Output feedback stabilization of Reaction-Diffusion PDEs with distributed input delay

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***Abstract*— This paper studies the boundary output feedback stabilization of reaction-diffusion PDEs in the presence of an arbitrarily long distributed input delay. The boundary control applies at the right boundary through a Robin boundary condition while the system output is selected as the left boundary Dirichlet trace. The actual control input applies to the boundary via a distributed delay spanning over a finite time interval. The proposed control strategy leverages a predictor feedback relying on Artstein's reduction method and is coupled with a finite-dimensional observer. Provided a structural controllability assumption, sufficient stability condition are derived and are shown to be always feasible provided the order of the observer is selected to be large enough.**

I. Introduction

The topic of stabilization of finite-dimensional systems in the presence of delays has been widely studied in the literature [1]. In the case of an arbitrarily long input delay, predictor feedback, which leverages the Artstein transformation [2], has emerged as the predominant control design method in both linear and nonlinear cases [3]. This paper focuses on the case of a distributed input delay, i.e. a delayed term of the form $\int_0^h \varphi(\sigma) u(t-\sigma)\,\mathrm{d}\sigma$ where $h > 0$ is the delay horizon, $\varphi \in L^2(0,h)$ is a given function, and $u(t) \in \mathbb{R}$ is the actual control input. Predictor feedback to compensate distributed input delays roots back to [2] and has been extended in a number of directions [4]–[12].

This paper focuses on the boundary output feedback stabilization of reaction-diffusion PDEs in the presence of an arbitrarily long distributed input delay. So far, this type of control design problem has solely been addressed in the context of an arbitrarily long discrete input delay. The state-feedback case has been addressed first in [13] using backstepping design and in [14] by leveraging spectral reduction methods [15]–[17] combined with predictor feedback. This latter approach has then been extended in a number of directions that include diagonal infinite-dimensional systems [18], robustness with respect to delay mismatches for time [19], [20] and spatially varying delays [21], and PI regulation control [22]. While all the above mentioned approaches embraced the case of a state-feedback, the possibility to address the case of an output feedback by coupling a predictor feedback with a finite-dimensional observer [23]–[31] has recently been demonstrated. Using the observer architecture from [30], the first step into that direction was reported in [32] in the specific and structurally limited setting of a Neumann boundary control and a bounded observation operator. Then a complete solution to this control design problem for general 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurement was reported in [33]. The dual problem, namely the case of an arbitrarily long output delay, was addressed in [34]. The problem of a state-delay was addressed in [35].

In contrast with the works mentioned in the previous paragraph, this paper addresses for the first time the case of a distributed input delay. More precisely, we consider general 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann measurement in the presence of an arbitrarily long distributed input delay. By adapting the procedures reported in [33] in the case of a discrete input delay, we demonstrate in this paper that an output feedback control strategy can always be designed in order to achieve the exponential stabilization of the plant provided a structural controllability assumption.

The paper is organized as follows. Notation and basic properties of Sturm-Liouville operators are presented in Section II. The problem setting studied in this paper is introduced in Section III. Then the control strategy is described in Section IV. The exponential stability assessment of the resulting closed-loop system is reported in Section V. A numerical illustration is carried out in Section VI. Finally, concluding remarks are formulated in Section VII.

II. Preliminaries

A. Notation

Real spaces $\mathbb{R}^n$ of dimension n are equipped with the Euclidean norm denoted by $\|\cdot\|$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors X and Y of arbitrary dimensions, we define the vector $\mathrm{col}(X,Y) = [X^\top, Y^\top]^\top$. $L^2(0,1)$ stands for the space of square integrable functions on $(0,1)$ and is equipped with the inner product $\langle f,g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ whose associated norm is denoted by $\|\cdot\|_{L^2}$. For any given integer $m \geq 1$, $H^m(0,1)$ stands for the m-order Sobolev space which is endowed with its usual norm denoted by $\|\cdot\|_{H^m}$. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that P is positive semi-definite (resp. positive definite).

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B. Properties of Sturm-Liouville operators

Reaction-diffusion PDEs are strongly related to the concept of Sturm-Liouville operators. We summarize in this subsection the key properties of these operators that will be intensively used in the sequel.

Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^1([0,1])$ and $q \in \mathcal{C}^0([0,1])$ with $p > 0$ and $q \geq 0$. The Sturm-Liouville operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \rightarrow L^2(0,1)$ is defined by

$$\mathcal{A}f = -(pf')' + qf$$

on the domain of definition

$$\begin{aligned} D(\mathcal{A}) = \{f \in H^2(0,1) \, : \, & c_{\theta_1} f(0) - s_{\theta_1} f'(0) = 0 \\ & c_{\theta_2} f(1) + s_{\theta_2} f'(1) = 0\}. \end{aligned}$$

Here we used the short notations $c_{\theta_i} = \cos \theta_i$ and $s_{\theta_i} = \sin \theta_i$.

The eigenvalues λ_n, $n \geq 1$, of the Sturm-Liouville operator $\mathcal{A}$ are simple, non negative (because $\theta_1, \theta_2 \in [0, \pi/2]$ and $q \geq 0$), and form an increasing sequence with $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$. The associated unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. The domain of the operator $\mathcal{A}$ can be characterized in function of the eigenstructures as follows:

$$D(\mathcal{A}) = \{f \in L^2(0,1) \, : \, \sum_{n \geq 1} |\lambda_n|^2 |\left\langle f, \phi_n \right\rangle|^2 < +\infty\}.$$

For any given $p_*, p^*, q^* \in \mathbb{R}$ so that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, we have

$$0 \leq \pi^2 (n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$$

for all $n \geq 1$; see [36] for details. Moreover with the further regularity $p \in \mathcal{C}^2([0,1])$, we have $\phi_n(\xi) = O(1)$ and $\phi_n'(\xi) = O(\sqrt{\lambda_n})$ as $n \rightarrow +\infty$ for any given $\xi \in [0,1]$; see also [36] or [37] for details. Assuming that $q > 0$, an integration by parts and the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$ show the existence of constants $C_1, C_2 > 0$ so that

$$C_1 \|f\|_{H^1}^2 \leq \sum_{n \geq 1} \lambda_n \left\langle f, \phi_n \right\rangle^2 = \left\langle \mathcal{A}f, f \right\rangle \leq C_2 \|f\|_{H^1}^2 \quad (1)$$

for all $f \in D(\mathcal{A})$. Combining (1) with the Riesz-spectral property of $\mathcal{A}$, we deduce that the series expansion $f = \sum_{n \geq 1} \left\langle f, \phi_n \right\rangle \phi_n$ is convergent in $H^2(0,1)$ norm for any $f \in D(\mathcal{A})$. Invoking the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$, we obtain that $f(0) = \sum_{n \geq 1} \left\langle f, \phi_n \right\rangle \phi_n(0)$ and $f'(0) = \sum_{n \geq 1} \left\langle f, \phi_n \right\rangle \phi_n'(0)$.

We conclude this section by defining for any integer $N \geq 1$ the quantity

$$\mathcal{R}_N f = \sum_{n \geq N+1} \left\langle f, \phi_n \right\rangle \phi_n$$

for all $f \in L^2(0,1)$.

III. Problem setting

Let the reaction-diffusion system with distributed input delay be described by

$$z_t(t,x) = (p(x) z_x(t,x))_x - \tilde{q}(x) z(t,x) \quad (2a)$$

$$c_{\theta_1} z(t,0) - s_{\theta_1} z_x(t,0) = 0 \quad (2b)$$

$$c_{\theta_2} z(t,1) + s_{\theta_2} z_x(t,1) = u_h(t) \triangleq \int_0^h \varphi(\sigma) u(t-\sigma) \, \mathrm{d}\sigma \quad (2c)$$

$$z(0,x) = z_0(x) \quad (2d)$$

for $t > 0$ and $x \in (0,1)$ where $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q} \in \mathcal{C}^0([0,1])$. Here $z(t, \cdot)$ is the state of the PDE at time t and z_0 is the initial condition. The command input $u(t) \in \mathbb{R}$ applies to the right boundary of the system through the introduction of a distributed input delay $u_h(t) \triangleq \int_0^h \varphi(\sigma) u(t-\sigma) \, \mathrm{d}\sigma$ for some delay $h > 0$ and with $\varphi \in L^2(0,h)$. For well-posedness assessment only, we further assume that there exists $h_m \in (0,h)$ so that $\varphi|_{[0,h_m]} = 0$. We also assume throughout the paper that $u(\tau) = 0$ for $\tau < 0$. Finally, restraining $\theta_1 \in (0, \pi/2]$, the system output is set as the left Dirichlet trace:

$$y(t) = z(t,0). \quad (3)$$

The objective is to achieve the output feedback stabilization of the plant described by (2-3).

