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Output Error Recursive Algorithms for Identification of Dual Youla-Kucera Models in Closed Loop Operation

Ioan Dore Landau and Bernard Vau

***Abstract*— Dual Youla-Kucera plant model parametrization is very useful for describing model uncertainties. Therefore it is interesting to develop recursive identification algorithms for identification of these type of plant model structures in closed loop operation for potential use in iterative tuning or adaptive control. Following the approach introduced in [1] closed loop output errors type recursive algorithms developed specifically for this type of model structure are presented. The algorithms assure global asymptotic stability in the deterministic environment and unbiased parameter estimation in the presence of noise when the plant model is in the model set. The algorithms will be applied for the identification in closed loop of a test bench for active noise control.**

I. INTRODUCTION

Dual Youla-Kucera (Y-K) model structure is a useful way of representing plant model uncertainties [2], [3] and evaluate robustness of different control schemes. See for example [4] among other publications. It also offers a way for handling large model uncertainties, provided that this structures can be identified recursively in closed loop operation in view of iterative tuning and adaptive control. The standard way of recursively identifying a dual parametrized Youla-Kucera plant model in closed loop operation is to use the so called "Hansen scheme" [2],[5] which transforms this operation in an open loop type identification scheme. This approach however does not take advantage of using dedicated methods for recursive identification in closed loop operation where the aim is to find a plant model estimate which allows to obtain the best closed loop predictor [1],[6]. The objective is also to get a better approximation in the frequency region close to the Nyquist point when the plant model is not in the model set. There are number of off-line identification procedure for identification of dual Youla Kucera parametrized models in closed loop ([7]).

The objective of this paper is to introduce recursive identification algorithms dedicated to the identification in closed loop operation of dual Youla Kucera parametrized models which try to minimize the closed loop (output) prediction error. The algorithms will be developed under the assumption that the controller is constant and known and when the plant model is in the model set. They will assure:

- global asymptotic stability for any initial parameter estimate and closed loop error prediction,
- unbiased parameter estimation in the presence of measurement noise

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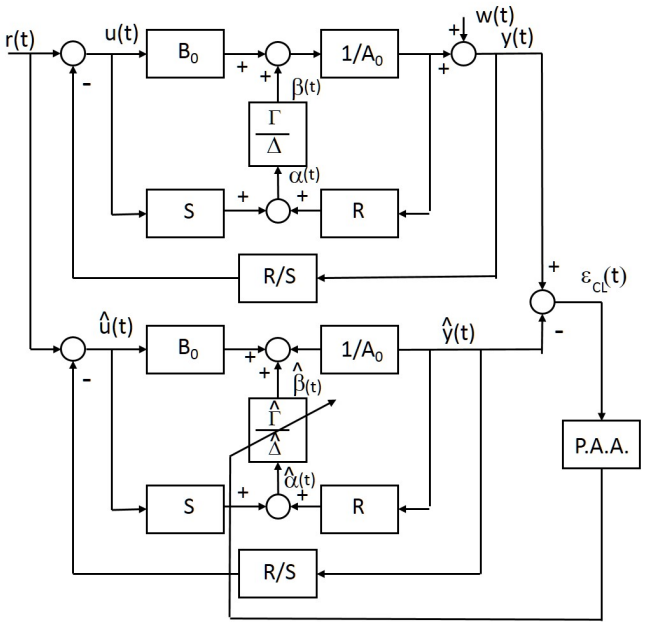


Fig. 1: Closed loop ouput error identification of dual Youla Kucera parametrized model

Identification in closed loop operation of a test bench for active noise control will illustrate the use of the algorithms and their performance. The paper is organized as follows: Basic equation will be presented in Section II. The algorithms will be developed and analyzed in Section III, IV and V under the hypothesis that the plant model is in the model set. The bias distribution in the frequency domain when the plant is not in the model set will be discussed in Section VI and Section VIII will present simulation results.

