A bilinear systems approach with input saturation to control the agreement value of multi-agent systems

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Daniel R. Alkhorshid[1], Eduardo S. Tognetti[1] and Irinel-Constantin Morărescu[2]

***Abstract*— While reaching an agreement in multi-agent systems (MAS) can be ensured by enforcing some connectivity properties between agents, the consensus depends on their initial conditions and the network topology. In this context, our main objective in this paper is to sway the consensus value of multi-agent systems towards a desired value. The asymptotic stability and maximization of the domain of attraction for the bilinear model representing the opinion dynamics in the presence of limited control action for a fixed and connected network are investigated. By using algebraic graph theory and linear matrix inequality (LMI), we provide sufficient conditions guaranteeing the convergence of agents toward the desired consensus. Furthermore, examples illustrate the effectiveness of the proposed method.**

I. Introduction

Multi-agent systems have a wide range of applications such as robotic teams, power grids, telecommunication networks, biology and opinion dynamics. In the MAS setup the agents have only a local limited view of the overall system, which means that they design decentralized control actions governing their behavior. The coherent behavior of the MAS is often described in terms of consensus i.e., the agents reach an agreement on certain variables of interest [1]–[4].

A very rich literature exists on the design of consensus algorithms for agents with linear dynamics, for which the behavior is well understood. In this setting, the literature considers fixed and time-varying communication topologies [5], [6], directed or undirected graphs [7], [8], synchronous or asynchronous information exchange [9], [10], delayed or immediate transmissions [2], [3], etc. It is not worthy that in all these cases the resulted agreement value depends on the initial conditions of the agents and the network topology describing their interactions. While in some applications this is convenient in some other such as power grids, traffic control or opinion dynamics, one needs to reach an agreement as close as possible to a desired a priori fixed value. In this case a supplementary centralized control action is required which in many applications is subject to saturation and global cost constraints.

In the last decade, many works focused on the consensus problem with saturated inputs (see for instance [11], [12]) or global control effort constraints (see [11], [13], [14]).

In this work, we focus on reaching a desired consensus on the network of interest by applying an external centralized control action to it. One considers both saturated inputs and the minimization of an overall cost corresponding to the global effort/budget of the external entity to control the agreement value. While different applications such as power grids or platooning can be modeled in this setup, our main motivation comes from the control of opinion dynamics over social networks.

Emergence of consensus in social networks is a controversial problem. Social studies pointed out that, in general, opinions tend to converge one toward another during interactions. Consequently, it is not surprising that consensus received a particular attention in opinion dynamics literature [15], [16]. It is also worth noting that some of the first mathematical models naturally lead to consensus [17], [18] while some others lead to network clustering [4], [19], [20]. To enforce consensus in social networks, some recent studies propose the control of one or a few agents, see [21], [22].

Besides these methods of controlling opinion dynamics towards consensus, we also find recent attempts to control the discrete-time dynamics of opinions such that as many agents as possible reach a certain set after a finite number of influences [23]. The closest work to the present manuscript is [11] in which the authors consider a discrete-time implementation (campaign influence) of the exogenous control action trying to sway the agreement value toward a desired one. Unlike [11] in this work we are considering a continuous-time influence of the external entity, a time-varying state dependent control input which is saturated and a total budget/energy which is continuously spent over time.

From a practical point of view, the problem considered in this manuscript is related to viral marketing in which sellers attempt to artificially create word-of-mouth advertising among potential customers. Social scientists and economists have well established the effectiveness of this trend [24], [25]. The control limitations imposed in the paper are related, on the one hand, to the fact that sellers have a limited power of influence on any potential customer and, on the other hand, they have a fixed given budget to influence customers' decisions. It is noteworthy that for simplicity, we consider the topology of the network of interest is fixed, meaning that the characteristics of each agent and the network itself would not change during time. The model representing the opinion dynamics is bilinear; hence an estimate of the domain of attraction of the origin is obtained based on a local quadratic Lyapunov function and its resulting (ellipsoidal)

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level sets [26]. As the states belong to the positive orthant, we propose a convex optimization problem, in terms of LMIs, exploiting properties of positive systems [27] to maximize the invariant region where the state trajectories must belong. To deal with the state dependency in the design conditions, we adopted the parametrization of the states as norm-bounded uncertainties. This approach is more appropriate than the polytopic approximation of the bilinear term [28] when dealing with high-order systems (large number of agents).

The paper is organized as follows. In section II we introduce the main problem along with the network rules and properties. Additionally, we introduce lemmas that are essential for further developments. In section III, we propose the main result of this paper in the form of a theorem and its mathematical justification along with it. In section IV we present two examples to display the effectiveness of the proposed method. Finally, the conclusion is made at the section V.

