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On the Stability of the Foschini-Miljanic Algorithm with Uncertainty over Channel Gains

Giannis Moutsinas, Themistoklis Charalambous and Weisi Guo

Abstract—Distributed power control in wireless networks faces challenges related to its stability. When perfect information of channel states and transmitting agents are available, previous work has shown that the stability conditions can be known. When there is uncertainty over the parameter space, stability is not well understood. In this work, we study the impact of parameter uncertainty and network structure on the stability and scalability of a well known distributed power control, namely the Foschini-Miljanic algorithm. More specifically, we derive probabilistic conditions with respect to the parameters of the channel distributions for which the system is stable. Furthermore, we study the effects of these parameters for different node distribution on the plane. Numerical examples validate our theoretical results.

Index Terms—Power control, uncertain channel conditions, stability, scalability, Foschini-Miljanic algorithm.

I. INTRODUCTION

Transmit power in wireless networks is a key ingredient in the management of interference, energy, and connectivity. When a device unilaterally increases its transmit power forsee successful transmissions over longer distances or higher data rates. The increased transmit power, however, result to co-channel interference to other transmitting devices in the network. As a result, these devices increase their power to maintain their connection and rate and, therefore, the battery of the device drains faster without necessarily any gain. Transmit power control in wireless networks has been extensively studied, and some of the results have had significant impact in wireless communication technology.

Early work in the field of power control for wireless networks [1], [2] proposed power balancing, which equalizes the Signal-to-Interference Ratio (SIR) in all the wireless links. These algorithms need global information about the network setting. This capacity improvement initiated extensive research on power control with focus on the design of distributed algorithms to meet a prefixed Signal-to-Interference-and-Noise Ratio (SINR) target (hard constraint), determined by the Quality of Service (QoS) requirements. Zander in [3], assuming a linear model of interference and negligible receiver noise, first proposed a distributed algorithm in which the transmitters constituting the network update their power levels in a distributed fashion to reach the greatest achievable SIR. Subsequently, Foschini and Miljanic in [4]

came up with a linear distributed algorithm, known as the Foschini-Miljanic (FM) algorithm, for both the continuous- and discrete-time systems, for the linear interference model that considers the thermal noise as well and adjust their power to reach a desired SINR.

The prefixed SINR target tracking ensures that a constant transmission rate can be sustained. If a feasible solution exists, then there exists a unique solution that minimizes transmit power in a pareto sense. But, if not, then the performance of the whole network degrades and the capacity is deteriorated. The target tracking approach is suitable for real-time, delay-sensitive applications like mobile phone services. These works set forth the introduction of power control algorithms in third generation (3G) CDMA-based cellular networks. While in fourth generation (4G) technologies used in wireless networks, they tried to avoid power control with orthogonal multiple access schemes, interestingly, power control has become essential to many candidate technologies for fifth generation (5G) networks (e.g., massive multiple-input multiple-output (MIMO) networks), since such networks consists of several small cells constituting a large network, in which orthogonal interference suppression mechanisms result in a poor resource allocation.

The seminal work by Foschini and Miljanic in [4] excited a furore of research for power control in both continuous- (e.g., [5]–[8]) and discrete-time (e.g., [9]–[15]). While it is assumed in the literature that power control operates in the discrete-time domain, the continuous-time counterpart has been widely studied as well, because it offers a simple model for analysis in complex networks or when integrated on advanced systems.

The aforementioned works mostly focus on power control with fixed channels or channels with slow fading. However, in several occasions, the channel experiences fast fading, and even though communication systems are often adaptive to such impairments, fading can change fast enough and no adaptation can be achieved. For the *discrete-time* power control case, there have been studies for power control with time-varying channels. For example, in [16] it is shown that the power control algorithms designed for fixed channel conditions may not work properly, since they fail to capture the dynamics of the time-varying channel. Additionally, a distributed power control algorithm based on stochastic approximation is proposed in [16] that eventually converges to the optimal power allocation. Long- and short-term fading wireless channel models are developed in [17], [18], in which iterative distributed power control algorithms are proposed,

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based on stochastic approximation; the FM algorithm constitutes a special case of their proposed algorithm by keeping a constant step size. The stability certificates of continuous-time power control algorithms, such as the FM algorithm, when there is uncertainty in the channel gains is underdeveloped. In [19], the authors propose a general framework for continuous time-power control algorithm under time-varying long term fading wireless channels and provide a sufficient conditions on the channels' attenuation coefficients for the existence of an optimal power. In [20] the conditions for which the FM algorithm is convergent are derived, even in the presence of bounded time-varying communication delays, and in the presence of topology changes arbitrarily between a fixed number of different configurations. While all these works aim at developing power control approaches, there are limited works on probabilistic stability guarantees (e.g., [21], [22]) and they do not discuss the scalability properties of the overall system. In this work, we aim at closing this gap.