II. BASIC EQUATIONS

The closed loop output error configuration for the identification of a dual Youla-Kucera parametrized plant models is shown in Figure 1

The initial available plant model (nominal model) is described by the transfer operator:

$$G_o(q^{-1}) = \frac{q^{-d_o} B'_o(q^{-1})}{A_o(q^{-1})} = \frac{B_o(q^{-1})}{A_o(q^{-1})} \qquad (1)$$

with:

$$\begin{aligned} d_0 &= \text{the plant pure time delay in} \\ &\quad \text{number of sampling periods} \\ A_o &= 1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A} = 1 + q^{-1} A_0^* \ ; \\ B_o &= b_1 q^{-1} + \cdots + b_{n_B} q^{-n_B} = q^{-1} B_0^* \ ; \\ B_o^* &= b_1 + \cdots + b_{n_B} q^{-n_B+1} \ ; \end{aligned}$$

where $A_0(q^{-1})$, $B_0(q^{-1})$, $B_0^*(q^{-1})$ are polynomials in the delay operator q^{-1} and n_{A_0}, n_{B_0} and $n_B - 1$ represent their orders[1]. The feedback fixed polynomial controller K is given by:

$$K(q^{-1}) = \frac{R(q^{-1})}{S(q^{-1})} \tag{2}$$

with:

$$\begin{aligned} S &= 1 + s_1 q^{-1} + \cdots + s_{n_S} q^{-n_S} = 1 + q^{-1} S^* \ ; \\ R &= r_0 + \cdots + r_{n_R} q^{-n_R} \ ; \end{aligned}$$

It will be assumed that S is an asymptotically stable polynomial. The unknown part of the plant model is described by the input output block Π:

$$\Pi(q^{-1}) = \frac{\Delta(q^{-1})}{\Gamma(q^{-1})} \tag{3}$$

where: with:

$$\begin{aligned} \Gamma &= 1 + \gamma_1 q^{-1} + \cdots + \gamma_{n_\Gamma} q^{-n_\Gamma} = 1 + q^{-1} \Gamma^* \ ; \\ \Delta &= \delta_1 q^{-1} + \cdots + \delta_{n_\Delta} q^{-n_\Delta} = q^{-1} \Delta^* \ ; \end{aligned} \tag{4}$$

The unknown plant model is described by:

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \tag{5}$$

and can be expressed as[2]

$$G(q^{-1}) = \frac{\Gamma B_0 + \Delta S}{\Gamma A_0 - \Delta R} \tag{6}$$

The closed loop output predictor can be described similarly except that one uses an estimation of Π:

$$\hat{\Pi}(q^{-1}) = \frac{\hat{\Delta}(q^{-1})}{\hat{\Gamma}(q^{-1})} \tag{7}$$

where:

$$\begin{aligned} \hat{\Gamma} &= 1 + \hat{\gamma}_1 q^{-1} + \cdots + \hat{\gamma}_{n_\Gamma} q^{-n_\Gamma} = 1 + q^{-1} \hat{\Gamma}^* \ ; \\ \hat{\Delta} &= \hat{\delta}_1 q^{-1} + \cdots + \hat{\delta}_{n_\Delta} q^{-n_\Delta} = q^{-1} \hat{\Delta}^* \ ; \end{aligned} \tag{8}$$

The input and output of Π will be denoted $\alpha(t)$ and $\beta(t)$ respectively and the input and output of $\hat{\Pi}$ will be denoted $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ respectively.

[1]The complex variable z^{-1} will be used for characterizing the system's behavior in the frequency domain and the delay operator q^{-1} will be used for describing the system's behavior in the time domain.