Remark 3.1: In this study, the parameters θ_i of the Robin boundary conditions of the plant (2) are restricted to $\theta_1 \in [0, \pi/2]$ and $\theta_2 \in (0, \pi/2]$. Note however that the results presented in this paper can easily be extended to the general case $\theta_1 \in (0, \pi) \cup (\pi, 2\pi)$ and $\theta_2 \in [0, 2\pi)$; see [29], [33] for details.

IV. Control design

A. Spectral reduction

In preparation of control design and stability analysis, we pick $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ so that

$$\tilde{q}(x) = q(x) - q_c, \quad q(x) > 0. \quad (4)$$

Since the PDE (2) is non-homogeneous due do the boundary distributed delayed input $u_h(t)$, we introduce the change of variable

$$w(t,x) = z(t,x) - \frac{x^2}{c_{\theta_2} + 2 s_{\theta_2}} u_h(t). \quad (5)$$

Hence, introducing $v_h = \dot{u}_h$, we obtain the following equivalent homogeneous representation:

$$\dot{u}_h(t) = v_h(t) = \int_0^h \varphi(\sigma) \dot{u}(t-\sigma) \, \mathrm{d}\sigma \quad (6a)$$

$$w_t(t,x) = (p(x) w_x(t,x))_x - \tilde{q}(x) w(t,x) \quad (6b)$$

$$+ a(x) u_h(t) + b(x) v_h(t) \quad (6c)$$

$$c_{\theta_1} w(t,0) - s_{\theta_1} w_x(t,0) = 0 \quad (6d)$$

$$c_{\theta_2} w(t,1) + s_{\theta_2} w_x(t,1) = 0 \quad (6e)$$

$$w(0,x) = w_0(x) \quad (6f)$$

with $a(x) = \frac{1}{c_{\theta_2}+2s_{\theta_2}}\{2p(x) + 2xp'(x) - x^2\tilde{q}(x)\}$, $b(x) = -\frac{x^2}{c_{\theta_2}+2s_{\theta_2}}$, and $w_0(x) = z_0(x) - \frac{x^2}{c_{\theta_2}+2s_{\theta_2}}u_h(0) = z_0(x)$. We introduce the coefficients of projection

$$z_n(t) = \langle z(t,\cdot), \phi_n \rangle, \, w_n(t) = \langle w(t,\cdot), \phi_n \rangle, \\ a_n = \langle a, \phi_n \rangle, \, b_n = \langle b, \phi_n \rangle.$$

The projection of the change of variable formula (5) gives

$$w_n(t) = z_n(t) + b_n u_h(t), \quad n \geq 1. \tag{7}$$

Then the projection of (6) into the Hilbert basis $(\phi_n)_{n\geq 1}$ gives (see e.g. [38], [39] for details)

$$\dot{u}_h(t) = v_h(t) \tag{8a}$$

$$\dot{w}_n(t) = (-\lambda_n + q_c)w_n(t) + a_n u_h(t) + b_n v_h(t). \tag{8b}$$

In view of (8) and invoking (7), the projection of (2) reads

$$\dot{z}_n(t) = (-\lambda_n + q_c)z_n(t) + \beta_n u_h(t) \tag{9}$$

with

$$\begin{aligned}\beta_n &= a_n + (-\lambda_n + q_c)b_n \\ &= p(1)\{-c_{\theta_2}\phi_n'(1) + s_{\theta_2}\phi_n(1)\} = O(\sqrt{\lambda_n})\end{aligned}$$

Finally, for classical solutions the Dirichlet measurement $y(t)$ expressed by (3) can be written as the series expansion:

$$y(t) = z(t,0) = w(t,0) = \sum_{n\geq 1} w_n(t)\phi_n(0). \tag{10}$$

B. Control strategy

We first fix $\delta > 0$, the desired exponential decay rate for the closed-loop system trajectories. It allows us to fix $N_0 \geq 1$ so that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. For an arbitrarily given $N \geq N_0 + 1$, that will be constrained later, we define the following observer dynamics which is introduced in order to estimate the N first modes of the plant in (original) z coordinates:

