# On Multiple-Delay Approximations of Multiple-Derivative Controllers

Yan Wan, Sandip Roy, Anton Stoorvogel, Ali Saberi

*Abstract*— **We study approximation of multiple-derivative output feedback for linear time-invariant (LTI) plants using multiple-delay approximations. We obtain a condition on the plant and feedback that yields an equivalence between the closed-loop spectra for the approximate feedbacks and the desired multiple-derivative feedbacks. On the other hand, we use a scalar example to illustrate that multiple-delay approximations of sufficiently high derivatives may in some cases yield closed-loop spectra that differ greatly from the dynamics upon derivative feedback (for instance, containing many and very large right half plane poles), while in other cases replicating the derivative feedback perfectly. Finally, through understanding this dichotomy, we present a condition for stabilizing a SISO relative degree-1 plant when a delay implementation of the first-order derivative is used in the output feedback control law.**

## I. Introduction

Control of linear time-invariant plants at its essence requires feedback of the output's derivatives, and so control schemes explicitly or implicitly must obtain approximations of output derivatives (see e.g. [10]). Typically, finite-dimensional filters (for instance, lead compensators) are used to obtain output derivatives. However, in recent years, feedback controllers in which derivatives are approximated directly from current and delayed output samples have gained some prominence [3]–[7], [9]. Specifically, these *multiple delay controllers* have been of interest as alternatives to the typical finite-state controllers for several reasons, including: 1) the need for new signal-based control schemes in applications where the traditional observer design fails (such as decentralized and adaptive control applications), 2) the simplicity of approximating derivatives with delay-differences in some application areas, and 3) the intrinsic presence of delays in many modern control systems. Thus, modeling and analyzing feedback control systems that use delay approximations for derivatives, and in turn *designing* delay-approximation schemes in controls, is important.

Differential equations with delays, and more specifically the closed-loop dynamics of control systems subject to delay, have been extensively studied [2]. However, the systems studied here are distinct from those studied in the delay literature, in that *multiply-delayed outputs are deliberately being used to approximate output derivatives and hence to implement desirable feedbacks*. This deliberate use of delays engenders new analyses—namely, efforts to equivalence the performance of the delay-based controller with a true derivative feedback control. It also forces study of delay systems in the case where the delays are made small, as is needed for accurate approximation of derivatives using delayed outputs. To the best of our knowledge, a systematic treatment of deliberate-delay-based output-feedback controllers has not been given: several works have addressed design for particular plants or particular controllers. Of interest to us, integrator-chains and relative degree one and two plants with certain high frequency gain constraints have been addressed in [3], [4], [7]. Moreover, motivated by their study of stabilizing uncertain steady states using difference feedback, Kokame and Mori in [6] studied the stability when using difference counterparts in a feedback that is only involved with a first-order derivative. Our earlier work [9] expanded on these efforts by showing that deliberate-delay-based controllers can *stabilize* a large class of LTI plants, but did not address more complicated controls goals such as pole placement (which we will address here); this previous effort also only gave a detailed proof of results for the SISO case, while we will fully study the MIMO case here. We note that the studies [7], [9] of deliberate-delay-based output feedback controls only consider approximation of sufficiently low output derivatives, in particular ones that are less than the relative degree of the plant. Meanwhile, several researchers have recognized in the state-feedback arena that deliberate-delay approximations of some higher derivative feedbacks may fail, while stability is achieved upon approximation of other feedbacks [3], [6]. This motivates the systematic study of higher derivative feedbacks pursued here.

In this article, we further the study of deliberate-delay control of LTI plants. We first show, in Section II, that the closed-loop spectrum of MIMO plants upon derivative feedback can be achieved asymptotically using deliberate-delay approximations, as long as the approximated derivatives are of sufficiently low order. In Section III, we demonstrate through a scalar example the phenomenon that approximation of higher-derivative feedbacks can in some cases yield unexpected and undesirable response characteristics (including spectra with poles far in the right half plane), while replicating derivative control exactly in other cases. Based on this understanding, we also give a more general result on delay approximation of first-derivative output feedback control of relative-degree-1 plant. These characteristics of the closed-loop spectrum and response are similar in flavor

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The first, second, and fourth authors are with the Department of Electrical Engineering, Washington State University, and can be reached at {sroy, ywan, saberi}@eecs.wsu.edu. The third author is with the Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, The Netherlands, and can be reached at A.A.Stoorvogel@utwente.nl. The first three authors contributed equally to this work, and are listed in order of increasing seniority.

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to characteristics of neutral-type delay differential equations, but also have some significant distinctions.

