Optimal Control of Motorway Tidal Flow*

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***Abstract*— When inbound and outbound traffic on a bi-directional motorway is unbalanced throughout the day a lane management strategy called tidal (reversible) flow lane control is usually applied. In this control case, the direction of one or more contraflow buffer lanes is reversed according to the needs of each direction. This paper introduces a basic dynamical model for tidal traffic flow and considers the minimum travel-time, minimum-time, and maximum throughput optimal control problems for efficient motorway tidal flow lane control. Lane management is effectuated by a control variable, indicating the number of lanes opened or closed in each direction of traffic. To derive the analytical form of optimal control, the Pontryagin's maximum principle is employed. The obtained optimal control is intuitively natural of bang-bang type, as also shown in a previous work by the authors [1]. It takes only the values ± 1 and switches between these values at most once. In other words, the optimal control strategy consists of switching between opening and closing in each direction of traffic one contraflow buffer lane. Of course it is an open-loop control, and thus the switch time (if applicable) depends on the initial conditions. In the case of the maximum throughput optimal control problem, semi-state feedback control is obtained and singular arcs might exist. Finally, cumulative arrival rate and output curves for both directions of traffic are used to provide a graphical interpretation of the minimum travel-time optimal control problem and obtained bang-bang control.**

I. INTRODUCTION

Reversible or tidal flow lane is a type of lane management for bi-directional motorways adopted in a number of countries, such as Australia, Brazil, Canada, China, New Zealand, Serbia, Spain, and UK. In case of flow disparity between the two directions of motorway traffic, the reversible lane can be assigned to one or other direction of traffic so as to better serve the direction currently with the highest demand. Unbalanced demand is typical in motorways connecting residential areas to business districts. Morning peaks usually experience higher demand toward the central area whereas evening peaks experience higher demand toward the residential area. In such cases, reversible lanes may be a suitable infrastructure to improve traffic conditions [1], [2].

The operation of tidal flow manages the capacity through an odd numbers of lanes. The direction of the middle lane is switched according to traffic demand by means of lane-use control signals located at overhead gantries that allow or forbid the use of the lane in one direction or another. In cases with more than three lanes, an extra lane may be used as a buffer zone to increase traffic safety. The operation of tidal lane is costly. Worldwide the most common practice for tidal lane operation is manual or fixed-time control [3], [4], [5]. Manual (or fixed-time) control may result in a bad timing of the lane reversal reducing the obtainable benefit and, as a matter of fact, may have a negative effect on system performance [6], [5]. A number of real-time tidal flow strategies have been proposed for urban arterials [7], for motorways or uninterrupted facilities [6], [8], [9], or both [10]. Recently, a simple and practicable policy for smooth and efficient tidal flow operations was proposed in [1]. This policy is based on certain properties of the triangular (or a generic concave) fundamental diagram of motorway traffic and relies on real-time measurements of density only. It is, thus, applicable in real-time to improve the bi-directional infrastructure efficiency. Except for [9] who proposed a simple logic-based strategy, all strategies proposed thus far are rather sophisticated or require advanced technologies.

This work first introduces a basic model describing tidal flow dynamics and then considers a number of problems, including the minimum travel-time, minimum-time, and maximum throughput optimal control problems for efficient motorway tidal flow lane control. Lane management is effectuated by a control variable, indicating the number of lanes opened or closed in each direction of traffic. The Pontryagin's maximum principle is employed to determine the required optimal control. The obtained analytical form of optimal control is intuitively natural of bang-bang type and conforms with previous results [1]. Of course it is an open-loop control, and thus the switch time (if applicable) depends on the initial conditions. In the first two problems, we show that singular control does not exist. In the case of throughput maximisation, semi-state feedback control is obtained and singular arcs might exist. This problem calls for the solution of a two-point boundary-value problem by appropriate iterative algorithms [11].

The rest of the paper is organised as follows. Section II provides some background on the basic optimal control problem and its solution via the Pontryagin's maximum principle. Section III-A develops a dynamical model of motorway tidal flow for reversible lane control. Then the following optimal control problems are presented: minimum total travel-time (Section III-B), minimum-time (Section III-C), and maximum throughput (Section III-D). Analytical solutions are derived via the Pontryagin's maximum principle. Finally, a graphical interpretation of the minimum travel-time optimal control problem is presented in Section III-E. Conclusions are given in Section IV.

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II. Preliminaries

Consider the nonlinear dynamical system

$$\dot{x}(t) = f\left[x(t), u(t)\right], \, t > 0, \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u(t) \in U = [-1, 1]^m \subset \mathbb{Z}^m$ is the control vector; the initial point $x_0 \in \mathbb{R}^n$ is given. We denote as $\mathcal{U}$ the space of continuous admissible controls u such that $u(t) \in U$,

$$\mathcal{U} = \{u : [0, \infty) \to \mathbb{R}^m \mid u \text{ is measurable}\}. \tag{2}$$

We also introduce the payoff functional

$$J[u(\cdot)] = \int_{t_0}^{t_f} \varphi\left[x(t), u(t)\right] dt, \tag{3}$$

where the initial and terminal times $t_0, t_f > 0$, and payoff function $\varphi : \mathbb{R}^n \times U \to \mathbb{R}$ are given. The optimal control problem is to determine $u^* \in \mathcal{U}$ such that:

$$J[u^*(\cdot)] = \max_{u \in \mathcal{U}} J[u(\cdot)], \tag{4}$$

subject to the system (1) to reach the origin.

