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*Abstract*— **In this paper, Input-to-State Stability conditions are proposed for disturbed linear polytopic discrete-time systems. The conditions allow to optimize, independently in a certain extent, both the bounds on the decay rate and on the ISS gain, two central quantites. Indeed, the decay factor characterizes the transient behaviour of the state while the ISS gain characterizes the sensitivity with respect to the disturbances. The conditions are expressed in terms of tractable Matrix Inequalities. It is shown that the conditions are more general than existing ones proposed so far in the literature and thus less conservative. The conditions hold both for analysis or synthesis purposes. Two illustrative examples addressing the problem of polytopic observer design are given.**

The concept of Input-to-State Stability (ISS) has been introduced in [18] for continuous-time systems and is now a popular tool to tackle problems related to disturbed nonlinear systems. Roughly speaking, ISS refers to the property that the state trajectory of a system which is stable when undisturbed remains bounded when the system is subjected to bounded disturbances. Such a property is not trivial because it can be shown that even tiny disturbances may destabilize a stable unforced system [15] [13].

A discrete-time counterpart of the ISS conditions stated in [18] has been proposed in [11]. A general treatment of ISS for nonlinear discrete-time systems can also be found in [9] [10]. Besides, ISS has been addressed for the special class of LPV discrete-time systems: a state feedback controller for norm-bounded disturbed systems is proposed in [14], the situation when the parameters are not exactly known is addressed in [6] for output-based feedback control purposes or in [16] for observer synthesis. The paper [12] presents conditions for global ISS and stabilization of discrete-time, possibly discontinuous, piecewise affine systems. The ISS framework is also particularly interesting for aperiodic control of sampled data systems. Most of the works are based on the guarantee that a closed-loop system with plant and controllers connected through a network is ISS with respect to measurements errors (see [19] for a valuable reference in the continuous-time case). In this context, the principle of event-triggered or self-triggered controls is quite appealing. The control law is updated once a triggering condition involving the norm of a measurement error is violated. For discrete-time systems, an ISS-based approach is described in [5] to compute an admissible inter-execution time of the control with an application to Model Predictive Control.

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When considering ISS, two typical quantities play a central role: the decay factor and the ISS gain. Roughly speaking, the decay factor characterizes the transient behaviour of the state while the ISS gain characterizes the sensitivity with respect to the disturbances. In synthesis perspective, it would be useful to fix *separately* both quantities. This is actually a challenging problem, and, up to now, there is no approach to achieve a perfect decoupling. For example, in [6], both the decay factor and the ISS gain depend on a common design parameter which results from the solution of an LMI-based optimization procedure. Hence, such an approach suffers from a lack of flexibility when it comes to synthesis. On the other hand, the paper [8] proposes an approach to get minimal ISS gain and transient bound for discrete nonlinear systems. The approach is based on dynamic programming but rests on heavy numerical schemes.

In this work, we derive ISS conditions allowing to scale, separately in a certain extent, the decay rate as well as the ISS gain. We focus on linear polytopic discrete-time systems, which include Linear Parameter Varying systems with polytopic description and switching linear systems. In Section I, some basic definitions are recalled including the notion of Input-to-State Stability (ISS) and ISS Lyapunov functions for polytopic systems. Section II is devoted to the ISS conditions in terms of Matrix Inequalities. A comparative study with existing conditions borrowed from the literature is carried out in Section III. Section IV is devoted to the problem statement for synthesis purposes. Finally, in Section V, two examples addressing the problem of polytopic observer design illustrate the efficiency of the approach.

*Notation*: $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$: the field of real numbers, the set of non-negative real numbers and the set of non-negative integers, respectively. $z^{(i)}$: the $i$th component of a real vector $z$. $z^T$: the transpose of $z$. $|z^{(i)}|$: the absolute value of the real number $z^{(i)}$. $\|z\| = \sqrt{z^T z}$: the Euclidean norm of $z$. $\{z\}$: a sequence of samples $z_k$, $z_{k+1}$, $\ldots$ without explicit initial and final discrete-time $k \in \mathbb{N}$. $\{z\}_{k_1}^{k_2}$: a sequence of samples $z_{k_1}, \ldots, z_{k_2}$. $l_2^n$: the Hilbert space of right-sided square summable real vector sequences of vectors of dimension $n$. $l_\infty^n$: the Banach space of right-sided bounded sequences $z$ of real vectors of dimension $n$. $\|z\|_2 = \sqrt{\sum_{k=0}^{\infty} z_k^T z_k}$: the norm on $l_2^n$. $\|z\|_\infty = max_i sup_k |z_k^{(i)}|$: the norm on $l_\infty^n$. $\mathbf{1}$: the identity matrix of appropriate dimension. $\mathbf{0}$: the zero matrix of appropriate dimension. $X^T$: the transpose of the matrix $X$. $X > 0$ $(X < 0)$: a positive definite (negative definite) matrix $X$. $X \geq 0$ $(X \leq 0)$: a positive semi-definite (negative semi-definite) matrix $X$. $\|X\| = \sqrt{\lambda_{max}(X^T X)}$: the spectral norm of the matrix $X$, where $\lambda_{max}$ is the largest eigenvalue of $X^T X$. $(\bullet)$: the blocks of a matrix induced by symmetry.