## II. A POLE EQUIVALENCE RESULT

Here, we give conditions under which delay approximations of multiple-derivative controllers achieve the same closed-loop performance as derivative feedback, in the sense that the closed-loop eigenvalues upon delay control either approach those upon derivative control, or move arbitrarily far to the left in the complex plane. In particular, we find that delay-based and derivative-based controllers can be made arbitrarily close as long as the derivatives being approximated are of sufficiently low order, namely such that the closed-loop system under derivative feedback would be strictly proper. Derivative feedback of this form is well known to permit stabilization and pole placement for a large class of LTI plants (see the classical literature on asymptotic timescale eigenstructure assignment, or ATEA, design [10]), and so we see that delay-based approximations are apt for stabilization and pole placement.

Specifically, let us consider a MIMO LTI system:

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u \\ y &= C_0 x \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Say that a multiple-derivative feedback controller

$$u(t) = \sum_{i=1}^{n} K_i y^{(i-1)}(t), \tag{2}$$

where $K_i \in \mathbb{R}^{m\times p}$, (which has transfer function $F(s) = \sum_{i=1}^{n} K_i s^{i-1}$) can be designed for this system, so that the closed-loop transfer functions $C_0(sI-A_0)^{-1}B_0K_is^{i-1}$ are strictly proper for $i=1,\ldots,n$ and the zeros of $sI-A_0-B_0F(s)C_0$ are in the open left half plane.

**Remark:** We stress that such multiple-derivative controllers can be designed for a wide class of LTI plants, including all minimum phase plants [9].

Let us consider approximating the derivative feedback using a multiple-delay scheme. In particular, we consider approximating $y^{(1)},\ldots,y^{(n-1)}$ by developing a polynomial interpolation of the observation at times $t-\varepsilon\tau_1,\ldots,t-\varepsilon\tau_n$, and computing the derivatives from the interpolation (see [7], [9] for background). With a little algebra, we can specify the transfer function $F_\varepsilon(s)$ of this approximate feedback (which is parametrized on $\varepsilon$) as follows. First defining

$$L_{i,\varepsilon}(s) = \frac{(i-1)!}{(-\varepsilon)^{i-1}} e_i' \begin{pmatrix} 1 & \tau_1 & \cdots & \tau_1^{n-1} \\ 1 & \tau_2 & \cdots & \tau_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \tau_n & \cdots & \tau_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} e^{-\varepsilon\tau_1 s} \\ \vdots \\ e^{-\varepsilon\tau_n s} \end{pmatrix},$$

where $0 \le \tau_1 < \tau_2 < \cdots < \tau_M$ and $e_i$ denotes the $i$th unit vector, we get

$$F_\varepsilon(s) = \sum_{i=1}^{n} K_i \begin{pmatrix} L_{i,\varepsilon}(s) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{i,\varepsilon}(s) \end{pmatrix} = \sum_{i=1}^{n} K_i \tilde{L}_{i,\varepsilon}(s). \tag{3}$$

We shall consider the closed-loop spectrum upon use of this approximate feedback control[1], as a function of $\varepsilon$. We obtain the following main result:

**Theorem 1** *For sufficiently small $\varepsilon$, the closed system upon application of the approximate feedback $F_\varepsilon(s)$ to the system* (1) *is also stable. In particular, n closed-loop poles approach the closed-loop poles when the true derivative feedback* (2) *is used, while the remaining (infinite number of) poles move arbitrarily far left in the complex plane as $\varepsilon$ is decreased.*

*Proof:* The proof consists of two steps. We will first show that the closed-loop system using the approximate feedback $F_\varepsilon(s)$ has exactly $n$ poles in the right half plane $\mathrm{Re}\, s \ge -1/\varepsilon$. Then we will show that these $n$ poles converge to the the $n$ poles of the closed-loop using derivative feedback as $\varepsilon \to 0$.

Let us look at the resulting closed loop system:

$$sI - A_0 - B_0F_\varepsilon(s)C_0 \tag{4}$$

and the associated zeros. We use the factorization: $C_0(sI-A_0)^{-1}B_0 = \frac{1}{q(s)}P(s)$, where $P$ is a polynomial matrix and $q(s) = \det(sI-A)$. Note that our earlier assumption guarantees that $P(s)K_i$ is a polynomial of order less than or equal to $n-i$ for $i=1,\ldots,n$. Next we note:

$$\begin{aligned} q(s)^{p-1}G_\varepsilon(s) &= q(s)^{p-1}\det(sI-A_0-B_0F_\varepsilon(s)C_0) \\ &= q(s)^p \det\left(I-(sI-A_0)^{-1}B_0F_\varepsilon(s)C_0\right) \\ &= q(s)^p \det\left(I-C_0(sI-A_0)^{-1}B_0F_\varepsilon(s)\right) \\ &= \det\left(q(s)I-P(s)F_\varepsilon(s)\right). \end{aligned}$$