[2]In order to simplify the writing the argument q^{-1} has been dropped out in many equations)

In the time domain the following input/output relationships will be used:

$$y(t) = \frac{\Gamma B_0 + \Delta S}{\Gamma A_0 - \Delta R} u(t) \tag{9}$$

$$u(t) = r(t) - \frac{R}{S} y(t) \tag{10}$$

where $r(t)$ is the external excitation.

$$\hat{y}(t) = \frac{\hat{\Gamma} B_0 + \hat{\Delta} S}{\hat{\Gamma} A_0 - \hat{\Delta} R} \hat{u}(t) \tag{11}$$

$$\hat{u}(t) = r(t) - \frac{R}{S} \hat{y}(t) \tag{12}$$

Note that:

$$\alpha(t) = S u(t) + R y(t) \tag{13}$$

$$\beta(t) = A_0 y(t) - B_0 u(t) = \frac{\Delta}{\Gamma} \alpha(t) \tag{14}$$

$$\hat{\alpha}(t) = S \hat{u}(t) + R \hat{y}(t) \tag{15}$$

$$\hat{\beta}(t) = A_0 \hat{y}(t) - B_0 \hat{u}(t) = \frac{\hat{\Delta}}{\hat{\Gamma}} \hat{\alpha(t)} \tag{16}$$

The closed loop output error is defined as:

$$\varepsilon_{CL} = y(t) - \hat{y}(t) \tag{17}$$

The closed loop poles for the nominal system G_0 are defined by:

$$P_0 = A_0 S + B_0 R \tag{18}$$

which is assumed to be a Hurwitz polynomial. Define:

$$\theta^T = [\gamma_1, \gamma_2 \ldots \delta_1, \delta_2, \ldots] \ ; \tag{19}$$

$$\hat{\theta}^T = [\hat{\gamma}1, \hat{\gamma}2 \ldots \hat{\delta}1, \hat{\delta}2 ...], \tag{20}$$

$$\varphi^T(t) = [-\hat{\beta}(t), -\hat{\beta}(t-1), \ldots \hat{\alpha}(t), \hat{\alpha}(t-1), \ldots] \tag{21}$$

For developing the identification algorithm it will be assumed that the plant model is in the model set. The algorithms will be analyzed first in this context and then in the case when the plant model is not in the model set.

III. Dual-CLOE Algorithm

In a deterministic context ($w = 0$ in Fig; 1), for the case of a constant estimated parameter vector $\hat{\theta}$ using the predictor given in Eq (11) one has the following result:

Lemma 1 For a constant estimated vector $\hat{\theta}$ in a deterministic environment ($w(t) = 0$) the closed loop output error is given by:

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0} (\theta - \hat{\theta})^T \varphi(t) \tag{22}$$

The proof of this lemma is given in appendix A.

In the presence of time varying parameter estimates one can define an *a priori* and and *a posteriori* filtered closed loop output error, i.e.

$$\varepsilon_{CL}^0(t+1) = \frac{S}{\Gamma P_0} (\theta - \hat{\theta}(t))^T \varphi(t) \tag{23}$$

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0} (\theta - \hat{\theta}(t+1))^T \varphi(t) \tag{24}$$

Eqs. (23)and (24) have the standard form for using parameter adaptation algorithms (PAA) of the form: [6], [8]

$$\hat{\Theta}(t+1) = \hat{\Theta}(t) + F(t)\phi(t)\nu(t+1) \tag{25}$$

$$\nu(t+1) = \frac{\nu^o(t+1)}{1+\phi^T F(t)\phi(t)} \tag{26}$$

$$F(t+1)^{-1} = \lambda_1 F(t)^{-1} + \lambda_2 \phi(t)\phi^T(t) \tag{27}$$

$$0 < \lambda_1 < 1 \quad 0 \leq \lambda_2 < 2, F_0 > 0^3 \tag{28}$$

Leading to the $Dual - CLOE$ algorithm with:
$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi(t)$ and $\nu(t) = \varepsilon_{CL}(t)$.