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u_h(t) \tag{11a}$$

$$\begin{aligned}\dot{\hat{z}}_n(t) &= (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u_h(t) \\ &\quad - l_n \left\{ \sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t) \right\}, \; 1 \leq n \leq N_0\end{aligned} \tag{11b}$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u_h(t), \; N_0 + 1 \leq n \leq N \tag{11c}$$

where $l_n \in \mathbb{R}$ are the observer gains. This observer dynamics is inspired by the pioneer work [30] and the recent development [33] embracing the case of a discrete input delay. Recall that the control input $u(t)$ acts on the system through the distributed input delay $u_h(t) = \int_0^h \varphi(\sigma)u(t-\sigma)\,\mathrm{d}\sigma$. In order to compensate this distributed input delay, we need to introduce a predictor component. To achieve this, we define $\hat{Z}^{N_0} = \begin{bmatrix}\hat{z}_1 & \dots & \hat{z}_{N_0}\end{bmatrix}^\top$, $A_0 = \mathrm{diag}(-\lambda_1 + q_c, \dots, -\lambda_{N_0} + q_c)$, and $\mathfrak{B}_0 = \begin{bmatrix}\beta_1 & \dots & \beta_{N_0}\end{bmatrix}^\top$. This allows the introduction of the following Artstein transformation [2], [40]:

$$\hat{Z}_A^{N_0}(t) = \hat{Z}^{N_0}(t) + \int_{t-h}^{t}\int_{t-s}^{h} e^{A_0(t-s-\sigma)}\mathfrak{B}_0\varphi(\sigma)\,\mathrm{d}\sigma\, u(s)\,\mathrm{d}s. \tag{12}$$

Therefore, the control is set as:

$$u(t) = K\hat{Z}_A^{N_0}(t), \quad t \geq 0 \tag{13}$$

where $K \in \mathbb{R}^{1\times N_0}$ is the feedback gain.

Remark 4.1: For well-posedness assessment, we make the assumption that there exists $h_m \in (0,h)$ so that $\varphi|_{[0,h_m]} = 0$. In that case, if we denote by

$$\psi(t) = \int_{t-h}^{t}\int_{t-s}^{h} e^{A_0(t-s-\sigma)}\mathfrak{B}_0\varphi(\sigma)\,\mathrm{d}\sigma\, u(s)\,\mathrm{d}s$$

the double integral appearing in (12), it can be observed that ψ is the unique solution to the ODE

$$\dot{\psi}(t) = A_0\psi(t) + \mathbf{B}_0 u(t) - \mathfrak{B}_0 u_h(t)$$

with initial condition $\varphi(0) = 0$, where u is given by (13) with $\hat{Z}_A^{N_0} = \hat{Z}^{N_0} + \psi$, $u_h(t) = \int_0^h \varphi(\sigma)u(t-\sigma)\,\mathrm{d}\sigma = \int_{h_m}^h \varphi(\sigma)u(t-\sigma)\,\mathrm{d}\sigma$ with $0 < h_m < h$, and $\mathbf{B}_0 = \int_0^h e^{-A_0\sigma}\mathfrak{B}_0\varphi(\sigma)\,\mathrm{d}\sigma$. Hence, a similar induction argument to the one employed in [34] in the case of a discrete input delay can be used to obtain the well-posedness of the closed-loop system trajectories using standard well-posedness results in the context of C_0-semigroups [41].

C. Truncated model

In order to introduce the main stability result, we first need to write a finite dimensional model that captures the dynamics (11-13) of the output feedback controller as well as the N first modes of the PDE as described in z coordinates by (9). To do so, let the error of estimation be defined by $e_n = z_n - \hat{z}_n$ for $1 \leq n \leq N$. Owing to (11a-11b) and invoking (7) and (10), we obtain that

$$\dot{\hat{z}}_n = (-\lambda_n + q_c)\hat{z}_n + \beta_n u_h + l_n \sum_{k=1}^{N}\phi_k(0)e_k + l_n\zeta \tag{14}$$

for all $1 \leq n \leq N_0$. Here we have defined the residue of measurement $\zeta = \sum_{n\geq N+1} w_n\phi_n(0)$. Hence, defining $E^{N_0} = \begin{bmatrix}e_1 & \dots & e_{N_0}\end{bmatrix}^\top$, the scaled error $\tilde{e}_n = \sqrt{\lambda_n}e_n$ as in [28], and $\tilde{E}^{N-N_0} = \begin{bmatrix}\tilde{e}_{N_0+1} & \dots & \tilde{e}_N\end{bmatrix}^\top$, we infer that

$$\dot{\hat{Z}}^{N_0} = A_0\hat{Z}^{N_0} + \mathfrak{B}_0 u_h + LC_0E^{N_0} + L\tilde{C}_1\tilde{E}^{N-N_0} + L\zeta$$

