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Mixed Frequency Structured AR Model Identification

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***Abstract*— This paper is concerned with identifiability of an underlying high frequency multivariate stable singular AR system from mixed frequency observations. Such problems arise for instance in economics when some variables are observed monthly whereas others are observed quarterly. In particular, this paper studies stable singular AR systems where the covariance matrix associated with the vector obtained by stacking observation vector, y_t, and its lags from the first lag to the p-th one (p is the order of the AR system), is also singular. To deal with this, it is assumed that the column degrees of the associated polynomial matrix are known. We consider first that there are given nonzero unequal column degrees and we show generic identifiability of the system and noise parameters. Then we extend the results to allow zero column degrees corresponding to fast components. In this case, we first show generic identifiability of the subsystem of the components with nonzero column degree. Then we show how to obtain those components of the parameter matrices of the components corresponding to zero column degree by regression.**

I. INTRODUCTION

When one is working with high-dimensional time series it is very common to encounter situations where some components of the time series are available at every point of time whereas others might be unavailable at some points of time, but still available periodically. This situation commonly arises in econometric modeling when one is dealing with GDP data and unemployment data or interest rates. In an econometric modeling context, the term mixed frequency is used to refer to the associated multivariate time series.

One way of handling mixed frequency data is the method of blocking [1], where the authors show that a tall blocked linear time-invariant system derived from an underlying unblocked linear system with one or more missing outputs is generically zero-free. In this paper, we follow a different approach; we specifically study the method of [2]. Reference [2] postulates that there exists an autoregressive model operating at the highest sampling frequency and shows that the parameters of this model are generically identifiable from those population second order moments which can be observed in principle[1]. Equivalently, such a model is identifiable on a superset of an open and dense subset of the parameter space.

It is worthwhile mentioning here that the covariance matrix associated with input sequences of an autoregressive model can be either regular or singular. The terms *regular autoregressive model* and *singular autoregressive model* are used to refer to the former and latter, accordingly. Moreover, with recent theoretical advances in the field of econometric modeling, see [3] and [4], singular autoregressive models have become more popular in this area. In econometric modeling and forecasting exercises using generalized dynamic factor models (GDFMs) [5], the latent variable, i.e. those parts of observed data remaining after removal of contaminating additive noise in the measurement, are modeled as an singular autoregressive model. In this paper we focus on this type of models.

This paper is in fact a continuation of our work in [2]. One should note that the results in [2] show identifiability only on a subset of the parameter space where Γ, the covariance matrix associated with the vector $Y_t = \left[\begin{array}{cccc} y_t^T & y_{t-1}^T & \cdots & y_{t-p}^T \end{array} \right]^T$, where y_t is the observation vector and p is the order of the AR system, is restricted to being nonsingular. Note that Γ is always nonsingular for regular AR models but can be singular for singular AR models. In this paper, we show generic identifiability for singular autoregressive models where the covariance matrix Γ is singular on a restricted parameter space , viz. following the reference [6], we assume that the column degrees of the associated polynomial matrix are known and consider the parameter space where the highest degree of each column of the associated polynomial matrix is bounded by its prescribed column degree. As in [2] we use a modified version of the extended Yule-Walker equations proposed by Chen and Zadrozny in [7] to obtain a sufficient condition for identifiability and then show that this condition generically holds. If we have identifiability, the system and noise parameters and thus all second moments of the observed process can be estimated consistently from mixed frequency data. Then linear least squares methods for forecasting and interpolating nonobserved variables can be applied.

The rest of this paper is organized as follows. First, the

Some of the proofs in this paper are omitted due to space limitations; they are available from the first author upon request and will be given in a full length version of this paper.

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[1] Second order moments which are observed 'in principle' are those which can be consistently estimated from sample statistics when the number of samples goes to infinity.

problem formulation is provided. Then we focus on possible scenarios and study them separately. Considering AR polynomials with prescribed column degrees, we distinguish two cases: the case where the AR polynomial matrix $A(q)$ has columns with unequal degree but there is no column with zero column degree and another case where columns of $A(q)$ are permitted to have column degree zero. The former case is studied in Section III and the latter is explored in Section IV. Finally, Section V concludes.

II. PROBLEM FORMULATION

Consider the following AR system

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + A_3 y_{t-3} + \ldots + A_p y_{t-p} + \nu_t, \quad (1)$$