A. Contributions

In this paper, we study the impact of parameter uncertainty and network structure on the stability and scalability of the FM power control algorithm. More specifically, the contributions of this work are as follows:

- we provide the stability certificates of the continuous- and discrete-time FM algorithms in a network with Rayleigh fading links. More specifically, we provide a probabilistic expression for the overall system to be stable. Note that this expression is similar to the probability density function of the signal-to-interference ratio, first provided in [21] and later in the seminal paper [22], in which the authors study the problem of minimizing the worst-case outage probability using power control.
- The effect of the parameters of the distributions of the Rayleigh fading links is studied, accounting for the scalability of the overall system for different node distributions on the plane, giving insights on how the overall system should be designed, such that the system remains stable as it scales up.
- Numerical results demonstrate the validity of our results and provide more insights on the robustness and scalability of wireless networks with interfering channels.

B. Organization

The rest of the paper is organized as follows. In Section II, we introduce the notation that will be used and review some preliminaries that are useful for the development of the results in this paper. In Section III, we give the system model, consisting of the network model and the channel model. Moreover, we review linear interference functions and the conditions for existence of feasible power levels. In Section IV, we review the FM algorithm. The main results are given in Section V. The validity of our results is justified by illustrative examples in Section VII. Finally, concluding remarks and directions for future work are given in Section VIII.

II. NOTATION AND PRELIMINARIES

A. Notation

Throughout the paper, vectors are written in bold lower case letters and matrices in capital letters. A^T and A^{-1} denote the transpose and inverse of matrix A respectively. By I we denote the identity of a squared matrix. $|A|$ is the elementwise absolute value of the matrix (i.e. $|A| \triangleq [|A_{ij}|]$), $A(<) \leq B$ is the (strict) element-wise inequality between matrices A and B . A matrix whose elements are nonnegative, called nonnegative matrix, is denoted by $A \geq 0$ and a matrix whose elements are positive, called positive matrix, is denoted by $A > 0$. $\lambda(A)$ denotes an eigenvalue of matrix A , and $\rho(A)$ denotes its spectral radius. $\text{diag}(x_i)$ the matrix with elements x_1, x_2, \dots on the leading diagonal and zeros elsewhere. We denote the probability density function (PDF) of an Exponential(α) variable by $f_{\text{exp}}(\alpha, x)$.

B. Gershgorin circle theorem

Let A be a complex $n \times n$ matrix with entries a_{ij} . For $i = \{1, \dots, n\}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ and let D_i be the closed disk of radius R_i centered at a_{ii} . Such disc is called a Gershgorin disk of matrix A .

Theorem 1 (Gershgorin [23]). *Let α be an eigenvalue of A , then there exists at least one $i \in \{1, \dots, n\}$ such that $\alpha \in D_i$.*

III. SYSTEM MODEL

The system model can be divided into two levels: level 1 describing the network as a whole; and level 2 describing the channels. Thus, we have the network model and the channel model. At the network level, the model concerns the general topology of the nodes and their characteristics. At the channel level, the model describes the assessment of the link quality between communication pairs and the interaction between the nodes in the network.

A. Network Model

Consider a network where \mathcal{T} denotes the set of transmitters and \mathcal{R} denotes the set of receivers in the network. The links are assumed to be unidirectional and each node is supported by omnidirectional antennae. For a planar network (easier to visualize without loss of generality), this can be represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where \mathcal{N} is the set of all nodes and \mathcal{L} is the set of the active links in the network. At each time instant, each node can act as a receiver or a transmitter only due to the half-duplex nature of the wireless transceiver. Each transmitter aims at communicating with a single node (receiver) only.

B. Channel model

The signal received at a receiving node j from a transmitting node i is given by $y_j = h_{ij}x_i + w_j$, where x_i is the signal transmitted by node i , h_{ij} denotes the channel coefficient for the link $i \rightarrow j$ and captures the effects of path-loss, shadowing and fading, and w_j captures the effects of receiver noise and other forms of interference at the receiving

node j . The quality of the wireless channels, h_{ij} , is degraded by Additive White Gaussian Noise (AWGN) and frequency non-selective Rayleigh block fading according to a complex Gaussian distribution with zero mean and variance σ_{ij}^2 for the link $i \rightarrow j$, i.e., $h_{ij} \sim \text{Rayleigh}(\sigma_{ij})$. The channel gains $g_{ij} \triangleq |h_{ij}|^2$ are, therefore, exponentially distributed, i.e., $g_{ij} \sim \text{Exp}(\sigma_{ij}^{-2}/2)$. Hence, all the g_{ij} 's are positive and can take values in the range $(0, 1]$.