Notation

The space of real matrices with dimension $n\times m$ is denoted by $\mathbb{R}^{n\times m}$, $\mathbb{R}^n_+$ denotes the space of vectors of dimension n with positive entries. For a matrix X, X^T and $X_\perp$ denote the transpose of X and any matrix whose columns form a basis for the null space of X, respectively. $X_{(i)}$ denotes the $i-$th row. If X is square, X^{-1} denotes the inverse of X; $\mathrm{He}\{X\}$ stands for $X+X^T$; and $X>0$ ($X<0$) indicates that matrix X is positive (negative) definite; $X\succ 0$ ($X\succeq 0$) denotes all the elements of X are positive (nonnegative). For a vector $v\in\mathbb{R}^n$, $\mathrm{diag}(v)=\mathrm{diag}(v_1,\dots,v_n)$ is a diagonal matrix composed with the elements of v, and for a matrix X, $\mathrm{diag}(X)$ is composed with the diagonal elements of X. The identity matrix of order n is denoted by I_n and the null $m\times n$ matrix is denoted by $0_{m,n}$ (or simply I and 0 if no confusion arises), $\mathbb{1}$ stands for a vector of ones of appropriate dimension. The symbol $\star$ denotes symmetric blocks in partitioned matrices, $\otimes$ denotes the Kronecker product, $\mathscr{L}_2^n[0,\infty)$ denotes the space of square integrable vectors of n functions over $[0,\infty)$.

II. Preliminaries

A. Problem formulation

In this work, we focus on the exogenous control of the consensus value in multi-agent systems. The study is mainly interpreted in term of opinion dynamics in social networks. Consequently, in the sequel we assume that the agents represent individuals that form a social network interacting over a fixed topology. The network is represented as a graph $\mathscr{G}$ in which each agent corresponds to a vertex belonging the set $\mathscr{V}=\{1,\dots,N\}$ associated with the N agents. An edge (i,j) in the graph indicates that agent j influences the opinion of agent i. Let us also consider the associated adjacency matrix $\mathscr{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$. The corresponding Laplacian matrix is $L=[l_{ij}]\in\mathbb{R}^{N\times N}$ defined by

$$\begin{cases} l_{ii}=\sum_{j=1}^N a_{ij}, & \forall i=1,\dots,N\\ l_{ij}=-a_{ij} & \text{if } i\neq j, \end{cases} \tag{1}$$

where $a_{ij}\neq 0$ means that agent i is influenced by agent j and $a_{ij}=0$ otherwise. For every agent $i\in\mathscr{V}$ we assign a time varying opinion $x_i(t)$ described by a scalar value normalized between 0 and 1, i.e., $x_i(t)\in[0,1]$. We consider that each opinion changes in time under the endogenous influence of neighbors' opinions on one hand, and exogenous control (persuasion) action of an external influencer.

In the sequel, $d\in\{0,1\}$ represents the opinion promoted by the external entity/influencer. The evolution in time of the states/opinions is described by the following dynamics:

$$\dot{x}_i(t)=\sum_{j=1}^N a_{ij}(x_i(t)-x_j(t))+(x_i-d)u_i(t),\quad \forall i\in\{1,\dots,N\},$$

where $u_i(t)\in[-\bar{u},\bar{u}]$, $\bar{u}\in(0,1)$ is a finite energy and bounded exogenous control action.

Here, we consider the desired opinion as extreme values of individual opinion, meaning expected binary behaviors (consume/ not consume, agree/ disagree, etc). Most problems can be put in this classification, while the case of intermediate desired opinion $d\in(0,1)$ (moderate consuming, for instance) is left for future researches. It is noteworthy that the rate of changes in every agent's opinion relies on both the granted budget J_u and their original opinion $x_0(t)$. The granted budget here embodies the limited bound of the control action to reach the desired opinion. It is evident that in real-world situations, not all agents are tempted equally (in a uniform manner) by the external control action [11].

Assumption 1 We suppose that the graph $\mathscr{G}$ representing the social network is weakly connected in the sense that it contains a directed spanning tree (i.e. a directed graph in which, except the root which is not influenced, each node is influenced by a single other node called parent).

Lemma 1 ([29]) Under Assumption 1 L has 0 as a simple eigenvalue associated with the right eigenvector $\mathbb{1}\in\mathbb{R}^N$, that is, $L\mathbb{1}=0$. All the other $N-1$ eigenvalues of L have positive real parts.