We introduce the Hamiltonian function $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}$, where

$$\mathcal{H}\left[x(t), p(t), u(t)\right] = \varphi\left[x(t), u(t)\right] + p^{\mathsf{T}}(t) f\left[x(t), u(t)\right] \tag{5}$$

for all $x, p \in \mathbb{R}^n$ and $u \in U$. The following theorem (maximum principle) provides *necessary conditions* for optimality in case that the final time t_f is free and the target state x_f is fixed.

Theorem 1 (Pontryagin's Maximum Principle): Assume u^* is optimal for (1), (3), and $x^* : \mathbb{R} \to \mathbb{R}^n$ is the corresponding state trajectory. Then there exists a function $p^* : [t_0, \tau] \to \mathbb{R}^n$, such that

$$\dot{x}^*(t) = \nabla_p \mathcal{H}\left[x^*(t), p^*(t), u^*(t)\right], \tag{6}$$

$$-\dot{p}^*(t) = \nabla_x \mathcal{H}\left[x^*(t), p^*(t), u^*(t)\right], \tag{7}$$

and

$$\mathcal{H}\left[x^*(t), p^*(t), u^*(t)\right] = \max_{u \in U} \mathcal{H}\left[x^*(t), p^*(t), u\right] \tag{8}$$

for $t_0 < t < \tau^*$. Also,

$$\mathcal{H}\left[x^*(t), p^*(t), u^*(t)\right] \equiv 0, \quad t_0 \leq t \leq \tau^*. \tag{9}$$

Proof: See [12]. ■

The last equality (9) holds because the final time t_f is free, and the right hand side of (1) does not explicitly depends on time t. Here τ^* denotes the first time the trajectory x^* hits the target point $x^*(\tau^*) = x_f$ and we call p^* the co-state. Note that this is an *open-loop control*, because $u^*(t)$ depends not only on the state $x^*(t)$ but also on the co-state $p^*(t)$ which has to be computed from the adjoint differential equation (7). State constraints, e.g., lower and upper bounds, can be readily handled with a modified form of Pontryagin's maximum principle or by introducing artificial variables in the system of state equations above and by taking the required Karush-Kuhn-Tucker optimality conditions [13].

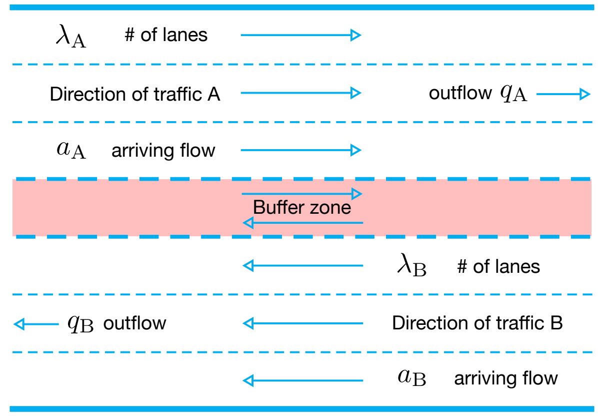


Fig. 1. A bi-directional motorway stretch with a contraflow buffer zone.

III. Optimal Control of Tidal Traffic Flow

A. Tidal Traffic Flow Dynamics

Consider a bi-directional motorway stretch with a contraflow buffer zone as shown in Fig. 1. The buffer lane is highlighted with red surface and literally permits tidal flow operations. The tidal (or reversible) lane switches direction at peak periods to provide extra capacity in one direction (either A or B). Typically, this changes to four lanes in one direction and two lanes in the opposite direction maintaining the one lane buffer for safety reasons. In case of limited infrastructure (e.g., three lanes with one buffer lane) or motorway stretches with lane speed control, the buffer lane can be assigned to the direction of traffic with severe congestion without any other lane closure. Under stationary traffic conditions and operations, we assume that both directions of traffic A, B in the motorway stretch will indicate nominal concave fundamental diagrams of traffic with number of lanes $\lambda_{\mathrm{A}} = \lambda_{\mathrm{B}} = \lambda > 1$.

A set of conservation equations describing the dynamic evolution of the traffic flow in the bi-directional motorway stretch read:

$$\dot{\varrho}_{\mathrm{A}}(t) = \frac{1}{\Delta}\left[a_{\mathrm{A}}(t) - q_{\mathrm{A}}(t)\left(\lambda_{\mathrm{A}} + u(t)\right)\right] \tag{10}$$

$$\dot{\varrho}_{\mathrm{B}}(t) = \frac{1}{\Delta}\left[a_{\mathrm{B}}(t) - q_{\mathrm{B}}(t)\left(\lambda_{\mathrm{B}} - u(t)\right)\right] \tag{11}$$

where ϱ_{A} and ϱ_{B} are the traffic densities (in veh/km) (state variables) at A and B, respectively; $u(t) \in [-1, 1] \subset \mathbb{Z}$ is the control, indicating the number of lanes opened or closed in each direction of traffic; a_{A} and a_{B} are the upstream arrival rates (in veh/h) at A and B, respectively; q_{A} and q_{B} are the outflows (in veh/h) at A and B, respectively; and, Δ is the length (in km) of the bi-directional motorway stretch. Given initial densities $\varrho(t_0) = \left[\varrho_{\mathrm{A}}(t_0) \; \varrho_{\mathrm{B}}(t_0)\right]^{\mathsf{T}}$ at t_0, we assume that the downstream infrastructure (off-ramps, exit-points, etc.) has enough capacity to discharge any desired outflows in both directions of traffic.

