S_s{}' D_s^{i\,\prime} + D_s^i S_s C_s^{i\,\prime} + \\ & D_s^i S_s W_s^{-1} S_s{}' D_s^{i\,\prime} + T S_s W_s^{-1} S_s{}' T') = \\ & Tr(C_s^i W_s C_s^{i\,\prime} + C_s^i S_s{}' D_s^{i\,\prime} + D_s^i S_s C_s^{i\,\prime} + \\ & D_s^i S_s W_s^{-1} S_s{}' D_s^{i\,\prime} + R S_s W_s^{-1} S_s{}') = J_s^{0,i}, \end{aligned}$$

$\forall\, i \in \Gamma$. ■

Remark 2: The extension of the full-order controller to uncertain systems is not immediate because $W_2^{0,i}$ depends on the state matrices A_{11}^i, A_{12}^i and B_1^i.

V. INDUSTRIAL APPLICATION

Here, experimental results concerning the robust steering control of the Eisenhüttenstadt hot strip mill of ArcelorMittal are presented [2], [7]. The rolling process consists in crushing a metal strip between two rolls in inverse rotation for obtaining a strip with constant and desired thickness. The lateral movement of the strip with reference to the mill axis, called strip off-centre (Z), may induce a decrease of the product quality and rolls damage. Then, this displacement must be reduced to improve the reliability and the quality of the process. Since the hot strip mill has a two-time scale dynamics, the singular perturbation method has been used to design a reduced controller. Moreover, a hot strip mill treats a set of very different products. Then, for each product parameter, a different uncertainty has to be considered and a robust controller is needed. Fig. 1 shows the results obtained using the reduced controller K_r designed using Theorem 2. The solid line shows the Z evolution for 10 different strips (thickness $\in [2, 3]\, mm$, width $\in [1250, 1600]\, mm$) whereas the dotted line shows the Z evolution of a strip rolled with the system in open loop and the following physical characteristics: thickness $= 2.02\, mm$, width $= 1510\, mm$. In the horizontal axis, we have the time (expressed in sample times, with $T_s = 0.05\, sec$). Notice that the same controller maintains the strip off-centre between -15 and $20\, cm$ for the whole set of treated products, whereas, when the system is not controlled, the strip off-centre varies between -30 and $50\, cm$.

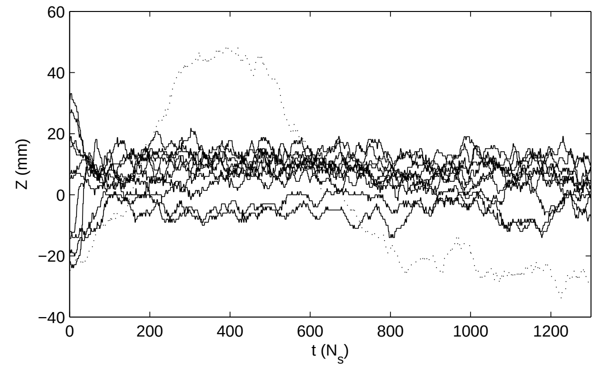

Fig. 1. Strip off-centre evolution

VI. CONCLUSION

In this article, a LMI solution for the LQ control design of singularly perturbed systems in the discrete-time case is proposed. In order to design the controller, a model representing the sampling of singularly perturbed continuous-time systems has been used. Then, results can be applied to the continuous-time systems controlled by digital devices.

The reduced controller can directly extended to systems with polytopic uncertainties. An example of industrial application, the robust steering control of hot strip mill, is presented.

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