C. Linear Interference Functions

The power level chosen by transmitter i is denoted by p_i and the intended receiver is also indexed by i . The interference power at the i th receiver consists of both the interference caused by other transmitters in the network $\sum_{j \in \mathcal{T}_{-i}} g_{ji} p_j$ (where \mathcal{T}_{-i} denotes all the transmitters j in the network that interfere with transmitter's i communications, i.e. $j \neq i, j \in \mathcal{T}$), and the thermal noise ν_i in node i 's receiver. Therefore, the interference at the receiver i , I_i , is given by $I_i(\mathbf{p}_{-i}) = \sum_{j \in \mathcal{T}_{-i}} g_{ji} p_j + \nu_i$, where \mathbf{p}_{-i} is the vector of powers levels of all transmitters except p_i , i.e., $\mathbf{p}_{-i} \triangleq [p_1 \dots p_{i-1} \ p_{i+1} \dots p_{|\mathcal{T}|}]$. The link quality is measured by the Signal-to-Interference-and-Noise Ratio (SINR), given by

$$\Gamma_i(\mathbf{p}) = \frac{g_{ii} p_i}{\sum_{j \in \mathcal{T}_{-i}} g_{ji} p_j + \nu_i}, \quad (1)$$

where \mathbf{p} is the vector of powers levels of all transmitters. Due to the unreliability of the wireless links, it is necessary to ensure QoS in terms of SINR in wireless networks. Hence, independently of nodal distribution and traffic pattern, a transmission from transmitter i to its corresponding receiver is successful (error free) if the SINR of the receiver is greater or equal to γ_i ($\Gamma_i(\mathbf{p}) \geq \gamma_i$), called the *capture ratio* which depends on the modulation and coding characteristics of the radio. Therefore,

$$\frac{g_{ii} p_i}{\sum_{j \in \mathcal{T}_{-i}} g_{ji} p_j + \nu_i} \geq \gamma_i. \quad (2)$$

Inequality (2) which depicts the QoS requirement of a communication pair i while transmission takes place is equivalent to the following condition:

$$p_i \geq \gamma_i \left(\sum_{j \in \mathcal{T}_{-i}} \frac{g_{ji}}{g_{ii}} p_j + \frac{\nu_i}{g_{ii}} \right). \quad (3)$$

In matrix form, for a network consisting of n communication pairs, this can be written as

$$\mathbf{p} \geq \Gamma G \mathbf{p} + \boldsymbol{\eta}, \quad (4)$$

where $\Gamma = \text{diag}(\gamma_i)$, $\mathbf{p} = [p_1 \ p_2 \ \dots \ p_n]^T$, $\eta_i = \gamma_i \nu_i / g_{ii}$ and

$$G_{ij} = \begin{cases} 0 & , \text{ if } i = j, \\ \frac{g_{ji}}{g_{ii}} & , \text{ if } i \neq j. \end{cases}$$

Let $C = \Gamma G$ such that

$$C_{ij} = \begin{cases} 0 & , \text{ if } i = j, \\ \gamma_i \frac{g_{ji}}{g_{ii}} & , \text{ if } i \neq j, \end{cases} \quad (5)$$

then (4) can be written as

$$(I - C)\mathbf{p} \geq \boldsymbol{\eta}. \quad (6)$$

Matrix C has strictly positive off diagonal elements and, since we are not considering isolated group of links that do not interfere with each other, it is reasonable to assume that C is irreducible [23]. By the Perron-Frobenius theorem [23], the spectral radius of C is a simple eigenvalue and its corresponding eigenvector is positive componentwise. The necessary and sufficient condition for the existence of a nonnegative solution to inequality (6) for every positive vector $\boldsymbol{\eta}$ is that $(I - C)^{-1}$ exists and is nonnegative. However, $(I - C)^{-1} \geq \mathbf{0}$ if and only if $\rho(C) < 1$ [24, Theorem 2.5.3], [25]. Therefore, the necessary and sufficient condition for (6) to have a positive solution \mathbf{p}^* for a positive vector $\boldsymbol{\eta}$ is that the Perron-Frobenius eigenvalue of the matrix C is less than 1. Hence, a network described by matrix C is feasible if and only if $\rho(C) < 1$.

IV. THE FOSCHINI-MILJANIC ALGORITHM

A. Continuous-time algorithm

The FM algorithm, defined by the following differential equation [4], is given by

$$\dot{p}_i(t) = k_i \left(-p_i(t) + \gamma_i \left(\sum_{j \in \mathcal{T}_{-i}} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{\nu_i}{g_{ii}} \right) \right), \quad (7)$$

where $k_i \in \mathbb{R}_+$ denotes the proportionality constant and γ_i denotes the capture ratio. The power control algorithm (7) can be written in matrix form as

$$\dot{\mathbf{p}}(t) = K(H\mathbf{p}(t) + \boldsymbol{\eta}), \quad (8)$$

where $K = \text{diag}(k_i)$ and $H = I - C$. For this differential equation, it is proven that the overall system converges to the optimal set of solutions, $\mathbf{p}^* > \mathbf{0}$, for any initial power vector, $\mathbf{p}(0) > \mathbf{0}$, provided $\rho(C) < 1$. Hence, the distributed algorithm (7) for each communication pair, leads to global stability of the system.