Let $x(t)=(x_1(t),\dots,x_N(t))^T\in\mathbb{R}^N$ and $u(t)=(u_1(t),\dots,u_N(t))^T\in\mathbb{R}^N$ be the vectors collecting the states and the control input of all agents. The collective dynamics is then described by

$$\dot{x}(t)=-Lx(t)+B(x(t)-\mathbb{1}d)u(t) \tag{2}$$

where the function $B(\cdot):\mathbb{R}^N\to\mathbb{R}^{N\times N}$ is given by $B(x)=\mathrm{diag}(x_1,x_2,\dots,x_N)$. We define the set of admissible values of the states as

$$\chi=\{x\in\mathbb{R}^N: x_i(t)\in[0,\ 1],\ i=1,\dots,N\}. \tag{3}$$

Let us define $x_{d_i}(t)=x_i(t)-d$ and $x_d(t)=x(t)-\mathbb{1}d$. From Lemma 1, the system (2) is rewritten as

$$\dot{x}_d(t)=-Lx_d(t)+B(x_d(t))u(t) \tag{4}$$

where $x_{d_i}\in[-d,\ 1-d]$, $i=1,\dots,N$. For $d=1$, one has $x_{d_i}\in[-1,\ 0]$. In the solution proposed in this paper, the case

$d=1$ is symmetric of $d=0$, then we adopt hereafter, without loss of generality, $d=0$ and $x_d=x$.

The following problem is addressed in this work.

Problem 1 To design a control law u_i that solves the following optimization problem.

$$\min_{u(t)} J_x = \int_0^\infty x(t)^T R x(t)\,dt \tag{5}$$

subjected to

$$|u_i(t)| \le \bar{u} \tag{6}$$
$$J_u = \int_0^\infty u(t)^T Q u(t)\,dt < \mu \tag{7}$$

where $\bar{u}$ and μ are given specifications, R and Q are positive definite matrices. The matrix Q penalizes the control effort required for synchronization. The conditions (6)–(7) represent constraints in the amplitude and in the energy of the control signal, respectively.

In simple language, when a system (for example an anti smoking campaign) is interested in encouraging the targeted social network (for example young population) to agree on a desired opinion (a behavior), it attempts to sway the network consensus as close as possible to the desired consensus (5), by using a granted budget which could be interpreted as a limited and bounded control action (6)–(7).

Finally, observe that the functions J_x and J_u can be rewritten as $\int_0^\infty z(t)^T z(t)\,dt$ and $\int_0^\infty y(t)^T y(t)\,dt$, respectively, where

$$\begin{aligned} z(t) &= R^{\frac{1}{2}} x(t) \\ y(t) &= Q^{\frac{1}{2}} u(t). \end{aligned} \tag{8}$$

B. Representation of the bilinear term and saturation model

The constraint (6) can be incorporated in the dynamics as $u(t) = sat(v(t))$ using the standard decentralized saturation function $\mathrm{sat}(v_{(\ell)}) = \mathrm{sign}(v_{(\ell)})\min(|v_{(\ell)}|, \bar{u})$, $\ell = 1,\ldots,N$, where v is an unbounded control signal to be designed. Then, one has

$$\dot{x}(t) = -Lx(t) + B(x(t))\mathrm{sat}(v(t)). \tag{9}$$

System (9) can be rewritten using the decentralized dead-zone nonlinearity $\psi(v) = v - sat(v)$

$$\dot{x}(t) = -Lx(t) + B(x(t))v(t) - B(x(t))\psi(v(t)). \tag{10}$$

We describe the bilinear product in (10) for $x \in \chi$ using a norm-bounded uncertainty, that is,

$$B(x) = B_0 + B_1\Delta(t), \tag{11}$$

with $B_0 = B(0.5\mathbb{1})$, $B_1 = 0.5$, and $\Delta(t) = \mathrm{diag}(\delta_1(t),\ldots,\delta_N(t)) \in \mathbb{R}^{N\times N}$, where $\delta_i(t)$, $i=1,\ldots,N$, are bounded Lebesgue measurable uncertainties belonging to the set

$$\mathscr{D} = \{\delta \in \mathbb{R} : \ \delta(t)^T\delta(t) \le 1\}.$$

Let us consider the following state feedback control law

$$v(t) = Kx(t), \qquad K = \mathrm{diag}(k_1,\ldots,k_N) \in \mathbb{R}^{N\times N}. \tag{12}$$

With the diagonal structure of the gain K, we impose the control action of each agent v_i to only depend on its opinion x_i. The closed-loop system formed by (8), (10) and (12) is given by

$$\begin{aligned} \dot{x}(t) &= (-L + (B_0 + B_1\Delta(t))K)x(t) \\ &\qquad - (B_0 + B_1\Delta(t))\psi(Kx(t)) \qquad \text{(13a)} \\ z(t) &= R^{\frac{1}{2}}x(t) \qquad \text{(13b)} \\ y(t) &= Q^{\frac{1}{2}}Kx(t) + Q^{\frac{1}{2}}\psi(Kx(t)). \qquad \text{(13c)} \end{aligned}$$

Observe that the controllability matrix formed from the pair $(-L, B(x))$ of system (9) loses rank for $x=0$ and the system (9) becomes uncontrollable. To circumvent this problem, we consider the interval $x_i \in [\varepsilon, 1]$, where ε is a positive scalar arbitrarily small, in the modeling of $B(x)$ in (11). This implies $B_0 = B((0.5+\varepsilon)\mathbb{1})$ and $B_1 = 0.5$.