## I. Problem statement

This section recalls some basics on ISS for general nonlinear and for linear polytopic discrete-time systems. All the definitions and theorems of this section are borrowed from [9] [10] [16].

### A. Preliminaries on ISS for discrete-time systems

**Definition 1:** A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$, and to class $\mathscr{K}_\infty$ if additionally $\varphi(s) \to \infty$ as $s \to \infty$.

**Definition 2:** A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(.,k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,.)$ is decreasing and $\lim_{k\to\infty} \beta(s,k) = 0$.
Consider the discrete-time nonlinear systems

$$x_{k+1} = f(x_k), \tag{1}$$

$$x_{k+1} = f_v(x_k, v_k), \tag{2}$$

with $x_k \in \mathbb{R}^n$ the state vector, $v_k \in \mathbb{R}^{d_v}$ an unknown disturbance input.

**Definition 3:** The system (1) is called Globally Asymptotically Stable (GAS) if there exists a $\mathscr{KL}$ function $\beta$ such that, for each $x_0 \in \mathbb{R}^n$, the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k),$$

for all $k \in \mathbb{N}$.

**Definition 4:** The system (2) is said to be Input-to-State Stable (ISS) with respect to $v_k$ if there exist a $\mathscr{KL}$ function $\beta$ and a $\mathscr{K}$ function $\gamma$ such that, for all input sequences $\{v\}$, for each $x_0 \in \mathbb{R}^n$, the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v\|_\infty), \tag{3}$$

for all $k \in \mathbb{N}$.

If $\beta$ can be chosen of the form $\beta(s,k) = ds\zeta^k$ for some $d \geq 0$ and $0 < \zeta < 1$, $\zeta$ is the *decay factor* for (1).

The function $\gamma$ is an *ISS gain* for (2).

### B. Linear polytopic systems

In this paper, we are concerned with linear polytopic discrete-time systems subjected to additive disturbances

$$x_{k+1} = A(\rho_k)x_k + v_k, \tag{4}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $v_k \in \mathbb{R}^n$ is the disturbance, and $A \in \mathbb{R}^{n\times n}$ is the dynamical matrix which depends on a time varying parameter vector $\rho_k$ assumed to lie in a bounded set $\Omega_\rho \subset \mathbb{R}^L$. The matrix $A(\rho_k)$ is assumed to admit a polytopic description

$$A(\rho_k) = \sum_{i=1}^{N} \xi^{(i)}(\rho_k) A_i, \tag{5}$$

where $\xi = \left[\xi^{(1)}, \ldots, \xi^{(N)}\right]$ belongs to the compact set $\mathscr{S}$

$$\mathscr{S} = \left\{ \mu \in \mathbb{R}^N : \mu^{(i)} \geq 0 \ \forall i \ , \sum_{i=1}^{N} \mu^{(i)} = 1 \right\}. \tag{6}$$

Owing to the convexity of $\mathscr{S}$, the matrices $\{A_1, \ldots, A_N\}$ are the vertices of a polytope.

**Remark 1:** We can also be concerned with the case when $\rho_k$ in (4) is not available but its estimate $\hat{\rho}_k \in \Omega_{\hat{\rho}}$ is. In such a case, we can define $\Delta A(\rho_k, \hat{\rho}_k) = A(\rho_k) - A(\hat{\rho}_k)$ and (4) still describes the situation by considering $\hat{\rho}_k$ instead of $\rho_k$ and $v_k = \Delta A(\rho_k, \hat{\rho}_k)x_k$. As a result, disturbances and uncertainties can be tackled with a same framework.
Deriving sufficient conditions to guarantee ISS is usually based on the notion of ISS Lyapunov function.

**Definition 5:** Let $d_1, d_2 \in \mathbb{R}_+$, let $a, b, c, l \in \mathbb{R}_+$ with $a \leq b$ and let $\alpha_1(s) = as^l, \alpha_2(s) = bs^l, \alpha_3(s) = cs^l$ and $\gamma \in \mathscr{K}$. A function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ which satisfies

$$\alpha_1(\|x_k\|) \leq V(x_k, \rho_k) \leq \alpha_2(\|x_k\|), \tag{7}$$

$$V(x_{k+1}, \rho_{k+1}) - V(x_k, \rho_k) \leq -\alpha_3(\|x_k\|) + \gamma(\|v_k\|), \tag{8}$$

for all $x_k \in \mathbb{R}^n$, all $v_k \in \mathbb{R}^n$ and all $\rho_k \in \Omega_\rho$ is called an ISS Lyapunov function for (4).

**Theorem 1:** [16] If the system (4) admits an ISS Lyapunov function as in Definition 5, then (4) is ISS with respect to $v_k$.