The next step is to apply a scaling: $\bar{s} = \varepsilon s$ and obtain:

$$\begin{aligned} \bar{G}_\varepsilon(\bar{s}) &= \varepsilon^{pn}q(s)^{p-1}G_\varepsilon(s) \\ &= \varepsilon^{pn}q(s)^{p-1}\det(sI-A_0-B_0F_\varepsilon(s)C_0) \\ &= \det\left(\varepsilon^n q(s)I - \varepsilon^n P(s)F_\varepsilon(s)\right) \\ &= \det\left(\varepsilon^n q(s)I - \varepsilon^n \sum_{i=1}^{n} P(s)K_i\tilde{L}_{i,\varepsilon}(s)\right) \\ &= \det\left(\bar{q}_\varepsilon(\bar{s})I - \sum_{i=1}^{n} \bar{P}_{i,\varepsilon}(s)\bar{L}_{i,\varepsilon}(s)\right) \end{aligned}$$

where $\bar{q}_\varepsilon(\bar{s}) = \varepsilon^n q(\varepsilon^{-1}\bar{s})$, $\bar{P}_{i,\varepsilon}(\bar{s}) = \varepsilon^{n-i}P(\varepsilon^{-1}\bar{s})K_i$, and $\bar{L}_{i,\varepsilon}(\bar{s}) = \varepsilon^i\tilde{L}_{i,\varepsilon}(\varepsilon^{-1}\bar{s})$.

We note that $\bar{q}_\varepsilon(\bar{s})$, $\bar{P}_{i,\varepsilon}(\bar{s})$ and $\bar{L}_{i,\varepsilon}(\bar{s})$ all depend polynomially on $\varepsilon$ and converge to $\bar{s}^n$, $P_{i,0}\bar{s}^{n-i}$ and 0 respectively when $\varepsilon \to 0$ where $P_{i,0}$ is some constant matrix (which might be zero). Hence $\bar{G}_0(\bar{s}) = \bar{s}^{pn}$ has exactly $pn$ zeros at the origin.

Next we note that for $\mathrm{Re}\,\bar{s} \ge -1$ there exists constants $N_1$, $N_2$ and $N_3$ such that $\|\bar{L}_{i,\varepsilon}(\bar{s})\| \le \varepsilon N_1$, $\|\bar{P}_{i,\varepsilon}(\bar{s})\| \le N_2|\bar{s}|^{n-i}$ and $|\bar{s}^n - \bar{q}_\varepsilon(\bar{s})| \le \varepsilon N_3|\bar{s}|^{n-1}$. But then

$$\left(\bar{q}_\varepsilon(\bar{s})I - \sum_{i=1}^{n} \bar{P}_{i,\varepsilon}(\bar{s})\bar{L}_{i,\varepsilon}(s)\right)v = 0$$

[1]We stress that these control problems that we are addressing here is not simply a delay-independent stability problem, in that not only the delays but also the parameters are changing with $\varepsilon$.

for some $v$ with $\|v\| = 1$ implies: $\bar{s}^n v = (\bar{s}^n - \bar{q}_\varepsilon(\bar{s}))\,v + \sum_{i=1}^n \bar{P}_{i,\varepsilon}(s)\bar{L}_{i,\varepsilon}(s)v$. But then $|\bar{s}|^n \le \varepsilon N_3 |\bar{s}|^{n-1} + \sum_{i=1}^n \varepsilon N_1 N_2 |\bar{s}|^{n-i}$ which clearly implies that for $\varepsilon$ small enough we must have that $|\bar{s}| \le 1$. Hence for $\varepsilon$ small enough, all zeros of $\bar{G}_\varepsilon(\bar{s})$ in $\mathrm{Re}\,\bar{s} \ge -1$ are also inside the unit circle. Next, an application of Hurwitz's theorem, see [1], implies that $\bar{G}_\varepsilon(\bar{s})$ has exactly $np$ eigenvalues inside the unit circle for $\varepsilon$ small enough and which converge to zero as $\varepsilon \to 0$. Hence we know that $\bar{G}_\varepsilon(\bar{s})$ for $\varepsilon$ small enough has exactly $np$ eigenvalues in $\mathrm{Re}\,s \ge -1$. This implies immediately that $G_\varepsilon(s)$ has, again for small $\varepsilon$, exactly $n$ eigenvalues in $\mathrm{Re}\,s \ge -1/\varepsilon$.