Let us finish this section with the following lemmas which are instrumental for further developments.

Lemma 2 (Petersen's Lemma [30]) Let $G = G^T \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{q\times n}$ be given matrices. For all $\Delta(t) \in \mathbb{R}^{p\times q}$ such that $\Delta(t)^T\Delta(t) \le I$, the inequality

$$G + M\Delta(t)N + N^T\Delta(t)^T M^T \le 0$$

holds if and only if there exists a scalar $\lambda > 0$ such that

$$G + \lambda M M^T + \frac{1}{\lambda} N^T N \le 0.$$

Lemma 3 ([31]) Consider a matrix $G \in \mathbb{R}^{N\times N}$ and define the region

$$\Pi = \left\{x \in \mathbb{R}^N : |v_i - G_i x| \le \bar{u}, \ i = 1,\ldots,N\right\}. \tag{14}$$

If $x \subseteq \Pi$ then the following relation holds

$$\psi(v)^T T(\psi(v) - Gx) \le 0 \tag{15}$$

for any matrix $T \in \mathbb{R}^{N\times N}$ diagonal and positive definite.

As detailed in the next section, the definition of positive systems will help construct the invariant region to which the initial conditions must belong.

Definition 1 ([27]) The system $\dot{x}(t) = Ax(t)$ is called positive (Metzlerian) system if for every initial condition $x(0) \in \mathbb{R}^n_+$, the states satisfy $x(t) \in \mathbb{R}^n_+$ for $t \ge 0$.

Lemma 4 ([27]) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is positive if and only if A is a Metzler matrix, that is, $A_{(ij)} \ge 0$ for $i \ne j$, and $B \succeq 0$.

III. Main results

In this section we present the main conditions to design the control law (12) and solve Problem 1. The control strategy must assure the trajectories to remain in χ and Π, where (15), used in the design conditions, holds. In other words, one must find an invariant region $\mathscr{S}$ such that if $x(0) \in \mathscr{S}$, then

$x(t) \in \chi \cap \Pi$ for all $t \geq 0$. We adopt the following candidate region for $\mathscr{S}$:

$$\mathscr{S} := \left\{x \in \mathbb{R}_{+}^{N} : x^T W^{-1} x \leq 1, \quad W = W^T > 0\right\}. \tag{16}$$

In general, $\mathscr{S}$ is not invariant even if $V(t) = x^T W^{-1} x$ is a Lyapunov function for the closed-loop system. Moreover, inclusion conditions for $\mathscr{S} \subset \chi$ are not easy to obtain.

Consider the following level curve of the Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$,

$$\mathscr{S}_a := \left\{x \in \mathbb{R}^{N} : x^T W^{-1} x \leq 1, \quad W = W^T > 0\right\}. \tag{17}$$

The set $\mathscr{S}_a$ can be obtained straightforwardly from stability conditions however the ellipsoid region is centered in the origin meaning $\mathscr{S}_a \nsubseteq \chi$, that is, even for $x(0) \in \mathscr{S}_a \cap \chi \cap \Pi$ the trajectories could try to escape from χ. To solve this problem we first propose an augmented space χ_a defined as

$$\chi_a = \{x \in \mathbb{R}^N : x_i(t) \in [-1,\ 1],\ i = 1, \ldots, N\}, \tag{18}$$

where $\mathscr{S}_a$ is contained, that is $\mathscr{S}_a \subseteq \chi_a$. Observe that χ_a is symmetric in all orthants in the state space allowing the maximization of $\mathscr{S}_a \cap \chi$. Finally, we propose to design the control law (12) such that the closed-loop system is positive (see Definition 1). Note that, $-L$ is a Metzler matrix and $B(x) \succeq 0$ for all $x \in \chi$, then, according to Lemma 4, system (2) is positive. Therefore, the set $\mathscr{S}$ is invariant. Fig. 1 illustrate sets χ, χ_a, $\mathscr{S}$, $\mathscr{S}_a$ and a trajectory $x(t)$.