B. Discrete-time algorithm

In the case of discrete-time FM algorithm, we have the difference equation

$$p_i(t+1) = (1 - k_i)p_i(t) + k_i \gamma_i \left(\sum_{j \neq i} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{\nu_i}{g_{ii}} \right). \quad (9)$$

Taking all the nodes together, (9) can be written in matrix form as

$$\mathbf{p}(t+1) = (I - KH)\mathbf{p}(t) + K\boldsymbol{\eta}, \quad (10)$$

and as long as k_i is chosen in the interval $(0, 1]$ the iterative algorithm (9) converges from any initial values for the power levels of the individuals transmitters, provided again that $\rho(C) < 1$.

V. PROBABILISTIC STABILITY CERTIFICATES

As mentioned in Section IV, both the continuous- and discrete-time FM algorithms are globally asymptotically stable if and only if $\rho(C) < 1$. Since for non-negative matrices, $\rho(C) \leq \|C\|_\infty$, a sufficient (but more conservative) condition guaranteeing stability of the system without requiring the knowledge of the whole matrix C is $\|C\|_\infty < 1$, i.e.,

$$g_{ii} > \gamma_i \sum_{j \neq i, j \in \mathcal{T}} g_{ji}, \quad \forall i \in \mathcal{T}. \quad (11)$$

This condition is equivalent to H being a diagonally dominant matrix with all main diagonal entries being positive, which is equivalent to the Gershgorin circle theorem. Thus, it provides an upper bound on the achievable target SINR levels in a given network. We can, therefore use the Gershgorin circle theorem to identify the probability of the worst case scenario, i.e., when at least one eigenvalue has the possibility to move to the unstable region for the FM algorithms (i.e., in the continuous-time the positive real half-plane, and in the discrete-time the unit disc).

Since the channel conditions are time-varying, the goal of this work is to characterize the conditions for stability (convergence) of the FM power control algorithms in probability. In Proposition 1, we give such a probabilistic condition that holds under some assumptions.

Proposition 1. *Consider n active connections where $g_{ij} \sim \text{Exponential}(\beta)$ when $i \neq j$ and $g_{ii} \sim \text{Exponential}(\alpha)$. Assume, without loss of generality, that γ_i has the same value for everyone and that $0 < k_i < 1$ (important only for the discrete time case). Under these assumptions, the probability $\Theta \triangleq P(z < 0)$, $z \in \mathbb{R}$, that equation (11) holds is given by*

$$\Theta = \left(\frac{1}{1 + \gamma\alpha/\beta} \right)^{n(n-1)}. \quad (12)$$

Proof. Let $w_i = \sum_{j \neq i} g_{ij}$. Since g_{ij} 's are $\text{Exponential}(\beta)$ random variables (RVs) and w_i is the sum of $n-1$ g_{ij} 's, then w_i is a $\text{Gamma}(n-1, \beta)$ RV (and since n is an integer, w_i is an $\text{Erlang}(n-1, \beta)$ RV), i.e., the PDF of w_i is

$$f_w(x) = \frac{\beta^{n-1}}{(n-2)!} x^{n-2} e^{-\beta x}. \quad (13)$$

Let $v_i = \gamma w_i$. We use the rule about transformation of RVs and we get that v_i is a $\text{Gamma}(n-2, \beta/\gamma)$ RV with PDF

$$f_v(x) = \frac{(\beta/\gamma)^{n-1}}{(n-2)!} x^{n-2} e^{-\beta x/\gamma}. \quad (14)$$

Then, we define $u_i = v_i - g_{ii}$. The PDF of z_i is given by

$$f_u(x) = \int_{-\infty}^{\infty} f_v(x+s) f_{\exp}(\alpha, s) ds. \quad (15)$$

Since both v_i and g_{ii} cannot be negative, the above relation gets simplified to

$$f_u(x) = \begin{cases} \int_0^{\infty} f_v(x+s) f_{\exp}(\alpha, s) ds, & \text{if } x \geq 0, \\ \int_{-x}^{\infty} f_v(x+s) f_{\exp}(\alpha, s) ds, & \text{if } x < 0. \end{cases}$$