## II. Main result

In this section, we give conditions which guarantee that (4) is ISS with respect to $v_k$ and we derive the corresponding ISS Lyapunov function. Furthermore, we explicit the decay factor and the ISS gain.

**Theorem 2:** If there exist positive definite symmetric matrices $P_i$, matrices $G_i$, real numbers $\mu > 0$, $\nu > 0$ and $\lambda \in ]0\ 1[$, fulfilling, $\forall (i,j) \in \{1 \cdots N\} \times \{1 \cdots N\}$, the Matrix Inequalities

$$\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ G_i A_i & G_i & G_i^T + G_i - P_j \end{bmatrix} > 0, \tag{9}$$

and

$$\begin{bmatrix} \lambda P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & (\nu-\mu)\mathbf{1} & (\bullet)^T \\ \mathbf{1} & \mathbf{0} & \nu\mathbf{1} \end{bmatrix} \geq 0, \tag{10}$$

then the system (4) is ISS with respect to $v_k$ and

$$\|x_k\| \leq \sqrt{\nu\lambda\mu}(1-\lambda)^{k/2}\|x_0\| + \nu\|v\|_\infty. \tag{11}$$

The decay factor is $\zeta = (1-\lambda)^{1/2}$ and the ISS gain is linear, with $\gamma(s) = \nu s$.
The corresponding ISS Lyapunov function is $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$, $V(x_k, \rho_k) = x_k^T \mathscr{P}_k x_k$ with $\mathscr{P}_k = \sum_{i=1}^N \xi^{(i)}(\rho_k) P_i$, that satisfies

$$V(x_{k+1}, \rho_{k+1}) - V(x_k, \rho_k) \leq -\frac{1}{\nu}\|x_k\|^2 + \mu\|v_k\|^2, \tag{12}$$

$$\frac{1}{\nu\lambda}\|x_k\|^2 \leq V(x_k, \rho_k) \leq \mu\|x_k\|^2, \tag{13}$$

for all $\xi \in \mathscr{S}$, all $x_k \in \mathbb{R}^n$ and all $v_k \in \mathbb{R}^n$.

**Proof 1 (Theorem 2):** In the proof, for sake of simplicity, $\xi^{(i)}(\rho_k)$ will be shortly denoted $\xi_k^{(i)}$. The proof comprises four main steps. For the first step of the proof, assume that (9) is feasible. Since $(P_j^{-\frac{1}{2}}G_i^T - P_j^{\frac{1}{2}})^T(P_j^{-\frac{1}{2}}G_i^T - P_j^{\frac{1}{2}}) > 0$, it holds, as shown in [3], that

$$G_iP_j^{-1}G_i^T > G_i + G_i^T - P_j. \tag{14}$$

This implies that, $\forall(i,j) \in \{1\cdots N\}\times\{1\cdots N\}$,

$$\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ G_iA_i & G_i & G_iP_j^{-1}G_i^T \end{bmatrix} > 0.$$

This is equivalent to, $\forall(i,j) \in \{1\cdots N\}\times\{1\cdots N\}$,

$$N\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ P_jA_i & P_j & P_j \end{bmatrix}N^T > 0,$$

with $N = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G_iP_j^{-1} \end{bmatrix}$. Hence, we have that, $\forall(i,j) \in \{1\cdots N\}\times\{1\cdots N\}$,

$$\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ P_jA_i & P_j & P_j \end{bmatrix} > 0.$$

Multiplying the above matrix inequality by $\xi_k^{(i)}$ and summing, multiplying by $\xi_{k+1}^{(j)}$ and summing yields

$$\begin{bmatrix} (1-\lambda)\mathscr{P}_k & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ \mathscr{P}_{k+1}A & \mathscr{P}_{k+1} & \mathscr{P}_{k+1} \end{bmatrix} > 0,$$

with $\mathscr{P}_k = \sum_{i=1}^N \xi_k^{(i)}P_i$ and $\mathscr{P}_{k+1} = \sum_{j=1}^N \xi_{k+1}^{(j)}P_j$, or equivalently, by Schur's formula,

$$\begin{bmatrix} (1-\lambda)\mathscr{P}_k & \mathbf{0} \\ \mathbf{0} & \mu\mathbf{1} \end{bmatrix} - \begin{bmatrix} A^T\mathscr{P}_{k+1} \\ \mathscr{P}_{k+1} \end{bmatrix}\mathscr{P}_{k+1}^{-1}\begin{bmatrix} \mathscr{P}_{k+1}A & \mathscr{P}_{k+1} \end{bmatrix} > 0.$$

Hence, for all $x_k \in \mathbb{R}^n$ and all $v_k \in \mathbb{R}^n$, it holds that

$$\begin{pmatrix} x_k^T & v_k^T \end{pmatrix} M \begin{pmatrix} x_k \\ v_k \end{pmatrix} \leq 0,$$

with $M = \begin{bmatrix} A^T\mathscr{P}_{k+1}A - (1-\lambda)\mathscr{P}_k & (\bullet)^T \\ \mathscr{P}_{k+1}A & \mathscr{P}_{k+1} - \mu\mathbf{1} \end{bmatrix}$. This implies that, for all $x_k \in \mathbb{R}^n$ and all $v_k \in \mathbb{R}^n$,

$$(Ax_k + v_k)^T\mathscr{P}_{k+1}(Ax_k + v_k) - (1-\lambda)x_k^T\mathscr{P}_kx_k \leq \mu v_k^Tv_k.$$