Next we choose a region $\mathscr{K}$ such that it contains all zeros of

$$sI - A_0 - B_0 \left( \sum_{i=1}^n K_i s^{i-1} \right) C_0. \tag{5}$$

We will show that $G_\varepsilon(s)$ has exactly $n$ zeros in $\mathscr{K}$ which converge to the zeros of (5) as $\varepsilon$ converges to zero. As already indicated in [7] $F_\varepsilon(s) = \sum_{i=1}^n K_i s^{i-1} + O(\varepsilon s)$. in the region $\mathscr{K}$. But this implies that inside the compact region $\mathscr{K}$, (4) converges uniformly to (5) as $\varepsilon \to 0$. This implies that $G_\varepsilon(s)$ has, for small enough $\varepsilon$, $n$ zeros inside $\mathscr{K}$ which converge to the zeros of (5) as $\varepsilon \to 0$. Since $G_\varepsilon(s)$ has exactly $n$ zeros in the region $\mathrm{Re}\,s \ge -1/\varepsilon$, we find that $G_\varepsilon(s)$ has no other zeros outside $\mathscr{K}$ in the region $\mathrm{Re}\,s \ge -1/\varepsilon$. This clearly implies that $n$ zeros converge to the zeros of (5) while the remaining zeros approach $-\infty$. This completes the proof of stability and the associated convergence of eigenvalues. ∎

Let us make one note about the above result. For SISO plants and MIMO uniform rank plants, the condition given in the above theorem reduces to the condition that the derivatives being approximated are strictly lower in order than the (common) relative degree of the plant; the theorem indicates that approximation of these derivatives can be used successfully in feedback control. More generally, the theorem indicates that output derivatives which can be written as linear functions of the concurrent state are amenable to multiple-delay approximation in feedback. We note that, often, many fewer than $n$ delays may be needed for approximation; a careful delineation of the number of delays needed requires the *special coordinate basis* for linear systems, see the study of asymptotic timescale and eigenstructure assignment (ATEA) in [10].

## III. Anatomy of higher derivative approximations: scalar examples

In Section II, we have shown that multiple-delay approximations to multiple-derivative controllers achieve equivalent performance in the limit of small delay, as long as the highest output derivative approximated is less than the relative degree of the plant (i.e., the closed loop transfer functions are strictly proper). In several domains including decentralized and adaptive control, one encounters the problem that multiple-derivative controllers involving derivatives up to and including the relative degree of the plant (i.e., controllers that make the closed-loop non-strictly proper) are needed (see e.g., [11]). We are thus motivated to understand the closed-loop dynamics when multiple-delay approximations for derivatives of order equal to the relative degree are used.

In this section, we expose the complexity of the dynamics when delay approximations for derivatives of order equal to the relative degree are used. It turns out in this case that the delay approximation does not always yield the dynamics achieved using the derivative controller, even in the limit of small delay. We show this interesting phenomenon using a canonical (scalar) example.

We consider the scalar system

$$\dot{x}(t) = u(t), \tag{6}$$

and consider delay approximation of the following stabilizing derivative-based controller

$$u = ax(t) + b\dot{x}(t), \tag{7}$$

where the gains $a$ and $b$ need to satisfy either $a > 0$ and $b > 1$ or $a < 0$ and $b < 1$ for stability. Specifically, we approximate the derivative term $\dot{x}(t)$ in the controller (7) as $\frac{x(t)-x(t-\Delta)}{\Delta}$, where $\Delta$ is a small time delay, in which case the delay-based controller is

$$u = \frac{b + a\Delta}{\Delta} x(t) - \frac{bx(t-\Delta)}{\Delta}. \tag{8}$$

We notice here that the derivative being approximated is in fact that of the full state (in this case, a scalar), and so the special output feedback structure used in Section II to prove equivalence is not in force here.

In Section III-A, we show that delay approximations to stabilizing derivative controllers can lead to instabilities that are not present if derivative control is used, and compare this effect with the instabilities observed in delay-differential equations of neutral type. Next, Section III-B identifies a class of derivative feedbacks (for the scalar plant) that are amenable to approximation by multiple-delay controllers. Using this insight into the dichotomy of approximation performance, we give conditions in Section IV on feedback controllers for relative-degree-1 plant with higher derivative feedbacks that allow approximation.

### A. Instabilities caused by delay approximations

Here we use a simple first-order system (6) to show that the delay approximation of certain derivative controllers may introduce ORHP poles. We find the pole locations of the closed-loop delay-based system, and present an interesting phenomenon: the unstable pole becomes larger with more accurate approximations. Let us present several results describing this possible instability caused by approximation.

The delay-based controller (8) may introduce ORHP poles, although the corresponding derivative controller stabilizes the system when $a > 0$ and $b > 1$. When (8) is used, the closed loop poles satisfy

$$s = a + \frac{b}{\Delta}(1 - e^{-\Delta s}). \tag{9}$$

As an example, when $a = 1$, $b = 10$, and $\Delta = 0.001$, the closed-loop system has a real root at $s \approx 10000$. With the decrease of $\Delta$, the real pole moves further to the right.