We also impose the following standard assumptions:

Assumption 1: The demand (arrival rates) $a_{\mathrm{A}}(t)$ and $a_{\mathrm{B}}(t)$, are non-negative constant (steady state) over the control interval $T \equiv t_f - t_0$, i.e., $a_{\mathrm{A}}(t) = a_{\mathrm{A}}$ and $a_{\mathrm{B}}(t) = a_{\mathrm{B}}$, $\forall \, t \in [t_0, t_f]$.

Assumption 2: The outflow (departure rates) $q_{\mathrm{A}}(t)$ and $q_{\mathrm{B}}(t)$, are described by concave (unimodal) fundamental diagrams $F(\varrho_{\mathrm{A}}(t))$ and $G(\varrho_{\mathrm{B}}(t))$ for the same number of nominal lanes λ. For the purpose of control, we assume stationary traffic conditions over the control interval $T \equiv t_f - t_0$, i.e., $q_{\mathrm{A}}(t) = q_{\mathrm{A}}$ and $q_{\mathrm{B}}(t) = q_{\mathrm{B}}, \forall\, t \in [t_0, t_f]$.

Given the initial state $\varrho(t_0) = [\varrho_{\mathrm{A}}(t_0)\ \varrho_{\mathrm{B}}(t_0)]^{\mathsf{T}}$, it will be desirable (i.e., for stability) the densities at A and B to be non-increasing during the transient period and zero at steady-state. Thus the time derivatives $d\varrho_{\mathrm{A}}(t)/dt$ and $d\varrho_{\mathrm{B}}(t)/dt$ in the system of differential equations (10)–(11) must be non-positive. From (10)–(11) the necessary condition for non-increasing densities reads:

$$\frac{a_{\mathrm{A}}}{q_{\mathrm{A}}} + \frac{a_{\mathrm{B}}}{q_{\mathrm{B}}} < 2\lambda,$$

where $\lambda > 1$ is the nominal number of lanes under stationary traffic conditions (e.g., $\lambda_{\mathrm{A}} = \lambda_{\mathrm{B}} = \lambda = 3$ in Fig. 1).

Now the system of conservation equations (10)–(11) can be re-written as,

$$\dot{\varrho}_{\mathrm{A}}(t) = d_{\mathrm{A}} - \frac{q_{\mathrm{A}}}{\Delta} u(t) \tag{12}$$

$$\dot{\varrho}_{\mathrm{B}}(t) = d_{\mathrm{B}} + \frac{q_{\mathrm{B}}}{\Delta} u(t) \tag{13}$$

where $d_{\mathrm{A}} = \frac{1}{\Delta}(a_{\mathrm{A}} - q_{\mathrm{A}}\lambda)$ and $d_{\mathrm{B}} = \frac{1}{\Delta}(a_{\mathrm{B}} - q_{\mathrm{B}}\lambda)$. The above system of equations will be used in the sequel for tidal flow lane control design.

B. Minimum Travel-Time Optimal Control Problem

Consider the minimum travel-time optimal control problem with final-time free. The total travel time (TTT) in the bi-directional motorway stretch over a time horizon t_f may be expressed as,

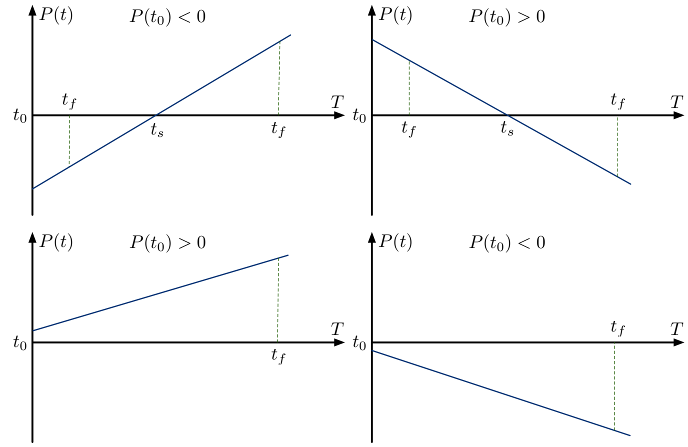
$$J = \Delta \cdot \int_{t_0}^{t_f} [\varrho_{\mathrm{A}}(t) + \varrho_{\mathrm{B}}(t)]\; dt, \tag{14}$$

where $\varrho_{\mathrm{A}}(t)$ and $\varrho_{\mathrm{B}}(t)$ are the traffic density (in veh/km) in direction A and B, respectively, at time t.

The tidal flow lane optimal control problem reads: For given initial state $\varrho_{\mathrm{A}}(t_0)$, $\varrho_{\mathrm{B}}(t_0)$ at time t_0, corresponding estimates (e.g., historical measurements) of demands a_{A}, a_{B} and outflows q_{A}, q_{B}, determine $u(t) \in [-1, 1] \subset \mathbb{Z}$, indicating the number of lanes opened or closed in each direction of traffic, so as to minimise (14) subject to the tidal traffic flow dynamics (12)–(13). We assume that the problem is well-possed, i.e. that there exists some control u that achieves the transfer from $\varrho(t_0) = [\varrho_{\mathrm{A}}(t_0)\ \varrho_{\mathrm{B}}(t_0)]^{\mathsf{T}}$ to the final state $\varrho(\tau) = [\varrho_{\mathrm{A}}(\tau)\ \varrho_{\mathrm{B}}(\tau)]^{\mathsf{T}} = [0\ \ 0]^{\mathsf{T}}$ in some time τ. This guarantees that an optimal control $u^* : [t_0, \tau] \to U$ exists.