This can be rewritten as

$$V(x_{k+1},\rho_{k+1}) - (1-\lambda)V(x_k,\rho_k) \leq \mu v_k^Tv_k, \tag{15}$$

or equivalently

$$V(x_{k+1},\rho_{k+1}) - V(x_k,\rho_k) \leq -\lambda V(x_k,\rho_k) + \mu\|v_k\|^2, \tag{16}$$

with $V(x_k,\rho_k) = x_k^T\mathscr{P}_kx_k = x_k^T(\sum_{i=1}^N \xi_k^{(i)}P_i)x_k$.
Now, let us proceed to the second step of the proof. On one hand, the feasibility of (9) implies that, $\forall(i,j) \in \{1\cdots N\}\times\{1\cdots N\}$,

$$\begin{bmatrix} \mu\mathbf{1} & (\bullet)^T \\ G_i & G_i + G_i^T - P_j \end{bmatrix} > 0,$$

which in turn implies that $\begin{bmatrix} \mu\mathbf{1} & (\bullet)^T \\ G_i & G_iP_j^{-1}G_i^T \end{bmatrix} > 0$ and so $G_iP_j^{-1}G_i^T - G_i\mu^{-1}G_i^T > 0$. Since for all $i \in \{1,\ldots,N\}$, $G_i$ is invertible, since for all $j \in \{1,\ldots,N\}$, $P_j$ is positive definite and $\mu > 0$, we infer that

$$P_j < \mu\mathbf{1}. \tag{17}$$

Multiplying respectively left and right by $x_k^T$ and $x_k$, we have that

$$V(x_k,\rho_k) \leq \mu\|x_k\|^2. \tag{18}$$

On the other hand, assume that (10) is feasible. Using Schur's formula, (10) can be rewritten as

$$\begin{bmatrix} \lambda P_i - \frac{1}{\nu}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\nu-\mu)\mathbf{1} \end{bmatrix} \geq 0. \tag{19}$$

The feasibility of (10) or equivalently of (19) for all $i \in \{1,\ldots,N\}$ implies that $\lambda P_i - \frac{1}{\nu}\mathbf{1} \geq 0$ and thus

$$P_i \geq \frac{1}{\nu\lambda}\mathbf{1}. \tag{20}$$

Multiplying respectively left and right by $x_k^T$ and $x_k$ gives

$$V(x_k,\rho_k) \geq \frac{1}{\nu\lambda}\|x_k\|^2. \tag{21}$$

As the third step of the proof, it's a simple matter to see that the consideration of (16) and (21) proves (12) while the consideration of (18) and (21) proves (13).
Finally, as the fourth step and to complete the proof, we proceed to explicitly compute the ISS gain and decay factor. Let us assume that (10) or equivalently (19) is feasible for all $i \in \{1,\ldots,N\}$. Multiplying successively (19) by $\xi_k^{(i)}$ and summing, and by $\xi_{k+1}^{(j)}$ and summing yields

$$\begin{bmatrix} \lambda\mathscr{P}_k - \frac{1}{\nu}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\nu-\mu)\mathbf{1} \end{bmatrix} \geq 0.$$

Hence, for all $x_k \in \mathbb{R}^n$ and all $v_k \in \mathbb{R}^n$, this implies that

$$\lambda V(x_k,\rho_k) \geq \frac{1}{\nu}\|x_k\|^2 - (\nu-\mu)\|v_k\|^2. \tag{22}$$

We also assume that the LMIs (9) are feasible. Thus, (15) holds and we get that

$$V(x_{k+1},\rho_{k+1}) \leq (1-\lambda)V(x_k,\rho_k) + \mu\|v_k\|^2. \tag{23}$$

Applying (23) successively leads to

$$\begin{aligned} V(x_k,\rho_k) &\leq (1-\lambda)^kV(x_0,\rho_0) + \\ &\quad \mu\textstyle\sum_{l=0}^{k-1}(1-\lambda)^{k-l-1}\|v_l\|^2 \\ &\leq (1-\lambda)^kV(x_0,\rho_0) + \frac{\mu}{\lambda}\|v\|_\infty^2. \end{aligned} \tag{24}$$

From (24) and (22), we get that:

$$\frac{1}{\nu}\|x_k\|^2 \leq \lambda(1-\lambda)^kV(x_0,\rho_0) + \mu\|v\|_\infty^2 + (\nu-\mu)\|v_k\|^2.$$