A. Stability Analysis

Using the Theorem 4.1 [6] or Theorem 3.3 [8] one can straightforwardly conclude that:
Lemma 2: Using the PAA given in Eqs. (25) through (27) with $\hat{\Theta}(t) = \hat{\theta}(t)$ and $\phi(t) = \varphi(t)$ and assuming that S is stable and the external excitation r(t) is bounded, one has :

$$\lim_{t\to\infty} \varepsilon_{CL}(t+1) = \lim_{t\to\infty} \varepsilon^0_{CL}(t+1) = 0 \tag{29}$$

together with the boundedness of $\varepsilon_{CL}(t)$, and $\varphi(t)$ for any initial conditions, provided that:

$$\frac{S(z^{-1})}{\Gamma(z^{-1})P_0(z^{-1})} - \frac{\lambda}{2}; \quad max\lambda_2(t) \leq \lambda_2 \leq 2 \tag{30}$$

is a *strictly positive real* (SPR) transfer function.

IV. Dual - FCLOE algorithm

Consider now Eq. (22), for the case with time varying estimated parameters. Neglecting the non-commutativity of the time-varying operators (but an exact algorithms can be derived - see [8]) one can consider to filter the observation vector as follows:

$$\varphi_f(t) = \frac{S}{\Gamma_0 P_0}\varphi(t) \tag{31}$$

where Γ_0 is an *a priori* estimation of Γ (Γ_0 =1 in the absence of *a priori* information or in the case of FIR models). In this case one gets:

$$\varepsilon_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}[\theta - \hat{\theta}(t+1)]^T\varphi_f(t) \tag{32}$$

This equation has the standard form for using the PAA given in Eqs. (25) through (27)leading to the $Dual - FCLOE$ algorithm with:
$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi_f(t)$ and $\nu(t) = \varepsilon_{CL}(t)$.
The stability condition in this case will be that the transfer function:

$$\frac{\Gamma_0(z^{-1})}{\Gamma(z^{-1})} - \frac{\lambda}{2}; \quad max\lambda_2(t) \leq \lambda_2 \leq 2 \tag{33}$$

is a *strictly positive real* (SPR) transfer function.
Remark: One can also consider to filter the regressor vector through the current estimation of $\hat{\Gamma}(t)$ but including a stability test ($Dual - AFCLOEalgorithm$).

[3] $\lambda_1(t)$ and $\lambda_2(t)$ allow to obtain various profiles for the evolution of the adaptation gain $F(t)$

A. The Noisy Case

In the presence of measurement noise $w(t)$, the output of the plant will be given by:

$$y((t+1) = -A * y(t) + B * u(t) + Aw(t+1) \tag{34}$$

One assumes that the measurement noise $w(t)$ is bounded and independent with respect to the external excitation $r(t)$ and that one uses a decreasing adaptation gain ($\lambda_1(t) \equiv 1, \lambda_2(t) > 0$). One can use for analysis, the ODE approach [9]. Following a similar path as for the proof of *Lemma 1* it can be shown that the closed loop output prediction error in the case of the $Dual - CLOE$ algorithm for a fixed value of the estimated parameter vector will be given by:

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0}(\theta - \hat{\theta})^T\varphi(t) + \frac{AS}{\Gamma P_0}w(t+1) \tag{35}$$

For the $Dual - FCLOE$ algorithm the expression of the adaptation error will take the form:

$$\varepsilon_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}(\theta - \hat{\theta})^T\varphi_f(t) + \frac{AS}{\Gamma P_0}w(t+1) \tag{36}$$

One observes that $\varphi(t)$, $\varphi_f(t)$ and $\varphi_{ff}(t)$ are independent with respect to $w(t+1)$ (by the way that tey are generated) and one can straightforwardly use Theorem 4.1 [8] One concludes that conditions (30) and (33) respectively assures asymptotic unbiased estimates under richness conditions.