Given the cost criterion (14) and equations (12)–(13), the Hamiltonian (5) is given by,

$$\begin{aligned}\mathcal{H}(\varrho, p, u) = \Delta\,[\varrho_{\mathrm{A}}(t) + \varrho_{\mathrm{B}}(t)] + p_{\mathrm{A}}(t)\left[d_{\mathrm{A}} - \frac{q_{\mathrm{A}}}{\Delta}u(t)\right] \\ + p_{\mathrm{B}}(t)\left[d_{\mathrm{B}} + \frac{q_{\mathrm{B}}}{\Delta}u(t)\right],\end{aligned} \tag{15}$$



(a) Case I: $q_{\mathrm{A}} > q_{\mathrm{B}}$ and $dP/dt > 0$ (b) Case II: $q_{\mathrm{A}} < q_{\mathrm{B}}$ and $dP/dt < 0$

Fig. 2. Form of the switch function $P(t)$ for different cases of outflow disparity. As can be seen four different cases of control are identified.

where ϱ and p are the vectors of states and co-states, respectively. Applying Pontryagin's maximum principle yields,

$$\begin{aligned}u^*(t) &= \arg \min_{u \in [-1,1]} \mathcal{H}(\varrho^*, p^*, u) \\ &= \arg \min_{u \in [-1,1]} \frac{1}{\Delta}\left[p^*_{\mathrm{B}}(t) q_{\mathrm{B}} - p^*_{\mathrm{A}}(t) q_{\mathrm{A}}\right] u(t),\end{aligned}$$

so that the co-state equations are (see (7)),

$$-\dot{p}_{\mathrm{A}}(t) = \frac{\partial \mathcal{H}}{\partial \varrho_{\mathrm{A}}(t)} = \Delta, \quad -\dot{p}_{\mathrm{B}}(t) = \frac{\partial \mathcal{H}}{\partial \varrho_{\mathrm{B}}(t)} = \Delta.$$

These equations imply that the co-states are linear functions of time t for $t_0 < t < t_f$ with solution

$$p_{\mathrm{A}}(t) = -\Delta t + c_1, \quad p_{\mathrm{B}}(t) = -\Delta t + c_2, \tag{16}$$

where c_1 and c_2 are constants of integration, which can be determined given $p(t_0) = [p_{\mathrm{A}}(t_0)\ p_{\mathrm{B}}(t_0)]^{\mathsf{T}}$. Since $u \in [-1, 1]$, we can easily minimise the Hamiltonian by choosing,

$$u^*(t) = \begin{cases} 1, & \text{if } p^*_{\mathrm{B}}(t) q_{\mathrm{B}} - p^*_{\mathrm{A}}(t) q_{\mathrm{A}} < 0 \\ -1, & \text{if } p^*_{\mathrm{B}}(t) q_{\mathrm{B}} - p^*_{\mathrm{A}}(t) q_{\mathrm{A}} > 0 \end{cases}.$$

The optimal control can be re-written in compact form,

$$u^*(t) = \operatorname{sgn}\left[p^*_{\mathrm{A}}(t) q_{\mathrm{A}} - p^*_{\mathrm{B}}(t) q_{\mathrm{B}}\right].$$

where $\operatorname{sgn}(\cdot)$ denotes the *sign function*[1] of a scalar $x \in \mathbb{R}$. As can be seen the optimal control depends on the co-state vector $p^*(t)$ and the outflows q_{A}, q_{B}. Defining the switch function $P(t) = p^*_{\mathrm{B}}(t) q_{\mathrm{B}} - p^*_{\mathrm{A}}(t) q_{\mathrm{A}}$ will give us some more insights on the form of $P(t)$ and optimal control u^*. Provided that both $p^*_{\mathrm{A}}(t)$ and $p^*_{\mathrm{B}}(t)$ are linear functions of time, $P(t)$ can cross the value 0 at most once and *singular control does not exist* over a finite time interval. Moreover,

$$dP(t)/dt = \Delta(q_{\mathrm{A}} - q_{\mathrm{B}}). \tag{17}$$

We now consider the following bi-directional motorway outflow cases (see Fig. 2):

[1] $\operatorname{sgn}(x) = 1$, if $x > 0$; $\operatorname{sgn}(x) = -1$, if $x < 0$; $\operatorname{sgn}(x) = 0$, if $x = 0$.

Case I: $q_{\mathrm{A}} > q_{\mathrm{B}}$.
Applying this to (17) results $dP(t)/dt > 0$. Now the number of zero crossings (switch point where $P(t_s) = 0$ and corresponding control switch) depends on the initial co-state $P(t_0)$ at time t_0, as shown in Fig. 2(a).
Case I-1: If $P(t_0) < 0$ then (see top subgraph in Fig. 2(a)):

- *A single switch point* at t_s exists if the initial traffic densities are such that $t_f > t_s$;

$$u^*(t) = \begin{cases} 1, & \text{for } t \in [t_0, t_s) \\ -1, & \text{for } t \in (t_s, t_f] \end{cases}.$$

- *No switch point* exists if the initial traffic densities are such that $t_f \le t_s$; $u^*(t) = 1$ for $t_0 \le t \le t_s \equiv t_f$.

Case I-2: If $P(t_0) > 0$ *no switch point* exists; $u^*(t) = -1$ for $t_0 \le t \le t_f$ (see bottom subgraph in Fig. 2(a)).