where the matrices are given by $C_0 = \begin{bmatrix}\phi_1(0) & \dots & \phi_{N_0}(0)\end{bmatrix}$, $\tilde{C}_1 = \begin{bmatrix}\frac{\phi_{N_0+1}(0)}{\sqrt{\lambda_{N_0+1}}} & \dots & \frac{\phi_N(0)}{\sqrt{\lambda_N}}\end{bmatrix}$, and $L = \begin{bmatrix}l_1 & \dots & l_{N_0}\end{bmatrix}^\top$. Computing now the time derivative of the Artstein transformation defined by (12) and using the control input (13), we infer that

$$\dot{\hat{Z}}_A^{N_0} = (A_0 + \mathbf{B}_0K)\hat{Z}_A^{N_0} + LC_0E^{N_0} + L\tilde{C}_1\tilde{E}^{N-N_0} + L\zeta \tag{15}$$

where $\mathbf{B}_0 = \int_0^h e^{-A_0\sigma}\mathfrak{B}_0\varphi(\sigma)\,\mathrm{d}\sigma$. Introducing now $\hat{Z}^{N-N_0} = \begin{bmatrix}\hat{z}_{N_0+1} & \dots & \hat{z}_N\end{bmatrix}^\top$, we obtain from (11c) that

$$\dot{\hat{Z}}^{N-N_0} = A_1\hat{Z}^{N-N_0} + \mathfrak{B}_1 u_h \tag{16}$$

where $A_1 = \operatorname{diag}(-\lambda_{N_0+1} + q_c, \dots, -\lambda_N + q_c)$ and $\mathfrak{B}_1 = \begin{bmatrix}\beta_{N_0+1} & \dots & \beta_N\end{bmatrix}^\top$. Finally, owing to (9) and (11b-11c), we infer that the error dynamics are given by

$$\dot{E}^{N_0} = (A_0 - LC_0)E^{N_0} - L\tilde{C}_1\tilde{E}^{N-N_0} - L\zeta, \quad (17a)$$
$$\dot{\tilde{E}}^{N-N_0} = A_1\tilde{E}^{N-N_0}. \quad (17b)$$

Hence, the introduction of the state vector

$$X = \operatorname{col}\left(\hat{Z}_A^{N_0}, E^{N_0}, \tilde{E}^{N-N_0}\right) \quad (18)$$

implies, in view of (15) and (17), that

$$\dot{X} = FX + \mathcal{L}\zeta \quad (19)$$

where

$$F = \begin{bmatrix} A_0 + \mathbf{B}_0 K & LC_0 & L\tilde{C}_1 \\ 0 & A_0 - LC_0 & -L\tilde{C}_1 \\ 0 & 0 & A_1 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}$$

which is completed with the dynamics (16). Defining the augmented vector $\tilde{X} = \operatorname{col}(X, \zeta)$, we obtain from (13) and (15) that

$$u = \tilde{K}X, \qquad v = \dot{u} = K\dot{\hat{Z}}_A^{N_0} = E\tilde{X} \quad (20)$$

with $E = K\begin{bmatrix}A_0 + \mathbf{B}_0 K & LC_0 & L\tilde{C}_1 & L\end{bmatrix}$ and $\tilde{K} = \begin{bmatrix}K & 0 & 0\end{bmatrix}$.

It is checked in [29] that the pairs $(A_0, \mathfrak{B}_0)$ and (A_0, C_0) both satisfy the Kalman condition. Then, as shown by the below lemma, The controllability property of the pair $(A_0, \mathbf{B}_0)$ holds if and only if $\int_0^h e^{(\lambda_n - q_c)\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$ for all $1 \leq n \leq N_0$.

Lemma 4.2: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $h > 0$, and $\varphi \in L^2(0,h)$. Define $\mathbf{B}_I = \int_0^h e^{-\mathbf{A}\sigma}\mathbf{B}\varphi(\sigma)\,\mathrm{d}\sigma$. Then the pair $(\mathbf{A}, \mathbf{B}_I)$ satisfies the Kalman condition if and only if the pair $(\mathbf{A}, \mathbf{B})$ satisfies the Kalman condition and $\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$ for all $\mu \in \operatorname{sp}\mathbf{A}$.

Proof: We use the Hautus test. Let $(\mathbf{A}, \mathbf{B})$ satisfies the Kalman condition and $\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$ for all $\mu \in \operatorname{sp}\mathbf{A}$. Assume that there exist $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$ so that $x \neq 0$, $x^*\mathbf{A} = \mu x^*$ and $x^*\mathbf{B}_I = 0$. Then $x^*e^{-\mathbf{A}\sigma} = e^{-\mu\sigma}x^*$, thus we have $0 = x^*\mathbf{B}_I = x^*\mathbf{B}\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma$. This implies that $x^*\mathbf{A} = \mu x^*$ and $x^*\mathbf{B} = 0$, hence $x = 0$ because $(\mathbf{A}, \mathbf{B})$ satisfies the Kalman condition. This is in contradiction with our initial assumption that $x \neq 0$. Hence $(\mathbf{A}, \mathbf{B}_I)$ does satisfies the Kalman condition.