V. Relaxation the positive real condition

A. Using Integral+proportional Adaptation

As shown in [8], [10], [1], is possible to relax the strictly positive real condition for stability by using an integral+proportional adaption algorithm:

$$\hat{\theta}_I(t+1) = \hat{\theta}_I(t) + F_I(t)\phi(t)\nu(t+1) \tag{37}$$

$$\hat{\theta}_p(t+1) = F_p(t)\phi(t)\nu(t+1) \tag{38}$$

$$F_I(t+1)^{-1} = \lambda_1(t)F_I(t)^{-1} + \lambda_2(t)\phi(t)\phi^T(t) \tag{39}$$

$$F_p(t) = \alpha(t)F_I(t)\,;\alpha(t) > -0.5 \tag{40}$$

$$\hat{\theta}(t+1) = \hat{\theta}_I(t+1) + \hat{\theta}_p(t+1) \tag{41}$$

provided that:

1) The adaptation gains F_I and F_P and the observation vector $\phi(t)$ satisfy the condition:

$$\phi^T(t)(F_P + \frac{1}{2}F_I)\phi(t) > K_{min} > 0 \ \ \forall t \tag{42}$$

2) The gain K_{min} is the minimum gain such that:

$$H_K(z^{-1}) = \frac{S/(\Gamma P_0) - \lambda/2}{1 + K[S/(\Gamma P_0) - \lambda/2]} \tag{43}$$

is a strictly positive real (a solution exists always in the discrete time).

B. Averaging

For small values of the adaptation gain on can use "averaging" [11] for analysing the stability of the system. Following the analysis provided in [12] stability of the system will be assured if the frequency region wher the SPR condition is violated, is samller than the frequency region where the SPR condition is verified.

VI. Bias distribution in the frequency domain

If the true system is not in the model set, the identified model will be necessarily biased. The bias distribution is usually evaluated in the frequency domain. Using the results of [13] it is possible to obtain the limit model expressions of $Dual-CLOE$ and $DualFCLOE$.

Lemma 3: The limit model for $Dual-CLOE$ algorithm is given by

$$\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \left|\hat{A}(e^{i\omega})\right|^2 \left|G(e^{i\omega}) - \hat{G}(e^{i\omega})\right|^2 \left|S_{yw}(e^{i\omega})\right|^2 \Psi_r(\omega) d\omega \quad (44)$$

where $S_{yw} = \frac{AS}{\Gamma P 0}$ is the output sensitivity function and $\Psi_r(\omega)$ is the PSD (Power Spectral Density) of the external excitation signal.
The proof is given in Appendix.

Lemma 4: The limit model of Dual-FCLOE algorithm is given by

$$\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \left|\frac{\hat{A}(e^{i\omega})S(e^{i\omega})}{\Gamma_0(e^{i\omega})P_0(e^{i\omega})}\right|^2 \left|G(e^{i\omega}) - \hat{G}(e^{i\omega})\right|^2 \left|S_{yw}(e^{i\omega})\right|^2 \Psi_r(\omega) d\omega \quad (45)$$

The proof is given in Appendix.

VII. Validation of he identified models

The techniques presented in [6] and [8] can be used also in this context.

VIII. Identification of a test bench for active noise control in closed loop operation

The objective of this section is to see if a reliable dual Youla-Kucera model can be obtained in closed loop operation. The case of feedback active noise attenuation is considered. The two physical configurations corresponding to G_0 and G are shown in Figure 2 and their corresponding Bode diagramms are presented in Fig. 3 (For more details see [14]).
The excitation signal on $r(t)$ is a PRBS (length: $2^{14}-1$ samples, without decimation). The system G has an order equal to 26 and the R-S controller has an order equal to 57 thus the full order Dual-Youla-Kucera filter has an order equal to 85. The first identification is carried out by using the Dual-FCLOE with $\Gamma_0 = 1$, the estimation being performed with the classical PAA (equations (25), (26), (27), (28)) with $\lambda_1 = \lambda_2 = 1$ (decreasing adaptation gain). One can observe