These integrals have known closed forms, so finally we get

$$f_u(x) = \begin{cases} \alpha \left(\frac{\beta}{\gamma\alpha + \beta} \right)^{n-1} \frac{\Gamma(n-1, (\alpha + \beta/\gamma)x)}{(n-2)!} e^{\alpha x}, & \text{if } x \geq 0, \\ \alpha \left(\frac{\beta}{\gamma\alpha + \beta} \right)^{n-1} e^{\alpha x}, & \text{if } x < 0, \end{cases}$$

where by Γ we denote the *upper incomplete gamma function* defined by $\Gamma(k, x) = \int_x^{\infty} s^{k-1} e^{-s} ds$. The probability that u_i is negative is

$$\mathbb{P}(u_i < 0) = \int_{-\infty}^0 f_u(x) dx = \left(\frac{\beta}{\gamma\alpha + \beta} \right)^{n-1}. \quad (16)$$

Finally, we define $z = \max\{u_1, \dots, u_n\}$. The probability of z being negative is equal to the probability of all u_i 's being negative, that is

$$\mathbb{P}(z < 0) = \left(\frac{\beta}{\gamma\alpha + \beta} \right)^{n(n-1)} = \left(\frac{1}{\gamma\alpha/\beta + 1} \right)^{n(n-1)}. \quad \square$$

If $z < 0$, then we know that H is diagonally dominant and all Gershgorin discs are contained in the negative real plane; thus, the system is stable. If $z > 0$ then this means that at least a part of one of the Gershgorin disks is in the positive real plane (while the eigenvalues may still all have negative real parts). However, notice that unlike the case $z < 0$, the case $z > 0$ does not guarantee that the system is unstable.

VI. EFFECT OF THE PARAMETERS

Equation (12) shows that the probability $\Theta = P(z < 0)$ depends only on n and the ratio $\phi = \gamma\alpha/\beta$. When ϕ is a constant then this probability goes to 0 as n grows. However, in some application it may be reasonable to assume that ϕ is a function of n , see for example Section VI-A. Then, the limit $\lim_{n \rightarrow \infty} \Theta$ depends on how fast $\phi(n)$ decays. We have the following proposition:

Proposition 2. *Let $c > 0$, $q > 0$ and $\phi(n) = cn^{-q} + o(n^{-q})$ for $n \in \mathbb{N}$, then we have the following cases:*

- 1) *If $q < 2$, then $\lim_{n \rightarrow \infty} \Theta = 0$.*
- 2) *If $q = 2$, then $\lim_{n \rightarrow \infty} \Theta = e^{-c}$.*
- 3) *If $q > 2$, then $\lim_{n \rightarrow \infty} \Theta = 1$.*

Proof. Recall that $\Theta(n) = (1 + \phi(n))^{-n(n-1)}$. It is straightforward to check that $x - x^2 \leq \log(1+x) < x$ holds for all $|x| \leq 1/2$. Let n be big enough so that $\phi(n) < 1/2$, then

$$\phi(n) - \phi(n)^2 \leq \log(1 + \phi(n)) \leq \phi(n).$$

By the assumption for $\phi(n)$ we have that $\phi(n)^2 = O(n^{-2q})$ which means that

$$\phi(n) - \phi(n)^2 = cn^{-q} + o(n^{-q}).$$

We exponentiate the above inequality and we get

$$e^{cn^{-q} + o(n^{-q})} \leq 1 + \phi(n) \leq e^{cn^{-q} + o(n^{-q})}.$$

Finally, we raise this relation to the power $-n(n-1)$. Thus, for the lower bound we have

$$\begin{aligned} (1 + \phi(n))^{-n(n-1)} &\geq e^{-(cn^{2-q} + cn^{1-q} + o(n^{2-q}))} \\ &= e^{-(cn^{2-q} + o(n^{2-q}))}. \end{aligned}$$

Similarly, for the upper bound we have

$$(1 + \phi(n))^{-n(n-1)} \leq e^{-(cn^{2-q} + cn^{1-q} + o(n^{2-q}))} \\ = e^{-(cn^{2-q} + o(n^{2-q}))}.$$

From the last two relation we see that there are 3 cases:

- 1) If $q < 2$, then the exponent of n is positive, so $\lim_{n \rightarrow \infty} (1 + \phi(n))^{-n(n-1)} = 0$.
- 2) If $q = 2$, then the exponent of e becomes $-c + o(1)$, so $\lim_{n \rightarrow \infty} (1 + \phi(n))^{-n(n-1)} = e^{-c}$.
- 3) If $q > 2$, then the exponent of n is negative, so $\lim_{n \rightarrow \infty} (1 + \phi(n))^{-n(n-1)} = 1$. \square

A. Closest neighbour connections

Let us assume that that we have n transmitters and n receivers. Each device aims to communicate with its closest neighbour. We assume that the parameter α is inversely proportional to a power of the distance, τ , the minimum distance and the parameter β is inversely proportional to the same power of the mean distance. It is known that τ depends on the environment and can be anything between 2 (in free space) and 6; see, e.g., [26].