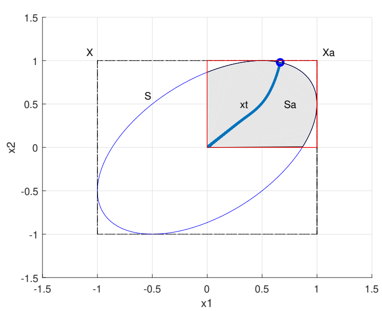

Fig. 1: Sets χ_a (black dashed box), χ (red box), $\mathscr{S}$ (gray region), $\mathscr{S}_a$ (blue curve), and trajectory $x(t)$ (bold blue line).

The control input energy/budget constraint (7) can prevent the trajectories from initial conditions belonging to $\mathscr{S}$ to reach the origin. From a practical point of view, making all individual opinions converge to the desired opinion can be costly and budget constraints will lead the opinions to converge to a value different from d. Therefore, we propose a method to find an invariant region $\mathscr{S}_u \subseteq \mathscr{S}$ such that for all $x(0) \in \mathscr{S}_u$ there is enough budget $J_u < \mu$ to make all opinions reach the origin. We also propose a method to limit investments ($u = 0$) for all initial opinions belonging to $\mathscr{S} \setminus \mathscr{S}_u$ when the budget μ is reached.

We can reformulate Problem 1 as:

Problem 2 To design a state feedback gain K such that the closed-loop system (13) is positive and to find

(i) an estimation of the domain of attraction $\mathscr{S} \subset (\chi \cap \Pi)$ such that for all initial conditions $x(0) \in \mathscr{S}$, the trajectories of the closed-loop system converge exponentially toward the origin with guaranteed cost J_x for all $\delta_i(t) \in \mathscr{D}$ and $|u_i(t)| \leq \bar{u}$;

(ii) an estimation of a region $\mathscr{S}_u$ such that for all initial conditions $x(0) \in \mathscr{S}_u \cap \mathscr{S}$, the trajectories of the closed-loop system converge exponentially toward the origin with guaranteed cost $J_u < \mu$ for all $\delta_i(t) \in \mathscr{D}$ and $|u_i(t)| \leq \bar{u}$;

(iii) a control law to limit investments ($u = 0$) for all initial opinions belonging to $\mathscr{S} \setminus \mathscr{S}_u$ when the budget $J_u = \mu$ is reached.

For the following results, the polyhedral set χ_a is represented by

$$\chi_a = \left\{x \in \mathbb{R}^N : \Omega x \leq \mathbb{1},\right\} \tag{19}$$

where $\Omega = I_N \otimes \begin{bmatrix}-1 & 1\end{bmatrix}^T \in \mathbb{R}^{2N \times N}$, $\mathbb{1} \in \mathbb{R}^N$, and $0 \in \chi_a$.

As methodology, following ideas presented in [11] where the maximum possible investment is used as soon as possible to minimize a cost function related to the convergence of x, we first present a result that solves (5)–(6) (Problem 2 (i)). After that, we propose an stability analysis condition that provides Lyapunov level curves that are upper bounds for the guaranteed cost J_u solving Problems 2 (ii) and (iii).

Theorem 1 Suppose that there exist diagonal positive definite matrices $W \in \mathbb{R}^{N \times N}$ and $S \in \mathbb{R}^{N \times N}$, a diagonal matrix $Z \in \mathbb{R}^{N \times N}$, a matrix $Y \in \mathbb{R}^{N \times N}$, and a scalar $\lambda > 0$, such that the following inequalities hold:

$$\begin{bmatrix} \mathrm{He}\{-LW + B_0 Z\} + \lambda I & \star & \star & \star \\ S B_0^T + Y & -2S & \star & \star \\ W & 0 & -R^{-1} & \star \\ Z & S & 0 & -4\lambda I \end{bmatrix} < 0 \tag{20}$$

$$\begin{bmatrix} W & \star \\ \Omega_{(i)} W & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \ldots, 2N \tag{21}$$

$$\begin{bmatrix} W & \star \\ Z_{(i)} - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \ldots, N. \tag{22}$$

Then, the state feedback gain $K = Z W^{-1}$ makes the closed-loop system (13a) asymptotically stable with $\mathscr{S} \subseteq \chi \cap \Pi$ the estimation of the domain of attraction of the origin with guaranteed cost $J_x \leq x(0)^T W^{-1} x(0)$.

Proof: First, consider the closed-loop system (13) and the Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$. The integral from 0 to ∞ of

$$\dot{V}(t) + z(t)^T z(t) < 0, \quad \forall x \in \chi, \tag{23}$$

implies $J_x < V(0)$. Condition (23) is equivalent to $\dot{V}(t) < -c x(t)^T x(t)$, for c the maximum eigenvalue of R, and thus it verifies the exponential stability of the origin.