This peculiar phenomenon motivates us to further characterize the locations and number of the ORHP poles. Interestingly, we can constrain the pole locations of the system upon use of the delay-based approximation to a contour in the complex plane. We show this result in Lemma 1.

**Lemma 1** *Consider the first order system* (6) *and a stabilizing derivative-based controller* (7)*. The poles of the closed-loop system using the corresponding delay-based implementation of the controller* (8) *are located on the contour* $(r-a-\frac{b}{\Delta})^2+q^2=\frac{b^2}{\Delta^2}e^{-2\Delta r}$.

*Proof:* Rewrite (9) with $s = r + qj$, where $r$ is the real part of a pole, and $q$ is the imaginary part. A little bit of algebra leads to $r = a+\frac{b}{\Delta}-\frac{b}{\Delta}e^{-\Delta r}\cos(\Delta q)$ and $q = \frac{b}{\Delta}e^{-\Delta r}\sin(\Delta q)$. Combining these two expressions and noticing that $\sin(\Delta q)^2+\cos(\Delta q)^2 = 1$, we obtain the condition. ■

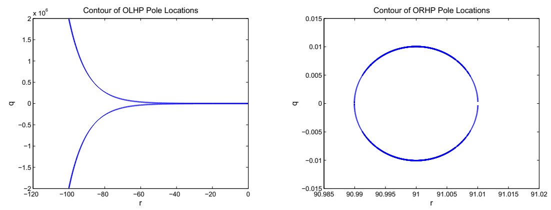

Fig. 1. The contour of poles when $a = 1$, $b = 9$ and $\Delta = 0.1$. a) the contour of the OLHP poles; b) the contour of the ORHP poles.

This result shows that the poles reside on a contour centered at $(a+\frac{b}{\Delta},0)$ with varying radius $\frac{b}{\Delta}e^{-\Delta r}$. An example of the contour when $a = 1$, $b = 9$ and $\Delta = 0.1$ is shown in Figure 1. When the delay $\Delta$ is small, $\Delta r$ roughly equals the constant $b$, hence the shape of the ORHP contour is roughly a circle. Also, with the decrease of $\Delta$, the ORHP contour shifts to the right with the radius roughly scaled with $\frac{1}{\Delta}$. On the other hand, the shape of the contour in the OLHP is determined by the dominant exponential term. Of interest to us, we note that the OLHP contour scales to the left as $\Delta$ decreases. Meanwhile, the OLHP contour changes significantly with $r$, due to the exponential term. We again stress that, as the delay $\Delta$ is decreased, the unstable pole becomes larger and larger.

The delay-based controller (8) for any fixed set of parameters is formally one of retarded type, and hence has a finite number of ORHP poles. We notice that if we instead use a controller with a *delayed-derivative* approximation for the scalar plant, i.e.

$$u(t) = ax(t)+b\dot{x}(t-\Delta) \tag{10}$$

the resulting closed-loop system is a neutral delay-differential equation. One characteristic of neutral delay-differential equations (see e.g. [2], [8]) is the presence of an "infinite root chain," i.e. of an infinite number of ORHP eigenvalues all with real part located in an interval. Besides the difference, the controllers in (10) and (8) may result in unstable dynamics that do not resemble that using the derivative controller (7). It is quite interesting to observe when using delay approximation, how this number of poles depends on the system parameters, to see whether the dynamics are similar to those of a neutral type system.

Let us describe the dependence of the number of ORHP poles on the delay $\Delta$ and on the gain $b$.

**Lemma 2** *Consider the first order system* (6) *and a stabilizing derivative-based controller* (7) *with $a > 0$ and $b > 1$. The corresponding delay-based implementation of the controller* (8) *introduces a finite number of poles in the ORHP. The number of ORHP poles can grow with the decrease of the time delay $\Delta$, but remains bounded. However, the number of ORHP poles grows unboundedly with $b$.*

*Proof:* First, we notice that when $a > 0$ and $b > 1$, the derivative-based controller (7) stabilizes the system. Moreover, from the proof of Lemma 1, we have $|\Delta q| \le be^{-\Delta r}$.

Next, we recall the well-known property that the closed-loop system using the delay-based controller—which is a retarded delay-differential equation for any fixed $\Delta$—has only a finite number of ORHP poles (see e.g., [2].)