Finally, by using again (18) and taking the square root, the inequality

$$\|x_k\| \leq \sqrt{\nu\lambda\mu}(1-\lambda)^{k/2}\|x_0\| + \nu\|v\|_\infty \tag{25}$$

is obtained. This inequality shows that (4) is ISS with respect to $v_k$. Furthermore, in view of Definition 4, we can infer that the ISS gain is linear with $\gamma(s)=\nu s$ and that the decay factor is $(1-\lambda)^{1/2}$. ■

It is worth pointing out that the Matrix Inequalities (9)-(10) are really tractable. Indeed, for the nonlinearity due to the product $\lambda P_i$ in (9), the range of $\lambda \in ]0,1[$ is known and bounded. Hence, a simple line search (gradient method or bisection), following the same line as [1] in a different context, can be used. A simple gridding can also be used. If so, (9)-(10) can be solved for every prescribed "gridded" $\lambda$ and boil down to Linear Matrix Inequalities. Such a remark is central when it comes to synthesis.

## III. Comparative study

In this section, we show that the conditions proposed in this paper are less conservative than the ones provided so far in the literature, and we recall the condition, proposed in [4] [7], which is known, to the best of our knowledge, as the less conservative one for polytopic LPV discrete-time systems.

**Theorem 3:** If there exist positive definite symmetric matrices $P_i$, matrices $G_i$, a real scalar $\sigma_e \geq 1$, fulfilling $\forall (i,j) \in (1,\ldots,N)\times(1,\ldots,N)$, the Matrix Inequalities

$$\begin{bmatrix} G_i^T+G_i-P_j & (\bullet)^T & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mathbf{1} & (\bullet)^T & (\bullet)^T \\ G_iA_i & \mathbf{1} & P_i & (\bullet)^T \\ G_i & \mathbf{0} & \mathbf{0} & \sigma_e\mathbf{1} \end{bmatrix} > 0, \quad (26)$$

then the system (4) is ISS with respect to $v_k$ and

$$\|x_k\| \leq \sqrt{\sigma_e}(1-\frac{1}{\sigma_e})^{k/2}\|x_0\| + \sigma_e\|v\|_\infty. \quad (27)$$

The proof is given in [4] [7]. It is shown that (26) ensures the existence of an ISS Lyapunov function $V:\mathbb{R}^n\times\mathbb{R}^L\to\mathbb{R}_+$, $V(x_k,\rho_k)=x_k^T\mathscr{P}_kx_k$ with $\mathscr{P}_k=\sum_{i=1}^N\xi_k^{(i)}(\rho_k)P_i$ which verifies for all $x_k\in\mathbb{R}^n$, all $v_k\in\mathbb{R}^n$ and all $\xi\in\mathscr{S}$ of (4), the following conditions

$$\|x_k\|^2 \leq V(x_k,\rho_k) \leq \sigma_e\|x_k\|^2, \quad (28)$$

$$V(x_{k+1},\rho_{k+1})-V(x_k,\rho_k) \leq -\|x_k\|^2+\sigma_e\|v_k\|^2. \quad (29)$$

The existence of $V$ fulfilling (28) and (29) is sufficient to obtain (27).

The following Proposition aims at showing that the conditions (9)-(10) of the proposed Theorem 2 are less conservative than (26) of Theorem 3.

**Proposition 1:** For the triplet $(\nu,\lambda,\mu)=(\sigma_e,\frac{1}{\sigma_e},\sigma_e)$, the inequality (25) reduces to the inequality (27). Furthermore, the solution of (26) is also a solution of (9)-(10).

**Proof 2:** The first part of Proposition 1 is straightforward and proved by a simple substitution. For the second part, we must first notice that (26) can be equivalently, by permuting rows and columns, rewritten as

$$\begin{bmatrix} P_i & (\bullet)^T & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \sigma_e\mathbf{1} & (\bullet)^T & (\bullet)^T \\ G_iA_i & G_i & G_i^T+G_i-P_j & (\bullet)^T \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} > 0. \quad (30)$$

By Schur's formula, (30) turns into

$$\begin{bmatrix} P_i-\mathbf{1} & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \sigma_e\mathbf{1} & (\bullet)^T \\ G_iA_i & G_i & G_i^T+G_i-P_j \end{bmatrix} > 0. \quad (31)$$

To prove that the feasibility of (31) implies the feasability of (9), it is sufficient to first prove that, whenever (31) is fulfilled, there always exists $\lambda=\frac{1}{\sigma_e}$ and $\mu=\sigma_e$ such that

$$\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ G_iA_i & G_i & G_i^T+G_i-P_j \end{bmatrix} - \begin{bmatrix} P_i-\mathbf{1} & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \sigma_e\mathbf{1} & (\bullet)^T \\ G_iA_i & G_i & G_i^T+G_i-P_j \end{bmatrix} \geq 0.$$