Using Lemma 3, the inequality (23) holds if $\dot{V}(t) + z(t)^T z(t) - 2\psi(v(t))^T T \psi(v(t)) + 2\psi(v(t))^T T G x(t) < 0$, and, considering (13) with $B(x)=B_0+B_1\Delta(t)$, the last inequality is rewritten as

$$\begin{bmatrix} x \\ \psi(v) \end{bmatrix}^T \begin{bmatrix} \mathrm{He}\{-W^{-1}L+W^{-1}B(x)K\}+R & \star \\ B(x)^T W^{-1}+TG & -2T \end{bmatrix} \begin{bmatrix} x \\ \psi(v) \end{bmatrix} < 0.$$

By pre- and post-multiplying the previous inequality by $\mathrm{diag}(W, T^{-1})$, one has

$$\begin{bmatrix} \mathrm{He}\{-LW+B(x)Z\}+W^T RW & \star \\ SB(x)^T+Y & -2S \end{bmatrix} < 0,$$

where $Z=KW$, $S=T^{-1}$, and $Y=GW$. Using the Schur complement lemma, one has

$$\begin{bmatrix} \mathrm{He}\{-LW+B(x)Z\} & \star & \star \\ SB(x)^T+Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0.$$

For all $x\in\chi$ we can replace $B(x)$ by (11) in the above inequality yielding

$$\begin{bmatrix} \mathrm{He}\{-LW+B_0 Z\} & \star & \star \\ SB_0^T+Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix} 0.5 \begin{bmatrix} Z & S & 0 \end{bmatrix} \right\} < 0.$$

Applying Lemma 2 for $\Delta(t)=\mathrm{diag}(\delta_1(t),\ldots,\delta_N(t))$, $\delta_i(t)\in\mathscr{D}$, and the Schur Complement, we recover (20).

Observe that (21) is equivalent to

$$\begin{bmatrix} W^{-1} & \star \\ \Omega_{(i)} & 1 \end{bmatrix} \geq 0,$$

which implies (see [32]) that $\mathscr{S}_a\subseteq\chi_a$ and, consequently, $\mathscr{S}\subseteq\chi$.

Finally, by pre-and-post multiplying inequality (22) by $\mathrm{diag}(W^{-1}, I)$, one has

$$\begin{bmatrix} W^{-1} & \star \\ K_{(i)}-G_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0$$

Considering the set Π in Lemma 3 and $\mathscr{S}_a$ in (17), the above inequality verifies $\mathscr{S}_a\subseteq\Pi$ [33] and $\mathscr{S}\subseteq\Pi$. ∎

Conditions of Theorem 1 are sufficient for the exponential stability of the closed-loop system (13a). However, it does consider the energy (budget) constraint in the control input (7). Additionally, the Lyapunov function $V(t)=x(t)^T W^{-1}x(t)$ with W obtained by Theorem 1 does not provide $\mathscr{S}_u$ and a condition that guarantees $J_u<\mu$ for all $x\in\mathscr{S}$. We propose the following result to compute level curves associated with the control input energy J_u and solve Problems 2 (ii) and (iii).

Theorem 2 If there exist diagonal positive definite matrices $P\in\mathbb{R}^{N\times N}$ and $S\in\mathbb{R}^{N\times N}$, a matrix $Y\in\mathbb{R}^{N\times N}$, and a scalar $\lambda>0$, such that the following inequalities hold

$$\begin{bmatrix} \mathrm{He}\{-LP+B_0KP\}+\lambda I & \star & \star & \star \\ SB_0^T+Y & -2S & \star & \star \\ KP & S & -Q^{-1} & \star \\ KP & S & 0 & -4\lambda I \end{bmatrix} < 0, \tag{24}$$

then, the closed-loop system (13a) has guaranteed cost $J_u\leq x(0)^T P^{-1}x(0)$.

Proof: Considering the Lyapunov function $V(t)=x(t)^T P^{-1}x(t)$, the integral from 0 to ∞ of $\dot{V}(t)+y(t)^T y(t)<0$ with respect to the closed-loop system (13a) implies $J_u<V(0)$. The rest of the proof follows the same lines of the proof of Theorem 1. ∎

Remark 1 The estimation of the region of initial conditions such that $J_u\leq\mu$ is given by $\mathscr{S}_u=\{x\in\mathbb{R}_+^N: x(0)^T P^{-1}x(0)\leq\mu\}$. For initial opinions belonging to $\mathscr{S}\setminus\mathscr{S}_u$, the following control law assures $J_u<\mu$:

$$u(t)=\begin{cases} sat(Kx), & x(t)^T P^{-1}x(t)>x(0)^T P^{-1}x(0)-\mu \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Condition (25) makes the trajectories of $x(0)\in\mathscr{S}\setminus\mathscr{S}_u$ be attracted to the boundary of the set $\{x\in\mathscr{S}: x(t)^T P^{-1}x(t)\leq x(0)^T P^{-1}x(0)-\mu\}$. Observe that when $u=0$ the system dynamics becomes $\dot{x}(t)=-Lx(t)$ which has a global uniformly exponentially stable attractor given by the consensus manifold.