where y_t is an $\mathbb{R}^n$ valued random variable. We assume that y_t consists of two parts; the fast components, y_t^f for $t \in \mathbb{Z}$, which are n^f-dimensional, and the slow components, y_t^s which are available for $t \in N\mathbb{Z}$ for some positive integer $N \geq 2$, which are n^s-dimensional, and $n^f + n^s = n$. The innovation ν_t, which is orthogonal to $y_{t-j}, j \geq 1$ is white noise and its covariance is $\mathbb{E}[\nu_t \nu_u^T] = \Sigma \delta_{tu}$ for all t, u, where δ_{tu} is the Kronecker delta, which is 1 for $t = u$ and 0 otherwise, $rank(\Sigma) = r < n$. (Note that the case $rank(\Sigma) = n$ implies that Γ is nonsingular and thus this case is already treated in [2].) We can write $\Sigma = bb^T$ where b is an $n \times r$ matrix. Accordingly, $\nu_t = b\varepsilon_t$, where $\mathbb{E}\left[\varepsilon_t \varepsilon_t^T\right] = I_r$. For given Σ, b is unique up to postmultiplication by an orthogonal matrix. Moreover, the AR polynomial matrix is $A(q) = I - A_1 q - \cdots - A_p q^p$ where $q y_t = y_{t-1}$ and $z = q^{-1}$, so q is the backward shift. Throughout we assume that the high frequency system (1) is stable, and that we restrict ourselves to the steady state and thus stationary solutions.

Now it is convenient to define the state variable x_t as below and rewrite the equation (1) in the state space form as

$$\underbrace{\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}}_{x_t} = \underbrace{\begin{bmatrix} A_1 & A_2 & \ldots & A_p \\ I & 0 & \ldots & 0 \\ & \ddots & \ddots & \\ 0 & 0 & I & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix}}_{x_{t-1}} + \underbrace{\begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathcal{B}} \epsilon_t. \quad (2)$$

In our previous work [2] we studied identifiability, i.e. whether the system parameters $(A_1, \ldots, A_p)$, and noise parameters Σ can be uniquely determined from those population second moments which can be observed in principle, these being $\gamma^{ff}(h) = \mathbb{E}\left[y_{t+h}^f \left(y_t^f\right)^T\right], \ h \in \mathbb{Z};\ \gamma^{fs}(h) = \mathbb{E}\left[y_{t+h}^f \left(y_t^s\right)^T\right], \ h \in \mathbb{Z};\ \gamma^{ss}(h) = \mathbb{E}\left[y_{t+h}^s \left(y_t^s\right)^T\right], \ h \in N\mathbb{Z}$. In reference [2], the main theorem showed identifiability of the system and noise parameters on a generic set. This generic set was the set of all stable AR systems where A_p was nonsingular and the eigenvalues of $\mathcal{A}$ had multiplicity one. The assumption that A_p is nonsingular implies the property that all column degrees in $A(q)$ are equal to p, which is restrictive. However, here we study the scenario where at least one column degree is less than p and thus A_p is singular (though nonzero). We assume to be given prescribed column degrees of $A(q)$. The parameters in the coefficient matrices of $A(q)$ not forced to be zero by the column degree restriction together with the $nr - \frac{r(r-1)}{2}$ free entries of Σ, or b, then define our parameter space.

In the following we consider two possible scenarios which may happen. First, we study the case where $A(q)$ has columns with unequal degree but there is no column with column degree zero. Second, a situation where columns of $A(q)$ are permitted to have column degree zero is explored. In the following section the former case is studied and in Section IV the latter case is investigated.

III. AR SYSTEMS WITH UNEQUAL COLUMN DEGREE IN $A(q)$

Throughout the paper, we assume for convenience and without loss of essential generality that the components of y_t are ordered such that the column degrees of $A(q)$, the AR polynomial, are decreasing, i.e., $p_i \geq p_j$ when $i < j$ where i and j denotes a column numbers in $A(q)$ and p_i and p_j are the corresponding column degrees. Also, in this section we further assume that $p_j > 0$.

Now, to deal with the situation stated above the following state space form is studied

$$\overline{x}_t = \overline{\mathcal{A}} \overline{x}_{t-1} + \overline{\mathcal{B}} \epsilon_t, \quad (3)$$

where $\overline{\mathcal{A}}$ is obtained from $\mathcal{A}$ by deleting columns corresponding to prescribed zero columns in $A_1, \ldots, A_p$ and corresponding rows. This has been called the quasi companion form in [6]. Here, $\overline{x}_t$ and $\overline{\mathcal{B}}$ are those entries of x_t and $\mathcal{B}$ respectively associated with $\overline{\mathcal{A}}$. Furthermore, it is easy to verify that

$$y_t = [\overline{A}_1 \overline{A}_2 \ \ldots \overline{A}_p] \overline{x}_{t-1} + b\epsilon_t. \quad (4)$$

where $[\overline{A}_1 \overline{A}_2 \ \ldots \overline{A}_p]$ is obtained by taking the first n rows of $\overline{\mathcal{A}}$. Note that $\overline{A}_1 = A_1$ since no column degree is prescribed to be zero; however, $\overline{A}_i, i \in \{2, \ldots, p\}$, may have fewer columns than A_i. Since we deleted only prescribed zero columns, identifiability of the system parameters is equivalent to identifiability of $(\overline{A}_1 \overline{A}_2 \ \ldots \overline{A}_p)$. In the following, we first show that the parameter matrices $\overline{A}_i$ and Σ are generically identifiable from those population second moments which can be observed in principle.