1) *Nodes distributed normally on the plane:* We assume that each device is located on the plane and each coordinate is a standard normal random variable. This means that each receiver has n transmitters in its vicinity and it tries to listen to its closest one. Let r_i denote the distance between the receiver and the i 'th transmitter. We want to calculate the expected mean distance and the expected minimum distance.

Lemma 2. *We have*

$$E[\text{mean}(\{r_i, \dots, r_n\})] = \sqrt{\frac{\pi}{2}}$$

and

$$E[\text{min}(\{r_i, \dots, r_n\})] = \sqrt{\frac{\pi}{2n}}.$$

This lemma implies that we expect for large n to have $\phi(n) = \gamma\alpha/\beta \approx c/n^{\tau/2}$. This means that if $\tau < 4$, the probability that the Gershgorin discs are contained in the negative real half plane as we increase n tends to zero. However if $\tau > 4$ the probability that the system will remain stable as we increase n tends to 1.

Proof of Lemma 2. Let x_1 and x_2 be standard normal random variables and define $y = x_1 - x_2$. By following the relation about the difference of random variables, we get that $f_y(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$. Then, we get the PDF of $z = y^2$ be using the relation for the transformation of random variables with $\phi(x) = x^2$ and we get $f_z(x) = \frac{e^{-x/2}}{\sqrt{2\pi x}}$. If z_1 and z_2 have the above PDF then we can find the PDF of $w = z_1 + z_2$ by using the relation for the sum of random variables and we get it is exponential with $f_w(x) = \frac{e^{-x/2}}{2}$. We use again the relation for the transformation of random variable to get the PDF of $r = \sqrt{w}$ and we get $f_r(x) = xe^{-x^2/2}$. The cumulative distribution function (CDF) of r is given by $F_r(x) = 1 - e^{-x^2/2}$. Since all r_i 's

are independent and expectation is a linear functional, we have $E[\text{mean}(\{r_i, \dots, r_n\})] = E[r] = \sqrt{\pi/2}$.

Let $\mu = \min(\{r_i, \dots, r_n\})$, then we need to observe that

$$1 - F_\mu(x) = \mathbb{P}(\mu > x) = \mathbb{P}(\min(\{r_i, \dots, r_n\}) > x) \\ = \prod_{i=1}^n \mathbb{P}(r_i > x) = (1 - F_r(x))^n.$$

Hence, we find $F_\mu(x) = 1 - (1 - F_r(x))^n = 1 - e^{-nx^2/2}$ and $f_\mu(x) = nxe^{-nx^2/2}$. Finally, we have $E[\mu] = \int_0^\infty xf_\mu(x)dx = \sqrt{\frac{\pi}{2n}}$. \square

2) *Nodes distributed uniformly on the unit square:* Let us assume now that we distribute the devices uniformly in the unit square. As before we choose a receiver and we define r_i to be the distance between it and the i 'th transmitter. Then for the expected mean and minimum distance we have the following result:

Lemma 3. *We have*

$$E[\text{mean}(\{r_i, \dots, r_n\})] \approx 0.5214$$

and for large n there exists $C > 0$ such that

$$E[\text{min}(\{r_i, \dots, r_n\})] \approx \frac{C}{\sqrt{n}}.$$

As it is the case of normally distributed points, we see that whether the system becomes unstable depends on τ .

Lemma 3. Let x_1 and x_2 be uniform $[0, 1]$ random variables and let $y = |x_1 - x_2|$, then the PDF of y is supported on $[0, 1]$ where it has the form $f_y(x) = 2 - 2x$. We define $z = y^2$ and we get $f_z(x) = \frac{1}{\sqrt{x}} - 1$. If z_1 and z_2 have the above PDF, then the PDF of $w = z_1 + z_2$ is given by

$$f_w(x) = \begin{cases} \int_0^x f_z(s)f_z(x-s)ds, & 0 \leq x \leq 1 \\ \int_{x-1}^1 f_z(s)f_z(x-s)ds, & 1 < x \leq 2 \end{cases}$$

which simplifies to

$$f_w(x) = \begin{cases} \pi - 4\sqrt{x} + x & , 0 \leq x \leq 1 \\ 2 \arccsc(\sqrt{x}) + 4\sqrt{x-1} \\ + 2 \arctan(\sqrt{x-1}) - x - 2 & , 1 < x \leq 2. \end{cases}$$