Conversely, assume that $(\mathbf{A}, \mathbf{B}_I)$ satisfies the Kalman condition. Let $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$ with $x \neq 0$ so that $x^*\mathbf{A} = \mu x^*$. Hence me must have $x^*\mathbf{B}_I \neq 0$, i.e., $x^*\mathbf{B}_I = x^*\mathbf{B}\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$. This implies that $\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$ for all $\mu \in \operatorname{sp}\mathbf{A}$. Finally, let $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$ so that $x^*\mathbf{A} = \mu x^*$ and $x^*\mathbf{B} = 0$. We infer that $x^*\mathbf{B}_I = x^*\mathbf{B}\int_0^h e^{-\mu\sigma}\varphi(\sigma)\,\mathrm{d}\sigma = 0$, hence $x = 0$. This shows that $(\mathbf{A}, \mathbf{B})$ satisfies the Kalman condition. ■

V. Main stability result

Theorem 5.1: Let $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, $\tilde{q} \in \mathcal{C}^0([0,1])$, and $\varphi \in L^2(0,h)$ for some $h > 0$ so that there exists $h_m \in (0,h)$ with $\varphi|_{[0,h_m]} = 0$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (4) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Assume that $\int_0^h e^{(\lambda_n - q_c)\sigma}\varphi(\sigma)\,\mathrm{d}\sigma \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + \mathbf{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist $P \succ 0$, $\alpha > 1$, $\beta, \gamma > 0$, and $q_1, q_2 \geq 0$ such that

$$\Theta_1 \preceq 0, \quad \Theta_2 \leq 0, \quad R_1 \leq 0, \quad R_2 \leq 0 \quad (21)$$

where

$$\Theta_1 = \begin{bmatrix} F^\top P + PF + 2\delta P + q_1 h\tilde{K}^\top\tilde{K} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta \end{bmatrix} + q_2 h E^\top E \quad (22a)$$

$$\Theta_2 = 2\gamma\left\{-\left(1 - \frac{1}{\alpha}\right)\lambda_{N+1} + q_c + \delta\right\} + \beta M_\phi \quad (22b)$$
$$R_1 = -q_1 e^{-2\delta h} + \alpha\gamma\|\mathcal{R}_N a\|_{L^2}^2\|\varphi\|_{L^2}^2 \quad (22c)$$
$$R_2 = -q_2 e^{-2\delta h} + \alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2\|\varphi\|_{L^2}^2 \quad (22d)$$

with $M_\phi = \sum_{n\geq N+1}\frac{|\phi_n(0)|^2}{\lambda_n} < +\infty$. Then there exists a constant $M > 0$ such that for any initial condition $z_0 \in H^2(0,1)$ so that $c_{\theta_1}z_0(0) - s_{\theta_1}z_0'(0) = 0$ and $c_{\theta_2}z_0(1) + s_{\theta_2}z_0'(1) = 0$, the trajectories of the closed-loop system composed of the PDE (2), the boundary Dirichlet measurement (3), and the controller (11-13) with null control in negative times ($u(\tau) = 0$ for $\tau < 0$) and zero initial condition for the observer ($\hat{z}_n(0) = 0$) satisfy

$$\|z(t,\cdot)\|_{H^1}^2 + \sup_{\tau\in[t-h,t]}|u(\tau)|^2 + \sum_{n=1}^N \hat{z}_n(t)^2 \leq Me^{-2\delta t}\|z_0\|_{H^1}^2$$

for all $t \geq 0$. Furthermore, the constraints (21) are always feasible for N selected large enough.