Now let us count the number of ORHP poles. When $q = 0$, the closed-loop system has a single real pole at $(a+\frac{b}{\Delta}-\frac{b}{\Delta}e^{-\Delta r},0)$. When $q \neq 0$, combining the expressions for $q$ and $r$ from the proof of Lemma 1 and eliminating $r$ from the expression, we obtain

$$\Delta q e^{a\Delta+b} = b\sin(\Delta q)e^{\Delta q\cot(\Delta q)}. \tag{11}$$

By introducing $q' = \Delta q$, we can see that the right side of (11) is simply an oscillating function of $q'$ with varying amplitude within the bound $|q'| < be^{-\Delta r}$, and the left side is a monotonic function of $q'$ with the scaling factor $e^{a\Delta+b}$. From consideration of the oscillating function on the right side with the observation that $\cot(\Delta q)$ is unbounded, one automatically sees that the number of solutions to (11) and hence the number of complex ORHP poles is between $\frac{be^{-\Delta r}}{\pi}-2$ and $\frac{be^{-\Delta r}}{\pi}+2$. Hence as $\Delta$ decreases, one sees that the number of solutions to (11) may increase but always below the bound $\frac{b}{2\pi}+1$. We also automatically recover from the lower bound that the number grows unboundedly with $b$. ■

We notice that our delay-based approximation, though formally yielding a retarded differential equation, has some resemblance to delayed-derivative approximation for large $b$, in terms of having highly unstable dynamics and a large number of ORHP poles.

The analysis of the example in this section demonstrates that the delay approximation when the stabilizing derivative controller has highest derivative term equal to the relative degree may cause instability. Such a system is different in terms of dynamics from both the delayed-derivative system and the delay approximation when we use one less degree in the derivative controller. The former system is a neutral system that has chains of infinite number of roots in certain right half planes. Meanwhile, the later controller guarantees that the dynamics using the approximation resembles the dynamics using the derivative controller. When a derivative equal to the relative degree is used in the control law, the resulting system remains retarded but may not represent the dynamics using the derivative controller anymore, and in fact show certain neutral type behavior, as we see when $b$ becomes large. This peculiar dynamics is cause by the fact that delay is used to approximate a quantity that is not part of the state of the system.

*B. A Pole Placement Result for Some Approximations*

We have shown that, unfortunately, delay approximations to derivative feedbacks of order equal to the relative degree of a plant can fail. Luckily, only some such approximations cause instability. In fact, in many cases, depending on the parameters of the stabilizing derivative controller, certain delay approximations can be shown to achieve equivalence to the derivative feedback in a pole-placement sense, as the delay $\Delta$ is made small. Here, we shall demonstrate this pole-equivalence result for the canonical scalar system (6) with $a < 0$ and $b < 0$ in (8). This simple example provides us with a means for implementing multiple-derivative controllers for decentralized systems. That is, through smart design of the stabilizing derivative controller, stability can be maintained using delay approximation.

Although this delay feedback only differs from the one considered in Section III-A in its sign, the resulting closed-loop dynamics are stable, and in fact the spectrum become equivalent to that in (7) as the delay $\Delta$ becomes small. Precisely, the following result holds:

**Theorem 2** *Consider the poles of the first-order system* (6) *using delay-feedback control* (8) *where $a < 0$ and $b < 0$. As $\Delta \to 0$, one pole approaches $\frac{a}{1-b}$, i.e. the single pole of the closed-loop system using derivative feedback control* (7)*. The remaining poles have real parts approaching $-\infty$.*

*Proof:* We notice that this closed-loop dynamics, which represents a negative derivative feedback to a single-integrator plant, has a single stable pole, at $\frac{a}{1-b}$. As in Section III-A, we can limit the poles to the contour given in Lemma 1. For $a < 0$ and $b < 0$, it is immediate that this contour is located entirely within the OLHP, so stability of the closed-loop follows.

What remains to be shown is that one pole approaches $\frac{a}{1-b}$, while the remaining poles move arbitrarily far left. To prove this, let us first show that one pole can be placed arbitrarily close to $\frac{a}{1-b}$ by choosing $\Delta$ small, while the remaining poles must be outside a circle centered the origin with radius increasing unboundedly as $\Delta$ becomes small.

We notice that the characteristic equation of the delay-feedback system is given by $s = a + \frac{bs}{1+q(s)}$, where $q(s) = \frac{s\Delta}{1-e^{-s\Delta}} - 1$, and we have written the differential equation in this form to highlight that the delay approximation performs a low-pass filtering. Notice that, roughly, $q(s) \approx 0$ for $|s| < \frac{1}{\Delta}$ and $q(s) \approx s\Delta$ for $|s| > \frac{1}{\Delta}$.