A. Modified Extended Yule-Walker Equations

Consider the system (4); then it is easy to see that

$$\mathbb{E}[y_t \overline{x}_{t-1}^T] = [\overline{A}_1 \ \overline{A}_2 \ \ldots \overline{A}_p] \overline{\Gamma}, \quad (5)$$

where $\overline{\Gamma} = \mathbb{E}[\overline{x}_{t-1} \overline{x}_{t-1}^T]$. Observe that provided $\overline{\Gamma}$ and $\mathbb{E}[y_t \overline{x}_{t-1}^T]$ are known and $\overline{\Gamma}$ is nonsingular, we can identify the parameters $\overline{A}_i$ easily using (5). However, we have difficulties in directly using (5) because y_t^s is not available at all times and consequently some entries of the matrices on both sides of (5) will be missing. In the rest of this subsection we first show how those population second moments, which can be observed in principle, can be used to determine the $\overline{A}_i$. Then, later we use these results to show generic identifiability of Σ.

To overcome the problem of missing covariance data we consider equation (4) and postmultiply both sides by $(y^f_{t-j})^T, j > 0$, the fast components, to obtain our extended Yule Walker equations as in [2]:

$$\begin{aligned}&\mathbb{E}\left[y_t\left((y^f_{t-1})^T,(y^f_{t-2})^T,\dots\right)\right]\\ =&[\overline{A_1},\dots,\overline{A_p}]\mathbb{E}\left[\overline{x}_{t-1}\left((y^f_{t-1})^T,(y^f_{t-2})^T,\dots\right)\right].\end{aligned} \tag{6}$$

From equation (3) we obtain for the second multiplicand on the right hand side

$$\begin{aligned}\mathbb{E}\left[\overline{x}_t y^T_{t-j}\right]&=\mathbb{E}\left[\left(\overline{\mathcal{A}}\overline{x}_{t-1}+\overline{\mathcal{B}}\overline{\epsilon}_t\right)y_{t-j}\right]^T=\\ \overline{\mathcal{A}}\mathbb{E}\left[\overline{x}_{t-1}y^T_{t-j}\right]&=\dots=\overline{\mathcal{A}}^j\mathbb{E}\left[\overline{x}_{t-j}y^T_{t-j}\right]=\overline{\mathcal{A}}^j\overline{\Gamma}\begin{pmatrix}I_n\\0\end{pmatrix}.\end{aligned} \tag{7}$$

We now can define $K=\overline{\Gamma}\begin{pmatrix}I_{n^f}\\0\end{pmatrix}$.

The rightmost matrix in (6) can be written as $(K,\overline{\mathcal{A}}K,\overline{\mathcal{A}}^2K,\dots)$. Now as $\overline{\mathcal{A}}\in\mathbb{R}^{(pn-s)\times(pn-s)}$, where s is the number of prescribed zero columns in $(A_1,\dots,A_p)$, using the well known Cayley-Hamilton theorem, it follows that this matrix has full row rank if and only if the following matrix which contains only the first $pn-s$ block columns has full row rank:

$$\mathcal{Z}^f=(K,\overline{\mathcal{A}}K,\overline{\mathcal{A}}^2K,\dots,\overline{\mathcal{A}}^{pn-s-1}K). \tag{8}$$

Now obviously the parameter matrices $\overline{A}_i$ are identifiable if the matrix $\mathcal{Z}^f$ has full row rank.

B. Generic Identifiability

In this paper we follow the same definition of generic identifiability as in reference [2]. Consider the parameter space associated with the system (4). Then a property is said to hold generically on the parameter space if it holds on a superset of an open and dense subset of the parameter space. Here, in what follows we first study the generic identifiability of the system parameters $(\overline{A}_1,\dots,\overline{A}_p)$ from those second moments which are observed in principle. Then later in Subsection III-B.2 the generic identifiability of noise parameters i.e. the entries of Σ, is examined.

1) Generic Identifiability of the System Parameters: In this subsection it is shown that $\mathcal{Z}^f$ has generically full row rank and thus we have generic identifiability of the system parameters $(\overline{A}_1\overline{A}_2\dots\overline{A}_p)$ for prescribed column degrees.