Case II: $q_{\mathrm{A}} < q_{\mathrm{B}}$.
Applying to (17) results $dP(t)/dt < 0$. Again the number of zero crossings depends on the initial co-state $P(t_0)$ at time t_0, as shown in Fig. 2(b).
Case II-1: If $P(t_0) > 0$ then (see top subgraph in Fig. 2(b)):

- *A single switch point* at t_s exists if the initial traffic densities are such that $t_f > t_s$;

$$u^*(t) = \begin{cases} -1, & \text{for } t \in [t_0, t_s) \\ 1, & \text{for } t \in (t_s, t_f] \end{cases}.$$

- *No switch point* exists if the initial traffic densities are such that $t_f \le t_s$; $u^*(t) = -1$ for $t_0 \le t \le t_s \equiv t_f$.

Case II-2: If $P(t_0) < 0$ *no switch point* exists; $u^*(t) = 1$ for $t_0 \le t \le t_f$ (see bottom subgraph in Fig. 2(b)).

The state trajectories can be found by substituting each of the allowed control values $u(t)$ into the system (12)–(13), and then solving the differential equations for $\varrho_{\mathrm{A}}(t)$ and $\varrho_{\mathrm{B}}(t)$ (given the initial time t_0 and state $\varrho(t_0)$):

- For $u(t) = 1$ the solution is $\varrho_{\mathrm{A}}(t) = \alpha\varrho_{\mathrm{B}}(t) + \beta$, where $\alpha = (d_{\mathrm{A}} - \frac{q_{\mathrm{A}}}{\Delta})/(d_{\mathrm{B}} + \frac{q_{\mathrm{B}}}{\Delta})$ and $\beta = \varrho_{\mathrm{A}}(t_0) - \alpha\varrho_{\mathrm{B}}(t_0)$;
- For $u(t) = -1$ the solution is $\varrho_{\mathrm{A}}(t) = \alpha'\varrho_{\mathrm{B}}(t) + \beta'$, where $\alpha' = (d_{\mathrm{A}} + \frac{q_{\mathrm{A}}}{\Delta})/(d_{\mathrm{B}} - \frac{q_{\mathrm{B}}}{\Delta})$ and $\beta' = \varrho_{\mathrm{A}}(t_0) - \gamma\varrho_{\mathrm{B}}(t_0)$.

The above state trajectories allow us to work on the state phase plane $(\varrho_{\mathrm{A}}, \varrho_{\mathrm{B}})$ for the two possible control inputs. Clearly in both control cases, the two traffic densities are linearly dependent (i.e., define a family of linear functions).

In the above cases, we can also determine the switch time t_s (if applicable) and the final time t_f (not shown here due to space limitations). Once t_s is calculated, the densities $\varrho(t_s)$ at switch time t_s can be calculated from (12)–(13) (given the initial time t_0 and state $\varrho(t_0)$). The co-states $p(t_s) = [p_{\mathrm{A}}(t_s)\ p_{\mathrm{B}}(t_s)]^{\mathsf{T}}$ at time t_s can be found from (16) with $P(t_s) = 0$ and $\mathcal{H}(t_s) = 0$, see (15) and the transversality condition (9) for free final time t_f.

Note that the derived analytical solution is an *open-loop control*, since $u^*(t)$ depends on the initial co-states $p(t_0)$ and states $\varrho(t_0)$. State-feedback control will be considered in an extended version of this paper by incorporating state constraints, e.g., minimum and maximum traffic densities. State constraints can be readily handled with an alternative form of Pontryagin's maximum principle involving an augmented Hamiltonian function and Lagrange or Karush-Kuhn-Tucker (KKT) multipliers. In this case, the assumption that both traffic densities discharge simultaneously, i.e. $[\varrho_{\mathrm{A}}(\tau)\ \varrho_{\mathrm{B}}(\tau)]^{\mathsf{T}} = [0\ 0]^{\mathsf{T}}$ for some time $\tau \le t_f$, can be relaxed to allow unbalanced dissipation. For instance, one of the traffic densities to dissipate much faster and to reach zero at time $\tau' < \tau$ while the other density remains positive. This will activate the state constraints (requirement of non-negative densities) and corresponding KKT multipliers, etc.

C. Minimum-Time Optimal Control Problem

Consider now the minimum-time optimal control problem with final-time free and cost criterion given by,

$$J = \int_{t_0}^{t_f} 1\, dt = t_f - t_0, \quad t_0 \ge 0,$$

which aims at minimising the time $t_f - t_0$ required to reach the final $\varrho(t_f)$ from the initial state $\varrho(t_0)$. If the final goal never steered to 0, we set $t_f = \infty$. The control objective is to steer ϱ_{A}, ϱ_{B} from the given initial state $\varrho(t_0)$ to a given final state $\varrho(t_f)$ in minimal time. Suppose that the problem is well-possed, i.e., there exists some control u that achieves the transfer from $\varrho(t_0)$ to $\varrho(t_f)$ in some time. This guarantees that a *time-optimal control* $u^* : [t_0, t_f^*] \to U$ exists.