Proof: We define the functional defined by

$$V(t) = V_0(t) + V_1(t) + V_2(t)$$

with

$$V_0(t) = X(t)^\top PX(t) + \gamma\sum_{n\geq N+1}\lambda_n w_n(t)^2 \quad (23a)$$
$$V_1(t) = q_1\int_{-h}^0\int_{t+\sigma}^t e^{-2\delta(t-s)}|u(s)|^2\,\mathrm{d}s\,\mathrm{d}\sigma \quad (23b)$$
$$V_2(t) = q_2\int_{-h}^0\int_{t+\sigma}^t e^{-2\delta(t-s)}|\dot{u}(s)|^2\,\mathrm{d}s\,\mathrm{d}\sigma. \quad (23c)$$

The computation of the time derivative of V gives

$$\begin{aligned}
&\dot{V}+2\delta V\\
&=\tilde{X}^\top \begin{bmatrix} F^\top P + PF + 2\delta P & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} \tilde{X}\\
&\quad + 2\gamma \sum_{n\geq N+1} \lambda_n \left\{(-\lambda_n + q_c + \delta) w_n + a_n u_h + b_n v_h\right\} w_n\\
&\quad + q_1 h |u(t)|^2 - q_1 \int_{-h}^{0} e^{2\delta\sigma} |u(t+\sigma)|^2 \,\mathrm{d}\sigma\\
&\quad + q_2 h |\dot{u}(t)|^2 - q_2 \int_{-h}^{0} e^{2\delta\sigma} |\dot{u}(t+\sigma)|^2 \,\mathrm{d}\sigma.
\end{aligned}$$

We note that $\int_{-h}^{0} e^{2\delta\sigma}|u(t+\sigma)|^2 \,\mathrm{d}\sigma \geq e^{-2\delta h}\int_{t-h}^{t}|u(\sigma)|^2 \,\mathrm{d}\sigma$ and $\int_{-h}^{0} e^{2\delta\sigma}|\dot{u}(t+\sigma)|^2 \,\mathrm{d}\sigma \geq e^{-2\delta h}\int_{t-h}^{t}|\dot{u}(\sigma)|^2 \,\mathrm{d}\sigma$. The use of Young's inequality gives for any $\alpha > 0$ that

$$\begin{aligned}
2\sum_{n\geq N+1} \lambda_n a_n u_h w_n &\leq \frac{1}{\alpha}\sum_{n\geq N+1}\lambda_n^2 w_n^2 + \alpha \|\mathcal{R}_N a\|_{L^2}^2 u_h^2,\\
2\sum_{n\geq N+1} \lambda_n b_n v_h w_n &\leq \frac{1}{\alpha}\sum_{n\geq N+1}\lambda_n^2 w_n^2 + \alpha \|\mathcal{R}_N b\|_{L^2}^2 v_h^2.
\end{aligned}$$

Moreover, we have $|u_h(t)|^2 = \left|\int_0^h \varphi(\sigma) u(t-\sigma)\,\mathrm{d}\sigma\right|^2 \leq \|\varphi\|_{L^2}^2 \int_{t-h}^{t} |u(\sigma)|^2 \,\mathrm{d}\sigma$ and, similarly, $|v_h(t)|^2 \leq \|\varphi\|_{L^2}^2 \int_{t-h}^{t} |\dot{u}(\sigma)|^2 \,\mathrm{d}\sigma$. Hence, the combination of the latter estimates and the use of (20) give

$$\begin{aligned}
&\dot{V}+2\delta V\\
&\leq \tilde{X}^\top \left\{ \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} + q_2 h E^\top E \right\} \tilde{X}\\
&\quad + 2\gamma \sum_{n\geq N+1} \lambda_n \left\{ -\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta \right\} w_n^2\\
&\quad + R_1 \int_{t-h}^{t} |u(\sigma)|^2 \,\mathrm{d}\sigma + R_2 \int_{t-h}^{t} |\dot{u}(\sigma)|^2 \,\mathrm{d}\sigma.
\end{aligned}$$

where $\Theta_{1,1} = F^\top P + PF + 2\delta P + q_1 h \tilde{K}^\top \tilde{K}$. Since the residue of measurement is expressed by $\zeta = \sum_{n\geq N+1} w_n \phi_n(0)$, Cauchy-Schwartz inequality implies that $\zeta^2 \leq M_\phi \sum_{n\geq N+1} \lambda_n w_n^2$. Therefore, we deduce that

$$\begin{aligned}
\dot{V}+2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{n\geq N+1} \lambda_n \Gamma_n w_n^2\\
+ R_1 \int_{t-h}^{t} |u(\sigma)|^2 \,\mathrm{d}\sigma + R_2 \int_{t-h}^{t} |\dot{u}(\sigma)|^2 \,\mathrm{d}\sigma
\end{aligned}$$

for any $\beta > 0$ where $\Gamma_n = 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta\right\} + \beta M_\phi$. Since $\alpha > 1$, we note that $\Gamma_n \leq \Gamma_{N+1} = \Theta_2$ for all $n \geq N+1$. Therefore, owing to the constraints (21), we infer that that $\dot{V} + 2\delta V \leq 0$, implying that $V(t) \leq e^{-2\delta t} V(0)$ for all $t \geq 0$. The claimed stability estimate now easily follows from the definition of the functional V, the inequalities (1), the definition of the distributed delayed input (2c), the Artstein transformation (12), the command input (13), and the dynamics (16).