Lemma 3.1: Let $\overline{\mathcal{A}}$ denote the block matrix defined above and let $A(z)$ denote the polynomial matrix $z^pI-A_1z^{p-1}-\dots-A_p$ with $z=q^{-1}$. Suppose that $\bar{\alpha}^T=[\bar{\alpha}^T_1\ \dots\ \bar{\alpha}^T_p]$ where $\bar{\alpha}_i$ has dimension equal to the number of columns of $\overline{A}_i$, is a left eigenvector of $\overline{\mathcal{A}}$ corresponding to eigenvalue λ. Then $\bar{\alpha}^T_1$ is in the left kernel of $A(\lambda)$ i.e. $\bar{\alpha}^T_1A(\lambda)=0$. Conversely, if $\bar{\alpha}^T_1\neq0$ is such that $\bar{\alpha}^T_1A(\lambda)=0$ for some λ, and $\underbrace{[\bar{\alpha}^T_2\,0\dots0]}_{\alpha_2}=\bar{\alpha}^T_1(\lambda I-A_1)$, $\underbrace{[\bar{\alpha}^T_3\,0\dots0]}_{\alpha_3}=\bar{\alpha}^T_1(\lambda^2I-\lambda A_1-A_2)$, ..., then $\bar{\alpha}^T=[\bar{\alpha}^T_1\ \bar{\alpha}^T_2\ \bar{\alpha}^T_3\ \dots\ \bar{\alpha}^T_p]$ is a left eigenvector of $\overline{\mathcal{A}}$ corresponding to eigenvalue λ; here, the number of zero entries in α_i, $i=2,\dots,p$, is equal to the number of prescribed zero columns in A_i.

Proof: Suppose that $\bar{\alpha}^T$ is a left eigenvector of $\overline{\mathcal{A}}$ associated with eigenvalue λ. Then it is evident that

$$\begin{aligned}\bar{\alpha}^T_1\overline{A}_1+[\bar{\alpha}^T_20\dots0]&=\lambda\bar{\alpha}^T_1,\\ \bar{\alpha}^T_1\overline{A}_2+[\bar{\alpha}^T_30\dots0]&=\lambda\bar{\alpha}^T_2,\\ &\vdots\\ \bar{\alpha}^T_1\overline{A}_{p-1}+[\bar{\alpha}^T_p0\dots0]&=\lambda\bar{\alpha}^T_{p-1},\\ \bar{\alpha}^T_1\overline{A}_p&=\lambda\bar{\alpha}^T_p.\end{aligned} \tag{9}$$

In the above equations, in the second summand on the left hand side, the $\bar{\alpha}_i$ are augmented with zeros so the above equalities can hold with dimensional consistency. Now if $\bar{\alpha}_1$ were zero then all $\bar{\alpha}_i$ would turn out to be zero which would be a contradiction. Thus, $\bar{\alpha}_1\neq0$. Now, let α_i denote the vector consisting of $\bar{\alpha}_i$ augmented with zeros if necessary to make its dimension equal to $dim\ y_t$. Note that we necessarily have $\bar{\alpha}_1=\alpha_1$. Then it is obvious that the following equations hold:

$$\begin{aligned}\bar{\alpha}^T_1A_1+\alpha^T_2&=\lambda\bar{\alpha}^T_1,\\ \bar{\alpha}^T_1A_2+\alpha^T_3&=\lambda\alpha^T_2,\\ &\vdots\\ \bar{\alpha}^T_1A_{p-1}+\alpha^T_p&=\lambda\alpha^T_{p-1},\\ \bar{\alpha}^T_1A_p&=\lambda\alpha^T_p.\end{aligned} \tag{10}$$

Thus, it is easy to obtain $\bar{\alpha}_1A(\lambda)=0$. Conversely, suppose that $\bar{\alpha}_1A(\lambda)=0$ holds for some nonzero λ. Then by defining $\bar{\alpha}_i$ according to the lemma statement, it can be easily verified that (9) holds and thus $\bar{\alpha}^T\overline{\mathcal{A}}=\lambda\bar{\alpha}^T$. ■

Theorem 3.2: The set $\mathcal{F}=\{[\overline{\mathcal{A}},\overline{\mathcal{B}}]|rank([zI-\overline{\mathcal{A}},\overline{\mathcal{B}}])=pn-s,\forall z\in\mathbb{C}\}$, where s is the number of prescribed zero columns in $(A_1,\dots,A_p)$, is open and dense in the set of all $\overline{\mathcal{A}},\overline{\mathcal{B}}$ satisfying the conditions described above and where $\overline{\mathcal{A}}$ corresponds to a stable $A(q)$.

Proof: The proof is omitted due to page limitation. ■

Lemma 3.3: Let $\overline{\mathcal{A}}$ and $A(z)$ denote the matrix and the polynomial matrix defined above, and let $\bar{c}$ be a right eigenvector of $\overline{\mathcal{A}}$ corresponding to eigenvalue $\lambda\neq0$. Partition $\bar{c}=[\bar{c}^T_1\,\bar{c}^T_2\ \dots\ \bar{c}^T_p]^T$ where $\bar{c}_i$ has the same number of entries as columns of $\overline{A}_i$. Then $\bar{c}_1\neq0$ is in the kernel of $A(\lambda)$.