Remark 2 A way to indirectly maximize the set $\mathscr{S}$ defined in (16) and minimize J_x, as presented in (5), is to maximize the trace of W (see [32]). Hence, (5)–(6) and the maximization of the estimate of the basin of attraction $\mathscr{S}$ are obtained by solving the following optimization problem:

$$\max \quad \mathrm{Trace(W)} \tag{26}$$

subjected to (20)–(22).

To obtain a tighter approximation for the upper bound for cost J_u in Theorem 2, we can also solve the following optimization problem:

$$\max \quad \mathrm{Trace(P)} \tag{27}$$

subjected to (24).

Remark 3 For $\varepsilon>0$ in $B_0=B((0.5+\varepsilon)\mathbb{1})$, we have that (23) holds for $x\in\chi\setminus\mathscr{B}_\varepsilon$, where $\mathscr{B}_\varepsilon=\{x\in\mathscr{S}: x^T x\leq\varepsilon\}$. This means that we can only assure the convergence of the trajectories to $\mathscr{B}_\varepsilon$. As long ε is sufficiently small, there is no practical implication since constraint (7) usually prevents $x_i(t)$ to reach d.

IV. Numerical Example

In this section, we present two numerical examples to illustrate the proposed results. The first example contains three agents that helps us in visualizing the domain of attraction along with the agents' trajectories. The second example involves twenty agents and is presented to show the effectiveness of the proposed method on a larger and realistic network.

Example 1 Consider the communication network from [13] described by a undirected graph $\mathscr{G}$, connected, with $N=3$ agents and Laplacian matrix is defined as:

$$L=\begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

We consider Problem 1 with $Q=10^{-1}I$, $R=10^{-1}I$, $\mu=0.675$, and $\bar{u}=0.9$. The design is performed by Theorems 1 and 2 adopting $B_0=B((0.5+\varepsilon)\mathbb{1})$, with $\varepsilon=0.1$. Theorem 1 yields $K=\mathrm{diag}(-1.4143,-1.4143,-1.3886)$ and the control law (25) is implemented from P obtained by Theorem 2. We observe in Fig. 2, depicting regions $\mathscr{S}$, $\mathscr{S}_a$, χ, and χ_a, that the estimation of the domain of attraction $\mathscr{S}$ (intersection of $\mathscr{S}_a$ and χ) encompasses most of χ. Fig. 3 illustrates $\mathscr{S}_u$ contained in $\mathscr{S}$ showing that for some initial conditions there is not enough budget to reach the origin due to (7). The trajectories of the agents is shown in Fig. 4 for the initial condition $x(0)=(0.2673,0.5345,0.8018)\in\partial\mathscr{S}$. We can see all trajectories converge as close as possible to the origin with respect to the energy constraint (7), and the control signal respects the input saturation (u_3 saturates in the initial instant).

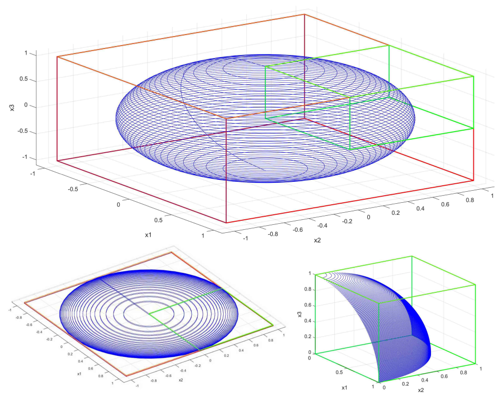

Fig. 2: Top figure: region $\mathscr{S}_a$ (blue) contained in χ_a (red); Bottom left figure: projection of $\mathscr{S}_a$ (blue) and χ_a (red) in the plane x–y; Bottom right figure: estimation of domain of attraction $\mathscr{S}$ (blue) contained in χ (green) for Example 1.