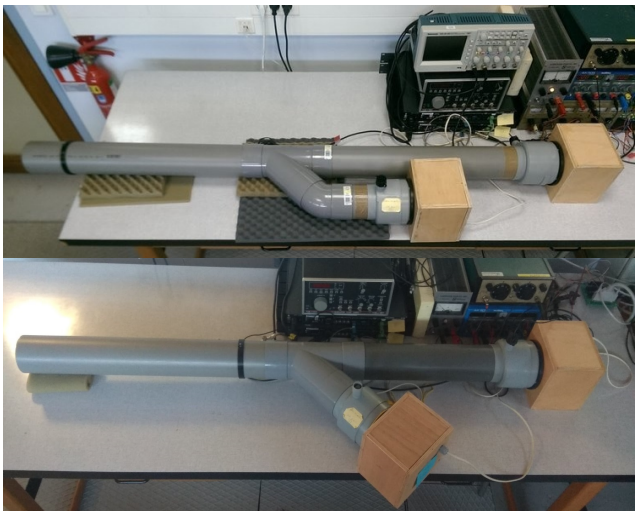


Fig. 2: Duct active noise control test bench (Photo): configuration G_o (top), configuration G (bottom)

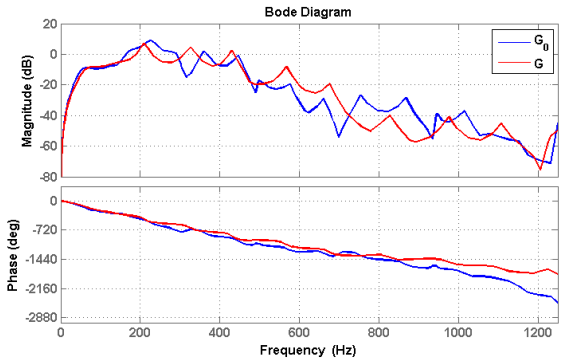


Fig. 3: Bode Diagram of G_0 and G

that the identification scheme converges, as shown in Fig. 4. The estimated parameters are represented as a function of time in Fig. 5. In order to improve the convergence, one can employ an Integral+Proportional adaptation as presented in Section V. An identification under the same conditions is carried out where $\alpha = 0.2$, Fig. 6 and 7 show that the convergence of the estimated parameters has been improved.

At last an identification with a reduced order Dual Youla-Kucera filter is performed (order 60) and Fig. 8 and 9 show that the algorithms performs well, even if a small bias appears in high frequencies, which is unavoidable.

IX. Conclusions

The paper has presented a set of recursive algorithms for identification in closed loop operation of plant models represented in dual Youla Kucera form. The algorithms have been analyzed in the deterministic and stochastic environment and have been evaluated on data from an acitve attenuation noise system. Further work will concern comparisons with the Hansen scheme.

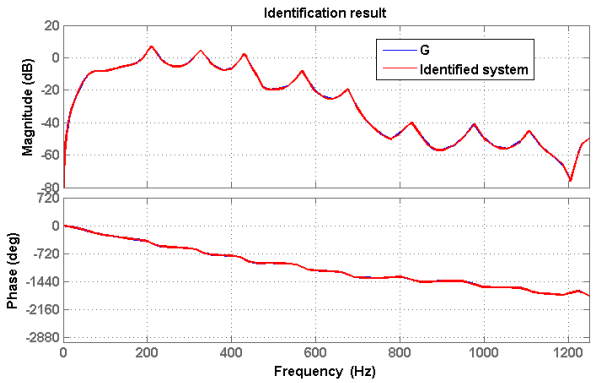


Fig. 4: Bode Diagram of G and of the estimated model for Dual-FCLOE (Dual Y-K filter of order 85, I adaptation)