Conversely, if $\bar{c}_1\neq0$ is such that $A(z)\bar{c}_1=0$ for some $\lambda\neq0$ and if $c_i=\lambda^{1-i}\bar{c}_1$ and $\bar{c}_i$ denotes the first n_i entries of c_i, where n_i is the number of columns in $\bar{A}_i$, then $\bar{c}=[\bar{c}^T_1\,\bar{c}^T_2\ \dots\ \bar{c}^T_p]^T$ is the right eigenvector of $\overline{\mathcal{A}}$ corresponding to eigenvalue $\lambda\neq0$.

Proof: The proof is omitted due to page limitation. ■

Theorem 3.4: Let $\overline{\mathcal{A}}$ be a matrix obtained from the procedure described above. Furthermore, let $\mathcal{E}_j$ denote a column vector of length equal to $dim\overline{\mathcal{A}}$ with one in the j-th position for $1 \leq j \leq dim\ y_t$ and zero elsewhere. Then the pair $(\overline{\mathcal{A}}, \mathcal{E}_j^T)$ is observable on a generic subset of the parameter space.

Proof: The proof is omitted due to page limitation. ■

A proof of a theorem like the following theorem was stated in [2]. Here, we modify the previous proof for our own purpose.

Theorem 3.5: The matrix $\mathcal{Z}^f$ has full row rank for a set of generic parameter matrices $\bar{A}_i$, $i = 1, 2, \ldots, p$.

Proof: Consider the system (3). Then observe that the following equality holds:

$$\overline{\Gamma} - \overline{\mathcal{A}}\,\overline{\Gamma}\,\overline{\mathcal{A}}^T = \overline{\mathcal{B}}\,\overline{\mathcal{B}}^T. \tag{11}$$

From (11) we obtain

$$(zI - \overline{\mathcal{A}})\overline{\Gamma}(z^{-1}I - \overline{\mathcal{A}})^T + \overline{\mathcal{A}}\,\overline{\Gamma}(z^{-1}I - \overline{\mathcal{A}}^T) + (zI - \overline{\mathcal{A}})\overline{\Gamma}\,\overline{\mathcal{A}}^T$$
$$= \overline{\mathcal{B}}\,\overline{\mathcal{B}}^T, \tag{12}$$
$$\overline{\Gamma} + (zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{A}}\,\overline{\Gamma} + \overline{\Gamma}\,\overline{\mathcal{A}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}$$
$$= (zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}\,\overline{\mathcal{B}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}, \tag{13}$$

where $\overline{\Gamma} = \mathbb{E}[\overline{x}_{t-1}\,\overline{x}_{t-1}^T]$. By pre- and postmultiplying (13) by $\mathcal{E}_1^T$ and $\mathcal{E}_1$ to get

$$\mathcal{E}_1^T\overline{\Gamma}\mathcal{E}_1 + \mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{A}}\,\overline{\Gamma}\mathcal{E}_1 + \mathcal{E}_1^T\overline{\Gamma}\,\overline{\mathcal{A}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}\mathcal{E}_1$$
$$= \mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}\,\overline{\mathcal{B}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}\mathcal{E}_1. \tag{14}$$

Using the results of Theorem 3.2 and Theorem 3.4, it is obvious that the pairs $(\overline{\mathcal{A}}, \overline{\mathcal{B}})$ and $(\overline{\mathcal{A}}, \mathcal{E}_1^T)$ respectively are reachable and observable respectively on a generic subset of the parameter space. Thus, $(\overline{\mathcal{A}}, \overline{\mathcal{B}}, \mathcal{E}_1^T)$ is minimal and $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}$ has McMillan degree equal to the dimension of $\overline{\mathcal{A}}$ i.e. $pn - s$. Since the McMillan degree remains unchanged under transposition and replacement of a variable by a Mobius transformation, $\overline{\mathcal{B}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}\mathcal{E}_1$ has the same McMillan degree. Furthermore, by the stability assumption of the underlying AR system and considering the genericity of the pair $(\overline{\mathcal{A}}, \overline{\mathcal{B}})$, we can conclude that there is no pole-zero cancelation in the product $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}\,\overline{\mathcal{B}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}\mathcal{E}_1$[2]. Thus, the McMillan degree of the product $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}\,\overline{\mathcal{B}}^T(z^{-1}I - \overline{\mathcal{A}}^T)^{-1}\mathcal{E}_1$ is equal to $2(np - s)$. Note that the nonconstant terms on the left hand side of (14) have the same McMillan degree and share no common poles. Therefore, $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{A}}\,\overline{\Gamma}\mathcal{E}_1$ has McMillan degree equal to $np - s$. Due to the fact that $(\overline{\mathcal{A}}, \mathcal{E}_1^T)$ is observable we can easily conclude that the pair $(\overline{\mathcal{A}}, \overline{\mathcal{A}}\,\overline{\Gamma}\mathcal{E}_1^T)$ is reachable; moreover, $\overline{\mathcal{A}}$ is nonsingular so, the reachability matrix, see [8], associated with the pair $(\overline{\mathcal{A}}, \overline{\mathcal{A}}\,\overline{\Gamma}\mathcal{E}_1^T)$ has the same rank as the reachability matrix associated with the pair $(\overline{\mathcal{A}}, \overline{\Gamma}\mathcal{E}_1^T)$. Hence, we can readily conclude that the pair $(\overline{\mathcal{A}},\ \overline{\Gamma}\mathcal{E}_1^T)$ is also reachable. Now recall the definition of K which is $K = \overline{\Gamma}\begin{pmatrix} I_{n^f} \\ 0 \end{pmatrix}$. Based on the definition of $\mathcal{E}_1^T$, it becomes obvious that we proved the conclusion of theorem for the case where $n^f = 1$ and it is trivial that the result can be generalized for an arbitrary value of n^f.