Rearranging, we obtain that the characteristic polynomial is $(s-a)q(s) + (1-b)s - a = 0$. Using this form, we shall prove that all poles within a circle in the complex plane whose radius increases unboundedly as $\Delta$ decreases can only lie within a small ball around $\frac{a}{1-b}$. In particular, consider a small ball of radius $\varepsilon > 0$ around the point $\frac{a}{1-b}$. For all $s$ outside this ball, notice that $|(1-b)s - a| \geq |(1-b)|\varepsilon$. Now consider a (large) circle of radius $f$. For any $f$, it is clear that the maximum value of $|q(s)|$ within this circle can be made arbitrarily small by choosing $\Delta$ sufficiently small; more specifically, it is easily seen that $|q(s)|$ within this circle can be upper bounded by $Kf\Delta$, for some positive constant $K$. In turn, we find that $|(s-a)q(s)|$ within the circle is upper bounded by $Kf(f+a)\Delta$. Thus, by choosing $\Delta < \frac{(1+b)\varepsilon}{Kf(f+a)}$, we can guarantee that $|(1+b)s + a| > |(s-a)q(s)|$ for all $s$ in the circle of radius $f$, for any $f$. Choosing $\Delta$ in this way, we guarantee that all roots of the characteristic equation within the circle of radius $f$ must be within the ball of radius $\varepsilon$ around $\frac{a}{1-b}$. By considering the characteristic equation, we can trivially check that there is indeed precisely one (real) pole within the ball, and that this pole has multiplicity 1.

We can use the contour on which the poles must be located to complete the proof. Notice that, for particular $a < 0$ and $b < 0$, the contour is located in the OLHP and further that as $\Delta$ is decreased the real part of the point on the contour for each possible imaginary value decreases (becomes more negative). This observation, together with the fact that $\Delta$ can be selected to exclude poles from inside a circle of arbitrary radius $f$ (except the one near $\frac{a}{1-b}$), shows that the remaining poles can be moved arbitrarily far left in the complex plane by choosing $\Delta$ sufficiently small. ■

The equivalence of the delay-feedback approximation with the derivative-based controller for $a < 0$ and $b < 0$ is heartening, because it suggests that some derivative controls of order equal to the relative degree of a plant can be implemented using multiple-delay approximations.

## IV. Designing derivative-approximation controllers for relative degree 1 plants

The anatomy of higher-derivative approximations introduced in the above sections shows that instabilities may result when approximating some derivative feedbacks, while approximations of other derivative feedbacks match the derivative feedback's performance. That is, approximation is possible when some feedback gains are applied to the higher-derivative terms, but not when other gains are used. This understanding motivates study of *which* feedback gains allow for use of higher-derivative approximations, for more general LTI plants. Here, let us present a first result in this direction. Specifically, let us show how delay-approximation controllers can be used to stabilize SISO LTI plants with relative degree 1. Formally, consider a SISO plant

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \tag{12}$$

that has relative degree 1, i.e. for which $cb$ is nonzero. Assume that there exist a derivative feedback $u = k_1 y + k_2 \dot{y}$

which stabilizes this system. Let us consider control of this plant by approximating this derivative feedback by

$$k_1 y(t) + k_2 \frac{y(t) - y(t - \varepsilon\tau)}{\varepsilon\tau}, \tag{13}$$

for any $\tau > 0$. We note that the derivative of the output to be approximated in feedback is equal to the relative degree of the plant, which equivalently means that 1) the closed-loop transfer function is not strictly proper and that 2) the derivative $\dot{y}$ is not simply a linear function of the concurrent state. Thus, we expect this feedback might not be easily approximated when certain gains $k_1$ and $k_2$ are used. The following theorem gives conditions under which the deliberate-delay approximation matches the derivative-feedback control.

**Theorem 3** *Consider the system* (12) *with relative degree* 1 *and the delay-based feedback* (13). *Provided $k_2$ and $cb$ have opposite sign, the closed loop system is asymptotically stable for all small enough $\varepsilon > 0$.*

*Proof:* The closed loop poles of the approximating feedback are the zeros of: $\det\left(sI - A - b\left[k_1 + k_2 \frac{1}{\varepsilon\tau}\left(1 - e^{-\varepsilon\tau s}\right)\right]c\right)$ which can be rewritten as: $g_\varepsilon(s) = \det\left(q(s) - p(s)\left[k_1 + k_2\frac{1}{\varepsilon\tau}\left(1 - e^{-\varepsilon\tau s}\right)\right]\right)$ where $c(sI - A)^{-1}b = \frac{p(s)}{q(s)}$, $q(s)$ is a monic polynomial of order $n$, and $p(s)$ is a polynomial of order $n-1$ whose leading coefficient equals $cb$. The next step is to apply a scaling: $\bar{s} = \varepsilon s$ and obtain: $g_\varepsilon(\bar{s}) = \varepsilon^n g_\varepsilon(s) = \det\left(\bar{q}_\varepsilon(\bar{s}) - \bar{p}_\varepsilon(\bar{s})\left[k_1\varepsilon + k_2\frac{1}{\tau}\left(1 - e^{\tau\bar{s}}\right)\right]\right)$, where $\bar{q}_\varepsilon(\bar{s}) = \varepsilon^n q(\varepsilon^{-1}\bar{s})$ and $\bar{p}_\varepsilon(\bar{s}) = \varepsilon^{n-1} p(\varepsilon^{-1}\bar{s})$.