Given this cost criterion and state equations (12)–(13), the Hamiltonian is given by,

$$\mathcal{H}[\varrho(t), p(t), u(t)] = 1 + p_{\mathrm{A}}(t)\left[d_{\mathrm{A}} - \frac{q_{\mathrm{A}}}{\Delta}u(t)\right] + p_{\mathrm{B}}(t)\left[d_{\mathrm{B}} + \frac{q_{\mathrm{B}}}{\Delta}u(t)\right].$$

Applying Pontryagin's maximum principle yields,

$$u^*(t) = \arg\min_{u\in[-1,1]} \frac{1}{\Delta}\left[p_{\mathrm{B}}^*(t)q_{\mathrm{B}} - p_{\mathrm{A}}^*(t)q_{\mathrm{A}}\right]u(t).$$

Since $u \in [-1, 1]$, we can easily minimise the Hamiltonian by choosing,

$$u^*(t) = \begin{cases} 1, & \text{if } p_{\mathrm{B}}^* q_{\mathrm{B}} - p_{\mathrm{A}}^* q_{\mathrm{A}} < 0 \\ -1, & \text{if } p_{\mathrm{B}}^* q_{\mathrm{B}} - p_{\mathrm{A}}^* q_{\mathrm{A}} > 0 \end{cases}.$$

As can be seen the optimal control depends on the co-state vector $p^*(t)$ and outflows q_{A}, q_{B}. From (7) the co-state equations are,

$$-\dot{p}_{\mathrm{A}} = \partial\mathcal{H}/\partial\varrho_{\mathrm{A}} = 0, \qquad -\dot{p}_{\mathrm{B}} = \partial\mathcal{H}/\partial\varrho_{\mathrm{B}} = 0,$$

which implies that the co-states are constant, i.e., $p_{\mathrm{A}}^*(t) = \overline{p}_{\mathrm{A}}$ and $p_{\mathrm{B}}^*(t) = \overline{p}_{\mathrm{B}}$ for $0 < t < t_f$, where $\overline{p}_{\mathrm{A}}$, $\overline{p}_{\mathrm{B}}$ appropriate constants. We this we see that,

$$u^*(t) = \operatorname{sgn}\left(\overline{p}_{\mathrm{A}}q_{\mathrm{A}} - \overline{p}_{\mathrm{B}}q_{\mathrm{B}}\right).$$

Note that neither $\overline{p}_{\mathrm{A}}$ or $\overline{p}_{\mathrm{B}}$ is a function of time. This means the optimal control has no switch point and solely relies on q_{A} and q_{B}. Moreover, for the time-optimal control problem the co-state vector $p^*(t)$ is nonzero $\forall\, t$. This holds because the cost J is everywhere nonzero. This condition is called the *nontriviality condition*, because with $p(t) \equiv 0$ all the statements of the maximum principle are trivially satisfied.

D. Throughput Maximisation

Consider now the problem of maximising the total bi-directional throughput with cost criterion given by,

$$J = \int_{t_0}^{t_f} \Big\{ F\left[\varrho_{\mathrm{A}}(t)\right] + G\left[\varrho_{\mathrm{B}}(t)\right] \Big\}\, dt, \tag{18}$$

where $q_{\mathrm{A}}(t) = F\left[\varrho_{\mathrm{A}}(t)\right]$ and $q_{\mathrm{B}}(t) = G\left[\varrho_{\mathrm{B}}(t)\right]$ (in veh/h) is the outflow (throughput) when the local density is $\varrho_{\mathrm{A}}(t)$ and $\varrho_{\mathrm{B}}(t)$ (in veh/km) at time t, respectively. Assume that F and G are two given concave fundamental diagrams. The problem of bi-directional throughput maximisation now reads: For given initial state $\varrho_{\mathrm{A}}(t_0)$, $\varrho_{\mathrm{B}}(t_0)$ at time t_0, corresponding estimates (e.g., historical measurements) of demands a_{A}, a_{B}, determine $u(t) \in [-1, 1] \subset \mathbb{Z}$, indicating the number of lanes opened or closed in each direction of traffic, so as to maximise (18) subject to the tidal traffic flow dynamics (10)–(11). Again, we assume that the final state vector $\varrho(t_f)$ is free and that the problem is well-possed. Here Assumption 2 is relaxed by assuming non-constant departure flows (i.e., the outflow in each direction of traffic is a function of the corresponding density via given fundamental diagram).

Given the cost criterion (18) and equations (10)–(11), the Hamiltonian (5) is given by,

$$\begin{aligned}
\mathcal{H}\left[\varrho(t), p(t), u(t)\right] &= F\left[\varrho_{\mathrm{A}}(t)\right] + G\left[\varrho_{\mathrm{B}}(t)\right] \\
&\quad + p_{\mathrm{A}}(t) \left\{ \frac{1}{\Delta}\left[a_{\mathrm{A}} - q_{\mathrm{A}}(t)\left(\lambda + u(t)\right)\right] \right\} \\
&\quad + p_{\mathrm{B}}(t) \left\{ \frac{1}{\Delta}\left[a_{\mathrm{B}} - q_{\mathrm{B}}(t)\left(\lambda - u(t)\right)\right] \right\},
\end{aligned}$$

where $q_{\mathrm{A}}(t)$ and $q_{\mathrm{B}}(t)$ are functions of F and G, respectively. Applying Pontryagin's maximum principle yields,

$$u^*(t) = \arg \max_{u \in [-1,1]} \frac{1}{\Delta}\left[p_{\mathrm{B}}^*(t) G(\varrho_{\mathrm{B}}^*) - p_{\mathrm{A}}^*(t) F(\varrho_{\mathrm{A}}^*)\right] u(t).$$