■

Hence, we have shown identifiability of $(\overline{A}_1\overline{A}_2\ \ldots \overline{A}_p)$ and thus of the system parameters $(A_1, \ldots, A_p)$.

2) Generic Identifiability of the Noise Parameters: In this part we show generic identifiability of the noise parameters Σ given that the entries of $(A_1, \ldots, A_p)$ are identifiable.

Theorem 3.6: The noise parameters Σ are generically identifiable from those population second moments which can be observed in principle.

The proof is analogous to the proof of Theorem 2 given in [2].

Note that for generic identifiability we only needed the subsystem $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ to be reachable, which is equivalent to $\bar{\Gamma}$ being nonsingular, whereas in [2] Γ had to be nonsingular for identifiability. Thus we have extended the results of [2] to the case where we have linear dependencies in x_t, but we have prescribed zero columns in $(A_1, \ldots, A_p)$. Also note that for prescribed column degrees $\bar{\Gamma}$ and $\mathcal{Z}^f$ are of the same (full) rank on a generic set.

IV. AR Systems with Zero Column Degree in $A(q)$

In the previous section we only considered AR systems whose columns of their AR polynomial matrix, $A(q)$ had unequal prescribed column degrees and no column degree was prescribed to be zero. In this section, we also permit column degrees of $A(q)$ to be prescribed to be zero. Here, we first define a subsystem from the AR system (1). We then discuss generic identifiability of this subsystem, which turns out to be an AR system, using ideas of the previous section. Finally, we end this section by explaining how to obtain the remainder of parameters.

Accordingly, define a subprocess y_t^r of y_t which consists of those components of y_t not corresponding to those columns of $A(q)$ with prescribed zero column degree.

Attention is first given to identifying those parameters associated with y_t^r. Then later we use a regression to obtain the rest of the parameters. Note that in general a marginalized AR process is not an AR process any more, but in the case of zero column degrees, the components of y_t^r are. This is explained in the following very straightforward lemma, for which the proof is omitted.

Lemma 4.1: Consider the AR process as defined in (1) and assume that its AR polynomial matrix $A(q)$ has one or more columns with zero degree. Then the process y_t^r obtained from deleting all components of y_t associated with columns of $A(q)$ with zero degree, is an AR process of the same order as (1).

[2]Observe that if the pair $(\overline{\mathcal{A}}, \overline{\mathcal{B}})$ is generic, so is the pair $(\overline{\mathcal{A}} + \overline{\mathcal{B}}F, \overline{\mathcal{B}})$, for any fixed but arbitrary F of the proper dimension. Moreover, while the poles of $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}$ and $\mathcal{E}_1^T(zI - \overline{\mathcal{A}} - \overline{\mathcal{B}}F)^{-1}\overline{\mathcal{B}}$ are generically different, their zeros are the same. Now suppose that there exists a z_0 which is both a zero of $\mathcal{E}_1^T(zI - \overline{\mathcal{A}})^{-1}\overline{\mathcal{B}}$ and the reciprocal of an eigenvalue of $\overline{\mathcal{A}}$. Then for almost all F the zeros of $\mathcal{E}_1^T(zI - \overline{\mathcal{A}} - \overline{\mathcal{B}}F)^{-1}\overline{\mathcal{B}}$ will be distinct to the reciprocal of the eigenvalues of $\overline{\mathcal{A}} + \overline{\mathcal{B}}F$ and there will be no pole-zero cancelation.

To illustrate our approach for dealing with the case where one or more columns of $A(q)$ have column degree zero, we first provide the example below. The main results are stated afterwards.