We note that $\bar{q}_\varepsilon(\bar{s})$ and $\bar{p}_\varepsilon(\bar{s})$ depend polynomially on $\varepsilon$ and converge to $\bar{s}^n$ and $p_0\bar{s}^{n-1}$ respectively when $\varepsilon \to 0$ where $p_0$ is some constant matrix (which might be zero).

Next we note that for $\mathrm{Re}\,\bar{s} \geq -1$ there exists constants $N_1$, $N_2$ and $N_3$ such that $\|k_1\varepsilon + k_2\frac{1}{\tau}\left(1 - e^{\tau\bar{s}}\right)\| \leq N_1$, $\|\bar{p}_\varepsilon(\bar{s})\| \leq N_2|\bar{s}|^{n-1}$, and $|\bar{s}^n - \bar{q}_\varepsilon(\bar{s})| \leq \varepsilon N_3|\bar{s}|^{n-1}$. But then $\bar{g}_\varepsilon(\bar{s}) = 0$ implies: $|\bar{s}|^n \leq \varepsilon N_3|\bar{s}|^{n-1} + N_1N_2|\bar{s}|^{n-1}$ which clearly implies that for $\varepsilon$ small enough we must have that $|\bar{s}| < N_4$ for some constant $N_4$.

Next, we note that

$$\bar{s}^n - \tfrac{k_2(cb)}{\tau}\bar{s}^{n-1}\left(1 - e^{-\tau\bar{s}}\right) \tag{14}$$

has $n$ zeros at the origin since $k_2(cb) \neq 1$. Moreover, this function has no zeros with $\mathrm{Re}\,\bar{s} > 0$. After all in that case

$$\bar{s} = \tfrac{k_2(cb)}{\tau}\left(1 - e^{-\tau\bar{s}}\right) \tag{15}$$

with $\mathrm{Re}(1 - e^{-\tau\bar{s}}) \geq 0$ and $k_2(cb) < 0$ yields that the right hand side lies in the open left half plane while $\bar{s}$ lies in the open right half plane which provides us with a contradiction. Thus, we have that $\bar{q}_\varepsilon(\bar{s})$ has no zeros in the ORHP. Next, we consider the possibility of imaginary axis zeros. If there were any, one would need $\tau\bar{s} = 2q\pi j$ for some integer $q$. Since in (15) the right hand side must be on the imaginary axis. However, this immediately yields $\bar{s} = 0$ if we go back to (15).

We find thus that (15) has exactly $n$ zeros in $\mathrm{Re}\,\bar{s} \geq 0$ and $|\bar{s}| \geq N_4$. Using that the function is analytic, we find that there exists $\delta > 0$ such that this function has exactly $n$ zeros in $\mathrm{Re}\,\bar{s} \geq -\delta$ and $|\bar{s}| \leq N_4$. But this implies that $\bar{g}_\varepsilon(s)$, which converges uniformly in the region $\mathrm{Re}\,\bar{s} \geq -\delta$ and $|\bar{s}| \leq N_4$ to (14), has exactly $n$ zeros in this region. Since we already established that this function did not have zeros with $|\bar{s}| \geq N_4$, we find that (14) has exactly $n$ zeros in $\mathrm{Re}\,\bar{s} \geq -\delta$.

Next we choose a region $\mathscr{K}$ in the open left half plane such that it contains all zeros of

$$\det\left(sI - A - b\left[k_1 + k_2 s\right]c\right) \tag{16}$$

which is possible since the derivative based feedback was stabilizing. We will show that $g_\varepsilon(s)$ has exactly $n$ zeros in $\mathscr{K}$ which converge to the zeros of (5) as $\varepsilon$ converges to zero. Similar to the proof of Theorem 1 we find: $k_1 + k_2\frac{1}{\varepsilon\tau}\left(1 - e^{-\varepsilon\tau s}\right) = k_1 + k_2 s + O(\varepsilon s)$. But this implies that inside the compact region $\mathscr{K}$, $G_\varepsilon(s)$ converges uniformly to (16) as $\varepsilon \to 0$. This implies that $g_\varepsilon(s)$ has, for small enough $\varepsilon$, $n$ zeros inside $\mathscr{K}$ which converge to the zeros of (5) as $\varepsilon \to 0$. Since $g_\varepsilon(s)$ has exactly $n$ zeros in the region $\mathrm{Re}\,s \geq -\delta/\varepsilon$, we find that $g_\varepsilon(s)$ has no other zeros outside $\mathscr{K}$ in the region $\mathrm{Re}\,s \geq -\delta/\varepsilon$. This clearly implies that $n$ zeros converge to the zeros of $g_\varepsilon(s)$ while the remaining zeros approach $-\infty$. This completes the proof of stability. ■

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