Since $u \in [-1, 1]$, we can again minimise the Hamiltonian:

$$u^*(t) = \begin{cases} 1, & \text{if } p_{\mathrm{B}}^*(t) G[\varrho_{\mathrm{B}}^*(t)] - p_{\mathrm{A}}^*(t) F[\varrho_{\mathrm{A}}^*(t)] > 0 \\ -1, & \text{if } p_{\mathrm{B}}^*(t) G[\varrho_{\mathrm{B}}^*(t)] - p_{\mathrm{A}}^*(t) F[\varrho_{\mathrm{A}}^*(t)] < 0 \end{cases}. \tag{19}$$

As can be seen the optimal control depends on the co-state vector $p^*(t)$ and the outflows $q_{\mathrm{A}}^*(t)$, $q_{\mathrm{B}}^*(t)$ as given by the corresponding fundamental diagrams $F[\varrho_{\mathrm{A}}^*(t)]$ and $G[\varrho_{\mathrm{B}}^*(t)]$, respectively. The optimality condition (7) yields,

$$\begin{aligned}
-\dot{p}_{\mathrm{A}}(t) &= \frac{\partial \mathcal{H}}{\partial \varrho_{\mathrm{A}}(t)} = \frac{\partial F[\varrho_{\mathrm{A}}(t)]}{\partial \varrho_{\mathrm{A}}(t)} \left(1 - \frac{\lambda + u(t)}{\Delta} p_{\mathrm{A}}(t)\right), \\
-\dot{p}_{\mathrm{B}}(t) &= \frac{\partial \mathcal{H}}{\partial \varrho_{\mathrm{B}}(t)} = \frac{\partial G[\varrho_{\mathrm{B}}(t)]}{\partial \varrho_{\mathrm{B}}(t)} \left(1 - \frac{\lambda - u(t)}{\Delta} p_{\mathrm{B}}(t)\right),
\end{aligned}$$

where $p_{\mathrm{A}}(t_f) = p_{\mathrm{B}}(t_f) = 0$ from the transversality condition (9) for free final time t_f. These conditions define a two-point boundary-value problem, since the boundary conditions required for solution are the initial state $\varrho(t_0)$ and the final co-states $p(t_f)$. The two-point boundary-value problem can be numerically solved by appropriate iterative algorithms [11]. This is a direction of future work.

Given that F and G are concave functions of ϱ_{A} and ϱ_{B}, respectively, different cases of bi-directional motorway initial densities can be distinguished and checked. **Case I:** One direction of traffic is congested and the other is uncongested, e.g., $\partial F(\varrho_{\mathrm{A}})/\partial \varrho_{\mathrm{A}} < 0$ and $\partial G(\varrho_{\mathrm{B}})/\partial \varrho_{\mathrm{B}} > 0$; **Case II:** Both directions of traffic are congested, $\partial F(\varrho_{\mathrm{A}})/\partial \varrho_{\mathrm{A}} < 0$ and $\partial G(\varrho_{\mathrm{B}})/\partial \varrho_{\mathrm{B}} < 0$; **Case III:** Both directions of traffic are uncongested, $\partial F(\varrho_{\mathrm{A}})/\partial \varrho_{\mathrm{A}} > 0$ and $\partial G(\varrho_{\mathrm{B}})/\partial \varrho_{\mathrm{B}} > 0$.

Define now the switch function $P(t) = p_{\mathrm{B}}^*(t) G[\varrho_{\mathrm{B}}^*(t)] - p_{\mathrm{A}}^*(t) F[\varrho_{\mathrm{A}}^*(t)]$ in (19). Singular control can be applied if $P(t) = 0$ holds over a finite time interval. Intuitively this includes cases (among others) where both directions of traffic are uncongested (singular arcs lying within the feasible region defined by the state constraints) or congested (singular arcs moving along *at least one* of the state constraints). For an alternative interpretation of the obtained control and singular arcs in (19), take $dP(t)/dt$ and check whether $dP(t)/dt = 0$ (i.e. $P(t)$ is constant over a finite time interval) and $P(t) = 0$ holds simultaneously, then cf. with equation (4) in [1] and discussion therein. Singular control means that a *do nothing policy* is applied (i.e. $u^* = 0$ in (19)) and both directions of traffic operate with $\lambda_{\mathrm{A}} = \lambda_{\mathrm{B}} = \lambda$.

E. Cumulative Demand and Service Diagrams

In this section, the minimum travel-time optimal control problem discussed in Section III-B is reconsidered. The idea here is to express the optimal control problem over the cumulative demand and service curves of each direction of traffic. Since the area between these curves represents aggregated delay or TTT, a graphical derivation and illustration of the optimal control is possible.

Let D_{A}, D_{B} and S_{A}, S_{B} be the cumulative demand and service curves, respectively. The quantities are the integrals over time of the arrival rates $a_{\mathrm{A}}(t)$, $a_{\mathrm{B}}(t)$ and service rates $q_{\mathrm{A}}(t)$, $q_{\mathrm{B}}(t)$ respectively, namely,

$$\begin{aligned}
D_i(t) &= \int_{t_0}^{t} a_i(\tau)\, d\tau, \quad i \in \{\mathrm{A}, \mathrm{B}\}, \\
S_i(t) &= \int_{t_0}^{t} q_i(\tau)\, d\tau, \quad i \in \{\mathrm{A}, \mathrm{B}\}.
\end{aligned}$$

Fig. 3 illustrates the cumulative demand and cumulative service curves over time for both competing directions of traffic (here $t_0 = 0$ is assumed). The area between the curves D_{A} and S_{A} (respectively, D_{B} and S_{B}) is a measure of the aggregate delay or total travel time to the drivers of the direction A (respectively, B) of the motorway stretch over time. Their vertical distance at any time t is the effective size of the vehicle queue $\varrho_{\mathrm{A}}(t) \cdot \Delta$ (respectively, $\varrho_{\mathrm{B}}(t) \cdot \Delta$) at A (respectively, B) at that time. The service rate (outflow) is governed by the fundamental diagram of traffic, reflecting the number of lanes assigned to its direction of traffic, i.e., the control is constrained $u(t) \in [-1, 1]$.