We define $r = \sqrt{w}$ and we get

$$f_r(x) = \begin{cases} 2x(\pi - 4x + x^2) & , 0 \leq x \leq 1 \\ 2x(2 \arccsc(x) + 4\sqrt{x^2-1} \\ + 2 \arctan(\sqrt{x^2-1}) - x^2 - 2) & , 1 < x \leq \sqrt{2}. \end{cases}$$

By integration we can get the CDF:

$$F_r(x) = \begin{cases} \pi x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4 & , 0 \leq x \leq 1 \\ \frac{4}{3}\sqrt{x^2-1}(2x^2+1) \\ + 2x^2 \arccsc(x) \\ - 2x^2 \arctan(\sqrt{x^2-1}) \\ + \frac{x^4}{2} - \frac{8x^3}{3} + \pi x^2 & , 1 < x \leq \sqrt{2}. \end{cases}$$

Then, we have $E[r] = \int_0^{\sqrt{2}} x f_r(x) dx \approx 0.5214$.

Just like before, we define $\mu = \min(\{r_i, \dots, r_n\})$ and it holds $F_\mu(x) = 1 - (1 - F_r(x))^n$, so we find $f_\mu(x) = n f_r(x)(1 - F_r(x))^{n-1}$ and $E[\mu] = \int_0^{\sqrt{2}} x f_\mu(x) dx$. This integral does not have a simple closed form from which we can deduce its asymptotic behaviour for large n . However we can deduce the asymptotic behaviour by analyzing the integrand. First we observe that the f_μ goes exponentially to 0 as we increase n for all $x > 1$. So for large n we can write

$$E[\mu] \approx \int_0^1 x f_\mu(x) dx = \int_0^1 n x f_r(x) (1 - F_r(x))^{n-1} dx.$$

Then, we notice that the distribution is unimodal for all n . This is due to the fact that $(1 - F_r(x))^{n-1} = 1$ at $x = 0$ and converges to 0 exponentially as we increase n for all $x > 0$. Moreover $f_r(x)$ is unimodal, $f_r(0) = 0$ and quadratically flat at $x = 0$. This means that there is only one maximum that converges to 0 and as n increases and the width of the peak decreases; see Fig. 1.

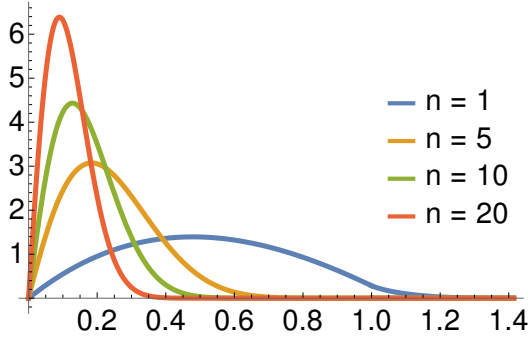


Fig. 1: The graph of f_μ for several values of n . Notice that when $n = 1$, $f_\mu = f_r$.

This means that we only need to analyze the behaviour for large n of f_μ close to 0. Notice that if we have already n points on the plane, then the number of them that are in a disk of a small radius R is equivalent to a 2d Poisson point process with parameter $n\pi R^2$. So when we add another point the probability of it being at a distance bigger than R from any of the existing points is the Poisson distribution evaluated at 0, i.e., $P(\mu \geq R) \approx e^{-n\pi R^2}$. Thus, we find that for big n and small x , $f_\mu(x) \approx 2\pi n x e^{-n\pi R^2}$. From this we can easily deduce that $E[\mu] \approx \frac{1}{2\sqrt{n}}$, which recovers the asymptotic behaviour of $E[\mu]$ up to a multiplicative constant. \square

VII. NUMERICAL EXAMPLES

We have checked numerically the validity of Θ in predicting the stability of the system. The results are shown in Fig. 2. The numerical results were obtained by the Monte Carlo method with 1000 trials. We notice that the probability the system is stable, denoted here by Π , which we calculate from Monte Carlo is always higher than what Θ predicts and this is because Θ is the probability that all Gershgorin discs are contained in the negative real half-plane. Of course this is a lower bound for Π and this is exactly what we