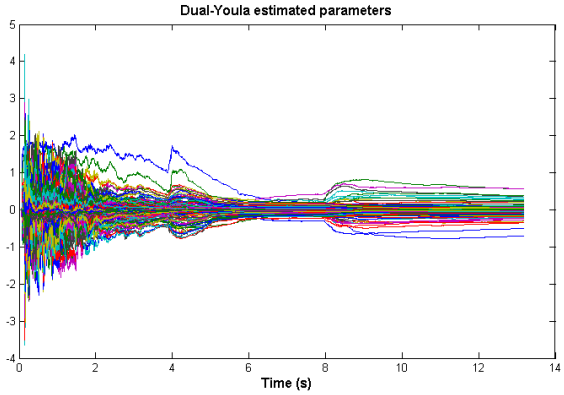


Fig. 5: Estimated parameters for Dual-FCLOE (Dual Youla-Kucera filter of order 85,I adaptation)

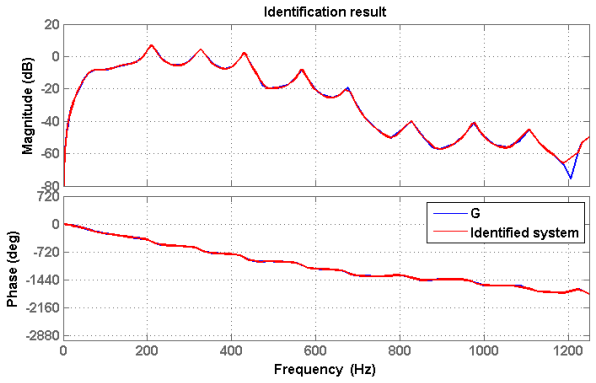


Fig. 6: Bode Diagram of G and of the estimated model for Dual-FCLOE (Dual Y-K filter of order 85, PI adaptation)

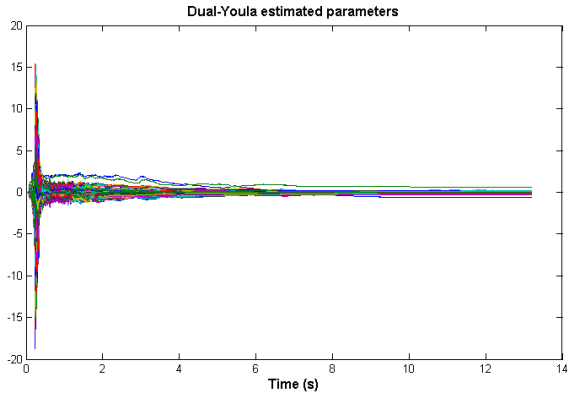


Fig. 7: Estimated parameters for Dual-FCLOE (Dual Y-K filter of order 85, PI adaptation)

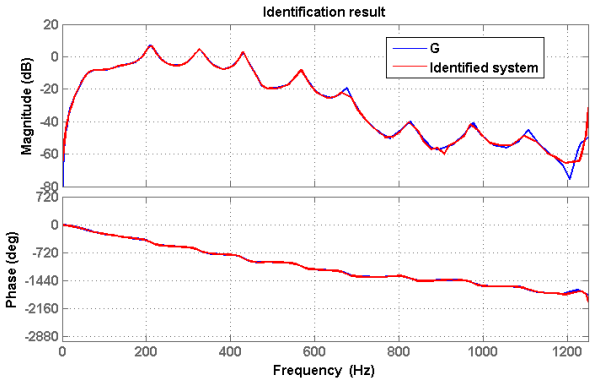


Fig. 8: Bode Diagram of G and of the estimated model for Dual-FCLOE (Dual Y-K filter of order 60, I adaptation)

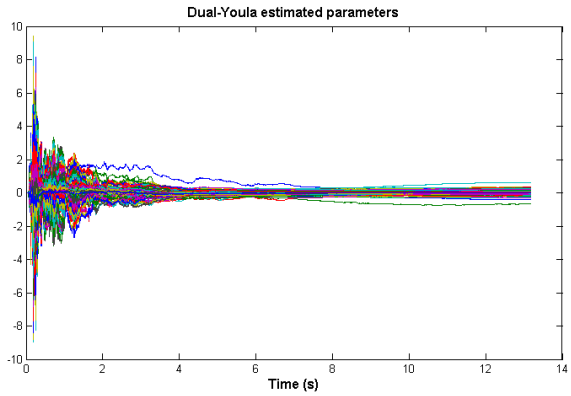


Fig. 9: Estimated parameters for Dual-FCLOE (Dual Youla-Kucera filter of order 85, I adaptation)