It is equivalent to show that there always exists $\lambda=\frac{1}{\sigma_e}$ and $\mu=\sigma_e$ such that $(1-\lambda)P_i \geq P_i-\mathbf{1}$ and $\mu-\sigma_e\geq 0$, or equivalently

$$\mathbf{1} \geq \lambda P_i \quad (32)$$

and

$$\mu \geq \sigma_e. \quad (33)$$

The inequality (33) is clearly fulfilled if $\mu=\sigma_e$. As for inequality (32), since (31) is assumed to be fulfilled, one has

$$\begin{bmatrix} \sigma_e\mathbf{1} & G_i^T \\ G_i & G_i^T+G_i-P_j \end{bmatrix} > 0, \quad (34)$$

which, following the same line of reasoning as in the first step of the proof of Theorem 2, is equivalent to

$$\begin{bmatrix} \sigma_e\mathbf{1} & P_j \\ P_j & P_j \end{bmatrix} > 0, \quad (35)$$

or equivalently

$$P_i < \sigma_e\mathbf{1}. \quad (36)$$

And yet, (36) implies (32) for $\lambda=\frac{1}{\sigma_e}$.

Next, it must be proved that, for $\lambda=\frac{1}{\sigma_e}$, $\mu=\sigma_e$ and $\nu=\sigma_e$, (10) is also fulfilled, or equivalently, by Schur's formula,

$$\begin{bmatrix} \lambda P_i-\frac{1}{\nu}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\nu-\mu)\mathbf{1} \end{bmatrix} \geq 0 \quad (37)$$

is fulfilled with $\lambda=\frac{1}{\sigma_e}$ and $\mu=\sigma_e$.

First, let us notice that the inequality $(\nu-\mu)\mathbf{1}\geq 0$ is obvious. Next, from (31), we can infer that

$$P_i > \mathbf{1}. \quad (38)$$

Besides, for $\nu=\sigma_e$ and $\lambda=\frac{1}{\sigma_e}$, it holds that $\lambda P_i-\frac{1}{\nu}\mathbf{1}=\frac{1}{\sigma_e}(P_i-1)$ and so, with (38), (37) is clearly fulfilled. That completes the proof. ■

Now, we go further by showing that the solution of (9)-(10) always gives a decay rate and an ISS gain lower than the solution of (26).

**Proposition 2:** There always exist $\delta_1\geq 0$, $\delta_2\geq 0$ and $\delta_3\geq 0$ such that $\lambda=\frac{1}{\sigma_e}+\delta_1$, $\mu=\sigma_e-\delta_2$ and $\nu=\sigma_e-\delta_3$ are solution of (9)-(10)

The interpretation of Proposition 2 is the following. $\delta_1>0$ corresponds to better decay rate (faster transient time) and $\delta_3>0$ corresponds to a better disturbance rejection. As a result, considering the optimal solution of Theorem 3, we can find out a better decay rate (if $\delta_1 > 0$) and a better ISS gain (if $\delta_3 > 0$).

**Proof 3:** Assume that (26) is feasible. Hence, (9) is also feasible with the triplet $(\nu,\lambda,\mu) = (\sigma_e, \frac{1}{\sigma_e}, \sigma_e)$ as previously shown. Then, there always exists a positive semi-definite matrix $P_\varepsilon = diag(\varepsilon_1 \mathbf{1}_n, \varepsilon_2 \mathbf{1}_n, \mathbf{0})P_i$ ($\mathbf{1}_n$ stands for the identity matrix of dimension $n$) such that

$$\begin{bmatrix} (1-\frac{1}{\sigma_e})P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \sigma_e \mathbf{1} & (\bullet)^T \\ G_i A_i & G_i & G_i^T + G_i - P_j \end{bmatrix} > P_\varepsilon,$$

which is equivalent to

$$\begin{bmatrix} (1-\frac{1}{\sigma_e}-\varepsilon_1)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \sigma_e \mathbf{1} - \varepsilon_2 P_i & (\bullet)^T \\ G_i A_i & G_i & G_i^T + G_i - P_j \end{bmatrix} > 0.$$

Hence, (9) is feasible with $\lambda = \frac{1}{\sigma_e} + \varepsilon_1$. Taking into account (36), (9) is feasible with $\mu = \sigma_e - \sigma_e \varepsilon_2$. As a result, $\delta_1 = \varepsilon_1 \geq 0$ and $\delta_2 = \sigma_e \varepsilon_2 \geq 0$.

For this special setting, the inequality (10), or equivalently (37), must still hold. Consider the upper left diagonal block with $\lambda = \frac{1}{\sigma_e} + \delta_1$ and consider $\delta_3$ defined as $\nu = \sigma_e - \delta_3$. The inequality is fulfilled whenever $\delta_3 \mathbf{1} \leq \sigma_e \mathbf{1} - \frac{1}{(\frac{1}{\sigma_e}+\delta_1)} P_i^{-1}$. Taking into account (38), it follows that $\delta_3 \leq \sigma_e - (\frac{1}{\sigma_e} + \delta_1)^{-1}$ is still admissible.