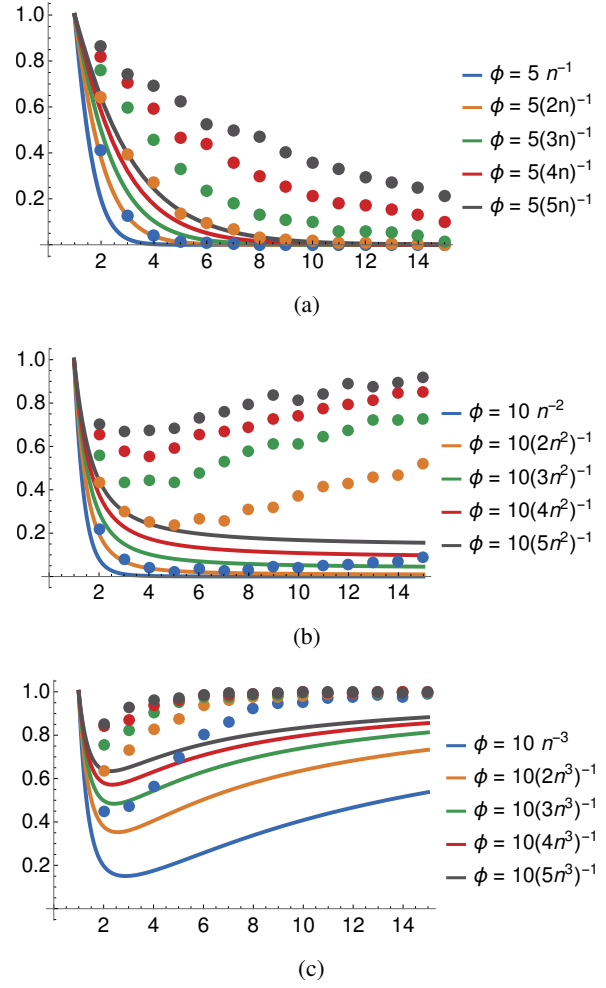


Fig. 2: Probability of the system being stable for different values of n is shown in the figures. Solid lines represent Θ and dots represent numerical values obtained by the Monte Carlo method.

observe. Notice that when the exponent of n is not -2 , Π and Θ have the same trend. Interestingly in the case of the exponent being -2 , Θ tends to some number between 0 and 1, but Π increases towards 1.

We have also checked numerically the distribution of the average and minimum distance between points on the plane. For each trial we have generated $n + 1$ random points. The last point generated was the one against which the distances were measured. We have performed 10000 trials for each n . Fig. 3 shows the average distance, the minimum distance and the minimum distance multiplied by \sqrt{n} . Notice that the x axis denotes the value of n in thousands. We see that, as expected, the average distance and the minimum distance multiplied by \sqrt{n} are independent of n .

VIII. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we studied the stability of the FM algorithm when the channel gains are uncertain. More specifically, we provided probabilistic conditions with respect to the

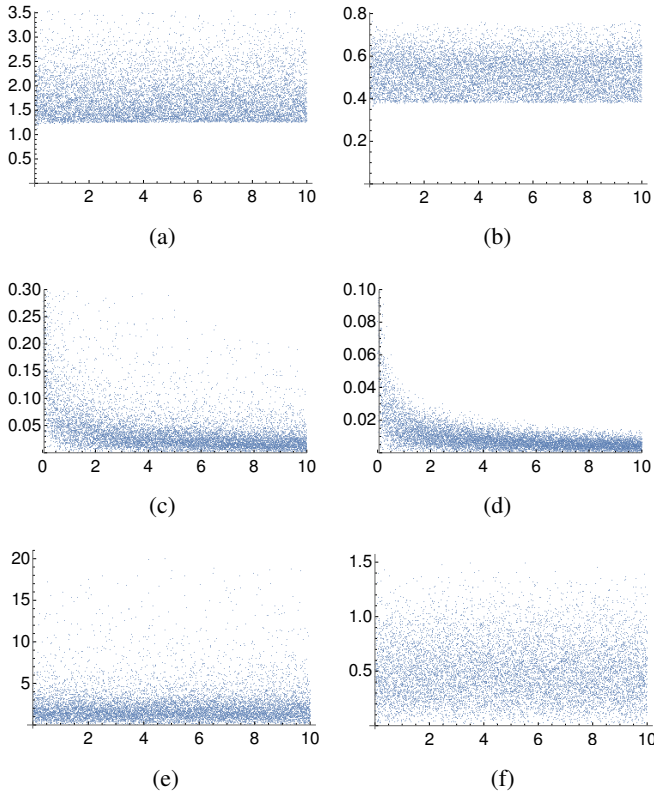


Fig. 3: The first column shows the graphs for the average and minimum distance in the case of normally distributed points. The second column shows the graphs for the average and minimum distance in the case of uniformly distributed points. The numbers in the x axis corresponds to thousands. In (a) and (b) the average distance with n neighbours is plotted against n , in (c) and (d) the minimum distance with n neighbours is plotted against n and in (e) and (f) the minimum distance scaled by \sqrt{n} .

parameters of the channel distributions for which the system is stable. Furthermore, we studied the effects of these parameters for different node distribution on the plane. The observations of this work open up the possibility of designing the system such that it is always stable as it scales up. We can do so by using localization algorithms to position the devices that need to communicate more often closer and by making sure that power of the interfering signals decay as fast as possible; this can be achieved, for example in MIMO systems, by using directional antennae.

Part of future work involves studying other distributions, such as the Nakagami-m distribution.

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