The total travel time is now given by (cf. with (14)),

$$\begin{aligned}
J &= \sum_{i \in \{\mathrm{A},\mathrm{B}\}} \int_{t_0}^{t_f} \left[D_i(t) - S_i(t)\right] dt \\
&\equiv \Delta \cdot \int_{t_0}^{t_f} \left[\varrho_{\mathrm{A}}(t) + \varrho_{\mathrm{B}}(t)\right] dt.
\end{aligned}$$

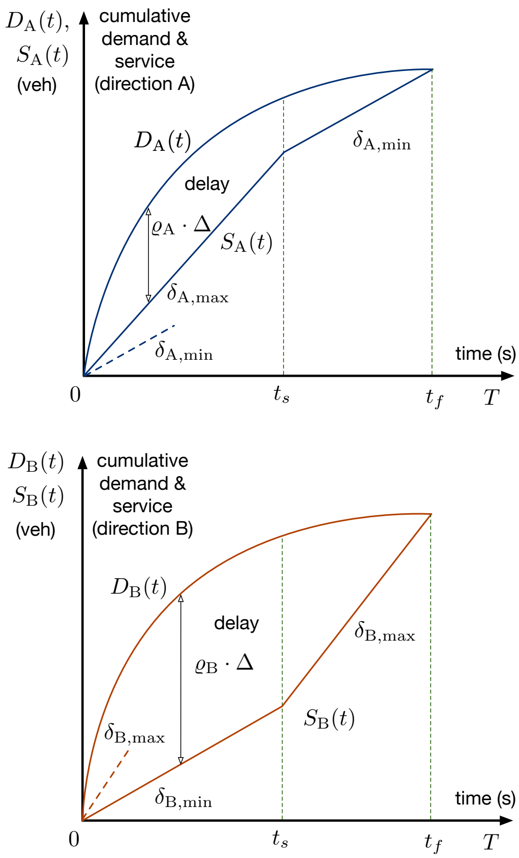


Fig. 3. Cumulative demand and service diagrams and optimal control.

The control objective is to minimise the above function subject to the aforementioned constrained control and the conservation equations (12)–(13). One can show that the total delay is minimised if both bi-directional vehicle queues are dissolved simultaneously, and thus there is some final time t_f where $D_{\rm A}(t_f) = S_{\rm A}(t_f)$ and $D_{\rm B}(t_f) = S_{\rm B}(t_f)$, i.e., $\varrho(t_f) = [\varrho_{\rm A}(t_f)\ \varrho_{\rm B}(t_f)]^{\sf T} = [0\ \ 0]^{\sf T}$.

Fig. 3 provides a graphical interpretation of the optimal control, where $\delta_{\rm A,min} = (\lambda_{\rm A} - 1)\, q_{\rm A}$, $\delta_{\rm A,max} = (\lambda_{\rm A} + 1)\, q_{\rm A}$ and $\delta_{\rm B,min} = (\lambda_{\rm B} - 1)\, q_{\rm B}$, $\delta_{\rm B,max} = (\lambda_{\rm B} + 1)\, q_{\rm B}$ are minimum and maximum allowable service rates at direction A and B, respectively. Note that these service rates are obtained from $u(t) \in [-1, 1]$ indicating the number of lanes opened or closed in each direction of traffic. The graphical solution suggests that if one direction of traffic is under critical conditions during peak hours and the other direction of traffic operates under free flow traffic conditions, then the buffer lane should switches direction to provide extra capacity (to the highest possible degree) to the crowded direction of traffic. At time t_s of the rush period, service should switch to maximum for the direction with the lower outflow. If the switch time t_s is selected properly (see Section III-B), both traffic densities will be served out at the same time, and with the minimum total delay to the users. Note that if both directions of traffic are serviced with maximum rates, i.e., at flow capacity of the fundamental diagram, then system's throughput is maximised. The graphical interpretation of the optimal control is intuitively natural of bang-bang type, as also shown by employing a different approach in [1]. The switch time is important as can be affected by the initial states and possible activation of state constraints. Note that the simultaneous dissipation of densities assumption can be relaxed by state-feedback control (see Section III-B).

IV. Conclusions

Reversible lanes are currently considered in a number of sites all over the world to manage tidal flow facilities and smart infrastructure, and thus better accommodate directional flow disparity. This paper tackled the minimum travel-time, minimum-time, and maximum throughput optimal control problems for efficient motorway tidal flow lane control. The obtained analytical form of optimal control is intuitively natural of bang-bang type. Of course it is an open-loop control, and thus the switch time (if applicable) depends on the initial conditions. In the case of the maximum throughput optimal control problem, semi-state feedback control was obtained in (19) wherein singular arcs might exist. On-going work considers state-feedback constrained control and simulation studies. State constraints can be handled with an alternative form of Pontryagin's maximum principle [13]. This will allow us to compare state-feedback control policies obtained from optimal control theory with the Kinematic Wave Theoretical (KWT) analysis of tidal flow in [1], [14].

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