Example 4.2: Consider the following AR(3) process

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + A_3 y_{t-3} + b\epsilon_t,$$

where $y_t \in \mathbb{R}^3$ and A_1, A_2 and A_3 have the following structure:

$$A_1 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & 0 \end{bmatrix} A_2 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & 0 \end{bmatrix} A_3 = \begin{bmatrix} \times & 0 & 0 \\ \times & 0 & 0 \\ \times & 0 & 0 \end{bmatrix}. \tag{15}$$

Based on the above discussion we define

$$x_t^r = \begin{bmatrix} y_t^r \\ y_{t-1}^r \\ y_{t-2}^r \end{bmatrix} = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \\ y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \\ y_{t-2}^{(1)} \\ y_{t-2}^{(2)} \end{bmatrix}, \tag{16}$$

where, $y_t^{(i)}$ denotes the i-th compenent of y_t. Accordingly, the state space equation is

$$x_t^r = \mathcal{A}_r x_{t-1}^r + \mathcal{B}_r \epsilon_t, \tag{17}$$

where

$$\mathcal{A}_r = \begin{bmatrix} \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \mathcal{B}_r = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{18}$$

The first two rows of $\mathcal{A}_r$ are $[\tilde{A}_1\ \tilde{A}_2\ \tilde{A}_3]$. Note that the process y_t^r can be obtained only by considering the first two rows of the equation (17).

$$y_t^r = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \epsilon_t = \tilde{A}_1 y_{t-1}^r + \tilde{A}_2 y_{t-2}^r + \tilde{A}_3 y_{t-3}^r + \bar{b}\epsilon_t, \tag{19}$$

where $b = [b_1^T\ b_2^T\ b_3^T]^T$, $\bar{b} = [b_1^T\ b_2^T]^T$, $b = [\bar{b}^T\ \hat{b}^T]^T$. For future reference, we note that of the 15 parameters appearing in the A_i, only 10 appear in $\mathcal{A}_r$. We will first deal with their identification.

Since there still exists a zero column in $\mathcal{A}_r$ we can further reduce the state. Thus, we define

$$\overline{x}_t^r = \begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \\ y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \\ y_{t-2}^{(1)} \end{bmatrix}, \tag{20}$$

and the related state space model will be of the form

$$\overline{x}_t^r = \overline{\mathcal{A}}_r \overline{x}_{t-1}^r + \overline{\mathcal{B}}_r \epsilon_t, \tag{21}$$

where

$$\overline{\mathcal{A}}_r = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \overline{\mathcal{B}}_r = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

The first two rows of $\overline{\mathcal{A}}_r$ are $[\breve{A}_1 \breve{A}_2 \breve{A}_3]$ and there holds:

$$y_t^r = [\breve{A}_1 \breve{A}_2 \breve{A}_3]\overline{x}_{t-1}^r + \bar{b}\epsilon_t. \tag{23}$$

Consider the generalized version of equation (23) for the AR(p) case:

$$y_t^r = [\breve{A}_1\ \breve{A}_2\ \dots \breve{A}_p]\overline{x}_{t-1}^r + \bar{b}\epsilon_t. \tag{24}$$

We also need to generalize the state space equation (21); thus, with a slight abuse of notation we define the state space model associated with (24) as:

$$\overline{x}_t^r = \overline{\mathcal{A}}_r \overline{x}_{t-1}^r + \overline{\mathcal{B}}_r \epsilon_t. \tag{25}$$

Then using the same procedure introduced in Subsection III-A, one can obtain the matrix below associated with the system (24)(the matrix K^r is defined below)

$$\mathcal{Z}^{r^f} = [K^r\ \overline{\mathcal{A}}_r K^r\ \dots\ \overline{\mathcal{A}}_r^{p(n-s_1)-s_2-1} K^r], \tag{26}$$

where s_1 is the number of prescribed zero columns in $A(q)$ and s_2 is the number of prescribed zero columns in $\mathcal{A}_r$.Since not all elements of y_t^r are available at all times, we consider those components of y_t^r that are observed at every time instant i.e. the fast components, denoting the associated vector by $y_t^{r^f}$ and then $K^r = \mathbb{E}[\overline{x}_t^r (y_t^{r^f})^T]$. It is apparent that the parameters in $\overline{A}_i$ can be determined if the matrix Z^{r^f} has full row rank. Similar to Subsection III-B we are interested in generic identifiability. We follow the same definition of generic identifiability, but the parameter space is now associated with (24).

The following result can be proved in a similar way as Theorem 3.5 and with some slight changes to the argument provided in the Subsection III-B.1.

Proposition 4.3: The matrix $\mathcal{Z}^{r^f}$ has full row rank for a set of generic parameter matrices $\breve{A}_i$, $i = 1, 2, \dots, p$.

Thus, from the above proposition, it readily follows that the parameter matrices $\breve{A}_i$ are generically identifiable from those population second moments which can be observed in principle.

Now similarly to the previous section, we study the generic identifiability of the noise parameters associated with the system (24). Let $\overline{\Sigma}$ be the noise covariance matrix corresponding to $\bar{b}$. Then using a similar argument as in the proof of Theorem III-B.2, one can prove that $\overline{\Sigma}$ is generically identifiable from those population second moments which can be observed in principle.

In the rest of this paper, we show how to obtain those system and noise parameters associated with the suppressed parts of the process y_t due to having columns with zero degree in $A(q)$. (In the previous example, there are 5 such parameters appearing the last rows of the A_i).