Finally, let us focus on the lower right diagonal block of (37) with $\nu = \sigma_e - \delta_3$ and $\mu = \sigma_e - \delta_2$. The inequality $(\nu - \mu)\mathbf{1} \geq 0$ is still feasible whenever $\delta_3 \leq \delta_2$. As a result, $\delta_3$ is clearly positive and must fulfill $\delta_3 = min(\sigma_e \varepsilon_2, \sigma_e - \frac{1}{(\frac{1}{\sigma_e}+\delta_1)})$ ■

## IV. Observer design

In this section, we consider a synthesis problem and more specifically an observer design. A quite similar treatment holds for control but is not addressed here due to the limited size of the paper. We consider LPV systems of the form

$$\begin{cases} x_{k+1} = A(\rho_k)x_k + Bu_k + w_k^d \\ y_k = Cx_k + Du_k + Hw_k^o, \end{cases} \tag{39}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, $y_k \in \mathbb{R}^p$ is the output vector. $A \in \mathbb{R}^{n\times n}$ is the dynamical matrix depending on the possibly time varying parameter vector $\rho_k \in \mathbb{R}^L$, $C \in \mathbb{R}^{p\times n}$ is the output matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix. Disturbances are splitted into the disturbance $w_k^d \in \mathbb{R}^n$ acting on the dynamics and the disturbance $w_k^o \in \mathbb{R}^{d_{wo}}$ acting on the output. The dependence of $A(\rho_k)$ with respect to $\rho_k$ is assumed to be polytopic. We should point out that the extension to LPV systems with time varying matrices $B$, $C$ and $D$ can be carried out with little effort (see a possible approach in [17] for example). For the purpose of state reconstruction of (39), we propose the following polytopic observer

$$\begin{cases} \hat{x}_{k+1} = A(\rho_k)\hat{x}_k + Bu_k + L(\rho_k)(y_k - \hat{y}_k) \\ \hat{y}_k = C\hat{x}_k + Du_k, \end{cases} \tag{40}$$

where $L$ is a time varying gain matrix depending on $\rho_k$

$$L(\rho_k) = \sum_{i=1}^{N} \xi^{(i)}(\rho_k) L_i, \quad \xi \in \mathscr{S}, \tag{41}$$

and where the $\xi^{(i)}(\rho_k)$ in (41) coincide, for every discrete time $k$, with the ones involved in the polytopic decomposition of $A(\rho_k)$.

It's a simple matter to derive, from (39) and (40), the reconstruction error $e_k = x_k - \hat{x}_k$

$$e_{k+1} = (A(\rho_k) - L(\rho_k)C)e_k + v_k, \tag{42}$$

with $v_k = w_k^d - L(\rho_k)w_k^o$.

We aim at computing the gain $L$ to guarantee both stability (with possible performance guarantees) and robustness with respect to $v_k$. To this end, restating Theorem 3 and Theorem 2 for synthesis purpose yields the following design problems. Let us stress that Matrix Inequalities (43) and (44) are obtained from the ones involved in Theorem 3 and Theorem 2 after, as usual, replacing $A_i$ by $A_i - L_i C$, changing the variable $F_i = G_i L_i$ which gives the solution $L_i = G_i^{-1} F_i$ (let us recall that $G_i$ is invertible).

*Problem 1* (Theorem 3)

$$\min_{\sigma_e, G_i, P_i, F_i \;\; i,j \in \{1,\ldots,N\}} J_{\sigma_e} = \sigma_e$$

$$\text{s.t.} \begin{bmatrix} G_i^T + G_i - P_j & (\bullet)^T & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mathbf{1} & (\bullet)^T & (\bullet)^T \\ G_i A_i - F_i C & \mathbf{1} & P_i & (\bullet)^T \\ G_i & \mathbf{0} & \mathbf{0} & \sigma_e \mathbf{1} \end{bmatrix} > 0. \tag{43}$$

It is clear that minimizing $J_{\sigma_e}$ does not allow to minimize independently the decay factor and the ISS gain, see (27).

*Problems 2* (Theorem 2)

$$\min_{\lambda,\nu,\mu,G_i,P_i,F_i \;\; i,j \in \{1,\ldots,N\}} J_{\lambda,\nu} = -\lambda, \nu$$

$$\text{or} \quad \min_{\lambda,\nu,\mu,G_i,P_i,F_i \;\; i,j \in \{1,\ldots,N\}} J_{\nu,\lambda=\lambda_0} = \nu$$

$$\text{or} \quad \min_{\lambda,\nu,\mu,G_i,P_i,F_i \;\; i,j \in \{1,\ldots,N\}} J_{\lambda,\nu=\nu_0} = -\lambda$$

$$\text{s.t.} \begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu \mathbf{1} & (\bullet)^T \\ G_i A_i - F_i C & G_i & G_i^T + G_i - P_j \end{bmatrix} > 0 \tag{44}$$

and

$$\begin{bmatrix} \lambda P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & (\nu-\mu)\mathbf{1} & (\bullet)^T \\ \mathbf{1} & \mathbf{0} & \nu \mathbf{1} \end{bmatrix} \geq 0. \tag{45}$$

If min $J_{\lambda,\nu}$ is considered, clearly, the decay factor and the ISS gain can be simultaneously minimized and can be different in magnitude. The consideration of min $J_{\nu,\lambda=\lambda_0}$ or min $J_{\lambda,\nu=\nu_0}$ corresponds respectively to the problems of minimizing the ISS gain for a prescribed decay rate $\lambda = \lambda_0$ or minimizing the decay rate for a prescribed $\nu = \nu_0$.