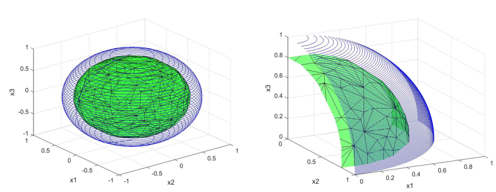

Fig. 3: Ellipsoids $\mathscr{S}_a$ (blue, left figure) and $\mathscr{S}$ (blue, right figure) covering $\mathscr{S}_u$ (green) for Example 1.

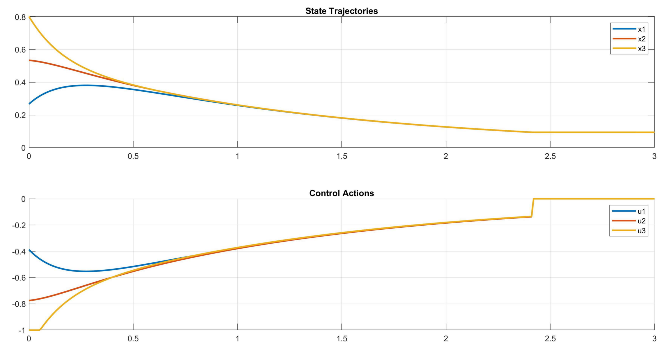

Fig. 4: Trajectories of the agents for $x(0)=(0.0673,0.5345,0.8018)$ and control signal for Example 1.

Example 2 Consider a graph with $N=20$ agents represented by a directed connected graph. The schematic of the network is depicted in Fig. 5.

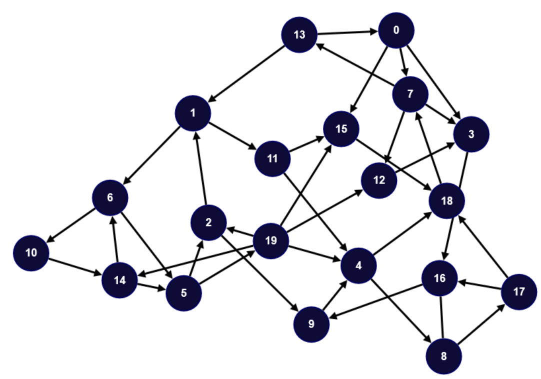

Fig. 5: The schematic of interested network in the form of a directed graph with 20 agents.

Theorem 1 is applied to design the state feedback gain K that solves Problem 1 with $Q=10^{-1}I$, $R=10^{-1}I$, $\mu=7.5$, and $\bar{u}=0.9$. We consider $\varepsilon=0.02$ in B_0 to solve Theorem 1and 2. The trajectories of the agents are illustrated in Fig. 6. Note that the total budget is over in $t=2.2$ and, from this time, the agents are attracted to the consensus manifold (at $t=7$, the maximum value of the agents is 0.0559). We also note that for some agents, the control action saturates in the initial instants showing that the maximum investment is applied for some individuals. It is evident that all the agents opinions are swayed toward the desired consensus over time, but due to budget constraints it is not possible to exactly reach the consensus at $d=0$.

V. Conclusion

In this paper, the problem of reaching a desired opinion (consensus) in a connected network have been discussed as an optimization problem. The proposed approach considers practical aspects such as finite budget in the form of input saturation and constraint in the energy of the control action. The dynamic representing individual opinions, normalized

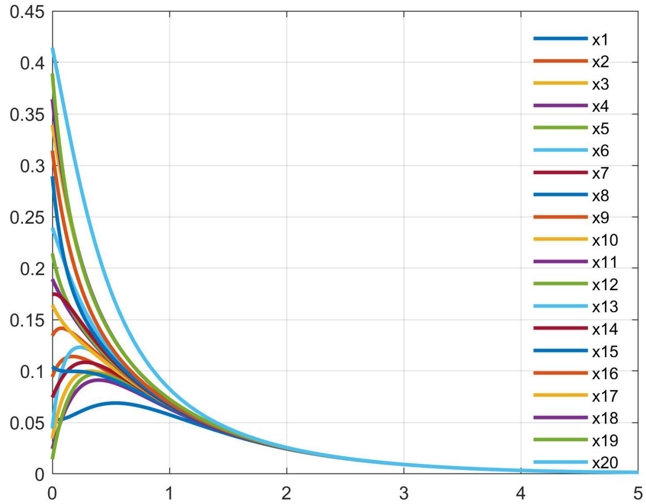

Fig. 6: Trajectories of the agents for some initial condition $x(0) \in \partial\mathscr{S}$ and control signal for Example 2.

in the interval $[0,1]$, is bilinear, yielding extra challenge in designing a stabilizing state-feedback control law. The main results are given as constructive conditions in terms of LMIs by exploiting some properties of positive systems and invariant ellipsoids and can be applied for large networks. Future works involve searching new invariant regions that maximize the domain of attraction of the desired opinion.

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