Proposition 4.4: Suppose that those columns of $A(q)$ with zero column degree are only associated with fast components of the process and we denote these components by $y_t^{\bar{r}}$. Then the system and noise parameters associated with $y_t^{\bar{r}}$, called $(\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_p)$ and $\hat{b}b^T$ respectively, are generically identifiable via regression.

Proof: We are starting from

$$y_t^{\bar{r}} = [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_p]\overline{x}_{t-1}^r + \hat{b}\epsilon_t, \tag{27}$$

where $y_t^{\bar{r}}$ contains all those components of y_t which we deleted when forming y_t^r and $\hat{A}_i$ is obtained from A_i by deleting all prescribed zero columns and taking only the rows associated with $y_t^{\bar{r}}$, $\hat{b}$ consists of the rows b corresponding to the components of $y_t^{\bar{r}}$. $\overline{x}_{t-1}^r$ is the vector of all components of $(y_{t-1}, \ldots, y_{t-p})$ which do not correspond to zero columns in $A(q)$. We first obtain the system parameters $\hat{A}_i$.

$$\mathbb{E}\left[y_t^{\bar{r}}\overline{x}_{t-1}^{rT}\right] = [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_p]\underbrace{\mathbb{E}\left[\overline{x}_{t-1}^r\overline{x}_{t-1}^{rT}\right]}_{\overline{\Gamma}^r} + \hat{b}\underbrace{\mathbb{E}\left[\epsilon_t\overline{x}_{t-1}^{rT}\right]}_{0}. \tag{28}$$

The matrix $\overline{\Gamma}^r$ is generically nonsingular, which can be seen as follows. Because of the stability assumption, we have that $\mathbb{E}\left[x_{t-1}^r x_{t-1}^{rT}\right] = \Gamma^r = \sum_{j=0}^{\infty}\mathcal{A}_r^j\mathcal{B}_r\mathcal{B}_r^T\mathcal{A}_r^{jT}$. As easily seen for $\overline{\Gamma}^r$ there similarly holds

$$\overline{\Gamma}^r = \sum_{j=0}^{\infty}\overline{\mathcal{A}}_r^j\overline{\mathcal{B}}_r\overline{\mathcal{B}}_r^T\overline{\mathcal{A}}_r^{jT} = (\overline{\mathcal{B}}_r, \overline{\mathcal{A}}_r\overline{\mathcal{B}}_r, \overline{\mathcal{A}}_r^2\overline{\mathcal{B}}_r, \ldots)\begin{pmatrix} \overline{\mathcal{B}}_r^T \\ \overline{\mathcal{B}}_r^T\overline{\mathcal{A}}_r^T \\ \overline{\mathcal{B}}_r^T\overline{\mathcal{A}}_r^{2T} \\ \vdots \end{pmatrix}. \tag{29}$$

Now the reachability matrices on the right hand side are of full rank since we showed in Subsection III-B.1 Theorem 3.2 that generically $(\overline{\mathcal{A}}_r, \overline{\mathcal{B}}_r)$ is reachable. Hence, $\overline{\Gamma}^r$ is generically full rank, so we have shown generic identifiability of $(\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_p)$.

It is left to show that the missing elements of Σ, viz. $\hat{b}b^T$ are generically identifiable. By postmultiplying both sides of (27) by y_t and taking expectations we obtain

$$\mathbb{E}\left[y_t^{\bar{r}}y_t^T\right] = [\hat{A}_1\ \hat{A}_2\ \ldots\ \hat{A}_p]\mathbb{E}\left[\overline{x}_{t-1}^r y_t^T\right] + \underbrace{\hat{b}\mathbb{E}\left[\epsilon_t y_t^T\right]}_{\hat{b}b^T}, \tag{30}$$

where we (generically) know all the covariances in $\mathbb{E}\left[\overline{x}_{t-1}^r y_t^T\right]$, which can be easily seen when we split y_t into y_t^r and $y_t^{\bar{r}}$. Thus, the term $\hat{b}b^T$ becomes available. ∎

V. CONCLUSIONS

In this paper, we have built on our work in [2] and have demonstrated that vector autoregressions with prescribed column degrees are generically identifiable on a restricted parameter space from covariance data in which significant information is missing, corresponding to the fact that some system outputs are not available every time instant. We were considering two cases. In case 1, we assumed to be prescribed nonzero unequal column degrees and we showed generic identifiability of the system and noise parameters. Then we showed generic identifiability for the case of zero column degrees by dividing the problem into two steps: In step 1 we only treated the subsystem of components corresponding to column degrees unequal to zero which had exactly the form of a system considered in case 1 and therefore had generic identifiability of this subsystem. Then we obtained the system parameters corresponding to components with zero column degree by regression and the noise parameters by taking expectations. Here, we only considered the scenario where those columns with zero degree are only associated with the fast part of process. Thus, as a part of our future works we will study a general scenario where columns with zero degree can be either associated with the fast part of the process or the slow part or both.

VI. ACKNOWLEDGEMENTS

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