We conclude the proof by assessing that the constraints (21) are always feasible provided the dimension $N \geq N_0 + 1$ of the observer is selected large enough. We first note that the matrix $F + \delta I$ is such that (i) the matrices $A_0 + \mathbf{B}_0 K + \delta I$ and $A_0 - LC_0 + \delta I$ are Hurwitz; (ii) $\|e^{(A_1+\delta I)t}\| \leq e^{-\kappa_0 t}$ for all $t \geq 0$ with $\kappa_0 = \lambda_{N_0+1} - q_c - \delta > 0$ a constant that is independent of N; and (iii) $\|L\tilde{C}_1\| \leq \|L\|\|\tilde{C}_1\|$ where L is independent of the dimension N of the observer while $\|\tilde{C}_1\| = O(1)$ as $N \to +\infty$. Hence, an approach similar to [28, Lemma in appendix] applied to the matrix $F + \delta I$ shows that the solution $P \succ 0$ to $F^\top P + PF + 2\delta P = -I$ is such that $\|P\| = O(1)$ as $N \to +\infty$. Moreover, $\|\mathcal{L}\|$ and $\|\tilde{K}\|$ are independent of the dimension N of the observer while $M_\phi = O(1)$ and $\|E\| = O(1)$ as $N \to +\infty$. We fix arbitrarily the value of $\alpha > 1$ and we set $\beta = \sqrt{N}$, $\gamma = 1/N$, $q_1 = e^{2\delta h}\alpha\gamma\|\mathcal{R}_N a\|_{L^2}^2 \|\varphi\|_{L^2}^2$, and $q_2 = e^{2\delta h}\alpha\gamma\|\mathcal{R}_N b\|_{L^2}^2 \|\varphi\|_{L^2}^2$. With this choice of parameters and invoking Schur complement, we deduce that the constraints (21) are satisfied for $N \geq N_0 + 1$ selected large enough. ■

VI. Numerical illustration

For numerical illustration we consider the PDE plant (2) with $p = 1$, $\tilde{q} = -5$, $\theta_1 = \pi/5$, and $\theta_2 = 0$ (corresponds to Dirichlet boundary control). The distributed delay is characterized by the function $\varphi(\sigma) = (\sigma+1)^2 1_{[0.1,+\infty]}(\sigma)$. In this configuration, the open-loop plant is unstable with one unstable eigenvalue.

We set $N_0 = 1$ and we place the feedback and observer gains so that we obtain the pole placement -4 in both cases. This gives the observer gain $L = 4.0832$ while the value of the feedback gain K depends on the value of the delay $h > 0$. The dimensions N of the observer so that Theorem 5.1 is applicable with exponential decay rate $\delta = 0.5$ are detailed in Tab. I for different values of the distributed input delay $h > 0$.

VII. Conclusion

This paper has addressed the topic of output feedback stabilization of general 1-D reaction-diffusion PDEs in the presence of an arbitrarily long distributed input delay. Provided a structural controllability assumption, the reported control design procedure is systematic in the sense that it always achieves the exponential stabilization of the closed-loop system, with prescribed exponential decay rate, provided the order of the observer is selected to be large enough.

To conclude, we mention here a number of direct extensions of the result presented in this paper. While the stability estimate derived in this paper holds for PDE trajectories evaluated in H^1 norm, similar conditions can be derived to obtained stability estimates in L^2 norm by adapting the arguments of [29]. Finally, while this paper was focused on the case of Dirichlet boundary measurement, the developed approach can be extended to Neumann boundary measurement by using the methods of [28], [29].

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Value of the delay	$h = 0.5$	$h = 1$	$h = 2$	$h = 3$	$h = 4$
Feedback gain	$K = -2.6830$	$K = -0.9321$	$K = -0.3166$	$K = -0.1704$	$K = -0.1131$
Dimension of the observer	$N = 2$	$N = 3$	$N = 5$	$N = 12$	$N = 31$

TABLE I

DESIGN PARAMETERS FOR DIFFERENT VALUES OF THE DISTRIBUTED DELAY $h > 0$ AND FOR $\delta = 0.5$.

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