## V. Examples

Let us compare, on two examples, the solutions of Problem 1 and Problems 2 to highlight the benefit of Theorem 2 compared to Theorem 3.

### A. Example 1

Let us consider the system investigated in [2], the airpath of a turbocharged Spark Ignition engine. Here, we focus on the subsystem from which the air flow trapped into the cylinders can be estimated. The state space realization of the discretized model obeys the form (39) with

$$A(\rho_k) = \begin{bmatrix} 1 & \rho_k \\ 0 & 1 \end{bmatrix}, \; B(\rho_k) = \begin{bmatrix} -\rho_k & \rho_k & \rho_k \\ 0 & 0 & 0 \end{bmatrix},$$

and $\rho_k$ is a time-varying parameter lying in the range $[\rho_{min} \quad \rho_{max}] = [-\ 3.3453 \quad -\ 0.0174]$ which clearly admits a polytopic description with two vertices $A_1$ and $A_2$ corresponding respectively to $\rho_{min}$ and $\rho_{max}$.

For the state reconstruction of $x_k$, a polytopic observer of the form (40) is proposed and (42) is obtained.

First, let us solve min $J_{\sigma_e}$. The optimal solution is given by $\sigma_e = 13401$ with observer gains $L_1 = [2.8856 \quad -\ 0.5637]^T$ and $L_2 = [1.0101 \quad -\ 0.5779]^T$.

Next, let us solve min $J_{\lambda,\nu}$. The best ISS gain corresponds to $\nu = 595.93$, $\lambda = 7.3\ 10^{-3}$ and $\mu = 595.92$. The gains are given by $L_1 = [2.4 \quad -0.42]^T$ and $L_2 = [1.007 \quad -0.42]^T$. As expected, $\nu < \sigma_e$ and $\lambda > 1/\sigma_e = 7.46\ 10^{-5}$, considering the discussion on conservativeness of (9)-(10).

Now, let us set $\nu = \sigma_e = 13401$. The solution of min $J_{\lambda,\nu=\sigma_e}$ is $\lambda = 0.02$ which is clearly greater that $1/\sigma_e = 7.46\ 10^{-5}$. It means that for the best ISS gain obtained with Theorem 3, with the new condition proposed in this paper, a much better decay rate can be obtained with Theorem 2.

### B. Example 2

Let us consider the example given in the paper [6] which addresses the issue of output-based controller design for discrete-time LPV systems with uncertain parameters. A separate design of state observers and input-to-state stabilizing state is proposed. Here, we only consider the observer synthesis step. The state space realization of the LPV system obeys the form (39) with

$$A(\rho_k) = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6+\rho_k \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [1\ 0\ 2], \quad D = 0,\ H = 0$$

and $\rho_k \in [0, 0.5]$. A polytopic description involving two vertices $A_1$, for $\rho_k = 0$, and $A_2$, for $\rho_k = 0.5$, can be obtained. The observer proposed in [6] is designed by solving min $J_{\sigma_e}$. The optimal solution is given by $\sigma_e = 5.8277$ with observer gains $L_1 = \begin{bmatrix} -0.0835 & -0.0011 & 0.3870 \end{bmatrix}^T$ and $L_2 = \begin{bmatrix} -0.0835 & -0.0011 & 0.7094 \end{bmatrix}^T$. Next, let us solve min $J_{\lambda,\nu}$. We get the solution $\nu = 3.0097$, $\lambda = 0.44$, $\mu = 3.0096$ that leads to the observer gains $L_1 = \begin{bmatrix} 0.0573 & 0.0013 & 0.2398 \end{bmatrix}^T$, $L_2 = \begin{bmatrix} 0.0418 & 0.0013 & 0.4498 \end{bmatrix}^T$. Again, we have that $\nu < \sigma_e$ and $\lambda > 1/\sigma_e = 0.1716$.

Now, let us set $\nu = \sigma_e = 5.8277$. The solution of min $J_{\lambda,\nu=\sigma_e}$ is $\lambda = 0.8$ which is clearly greater that $1/\sigma_e = 0.1716$. It means that for the best ISS gain obtained with Theorem 3, with the new condition proposed in this paper, a much better decay rate can be obtained with Theorem 2..

## VI. Conclusion

We have derived ISS conditions for disturbed linear polytopic discrete-time systems. The conditions provides scalable decay rate and ISS gain. They are expressed in terms of tractable Matrix Inequalities. A comparative study has shown that the conditions are more general than the ones available so far. Benefiting from an increase of flexibility, the conditions are suitable for reducing the bounds for analysis purposes but also to improve the performances for control or state reconstruction perspectives. A similar treatment applies in the context of uncertain systems.

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