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X. APPENDIX A: PROOF OF LEMMA 1

Proof: From Eq. (9) one gets:

$$\Gamma[A_0 y(t+1 - B_0 u(t+1] = \Delta[Su(t+1) + Ry(t+1] \quad (46)$$

which can be alternatively written as:

$$y(t+1) = -(\Gamma A_0)^* y(t) + \Gamma B_0 u(t+1) + \Delta[Su(t+1) + Ry(t+1)] \quad (47)$$

where:

$$\Gamma A_0 = 1 + q^{-1}(\Gamma A_0)^* \quad (48)$$

The output of the closed loop predictor will be given by:

$$\hat{y}(t+1) = -(\hat{\Gamma} A_0)^* \hat{y}(t) + \hat{\Gamma} B_0 \hat{u}(t+1] + \hat{\Delta}[S\hat{u}(t+1) + R\hat{y}(t+1)] \quad (49)$$

where:

$$\hat{\Gamma} A_0 = 1 + q^{-1}(\hat{\Gamma} A_0)^* \quad (50)$$

Eq. (47) can be rewritten:

$$\begin{aligned} y(t+1) = &-(\Gamma A_0)^* y(t) + \Gamma B_0 u(t+1) + \\ &\Delta[Su(t+1) + Ry(t+1)] \\ &\pm[\Gamma A_0)^* \hat{y}(t) + \Gamma B_0 \hat{u}(t+1) + \Delta[S\hat{u}(t+1) + R\hat{y}(t+1)] \end{aligned} \quad (51)$$

taking into account Eq.(17) , Eq. (51) can be expressed as:

$$\begin{aligned} y(t+1) = &-(\Gamma A_0)^* \varepsilon_{CL}(t) - \Gamma B_0 \frac{R}{S} \varepsilon_{CL}(t) \\ &-(\Gamma A_0)^* \hat{y}(t) + \Gamma B_0 \hat{u}(t) + \Delta[S\hat{u}(t+1) + R\hat{y}(t+1)] \\ &+\Delta[R\varepsilon_{CL}(t) - S\frac{R}{S}\varepsilon_{CL}(t)] \end{aligned} \quad (52)$$

Note that the last term in Eq. (52) is zero. Subtracting now Eq. (49) from Eq. (52) one gets:

$$\begin{aligned} &[1 + q^{-1}(\Gamma A_0)^* + q^{-1}\frac{\Gamma B_0 R}{S}]\varepsilon_{CL}(t+1) \\ &= -[(\Gamma A_0)^* - (\hat{\Gamma} A_0)^*]\hat{y}(t) + (\Gamma - \hat{\Gamma}) B_0 \hat{u}(t+1) \\ &\quad +(\Delta - \hat{\Delta})[Su(\hat{t} + 1 + R\hat{y}(t+1)] \end{aligned} \quad (53)$$

Taking into account that:

$$(\Gamma A_0)^* - \hat{\Gamma} A_0)^* = (\Gamma^* - \hat{\Gamma}) A_0 \quad (54)$$

$$(\Gamma - \hat{\Gamma}) = q^{-1}(\Gamma^* - \hat{\Gamma}^*) \quad (55)$$

and also Eq. (18) , Eq. (53) can be rewritten as:

$$\begin{aligned} \frac{\Gamma P_0}{S}\varepsilon_{CL}(t+1) = &(\Gamma^* - \hat{\Gamma}^*)[A_0 \hat{y}(t) - B_0 \hat{u}(t)] \\ &+(\Delta^* - \hat{\Delta}^*)[S\hat{u}(t+1) + R\hat{y}(t+1)] \end{aligned} \quad (56)$$

Taking also into account Eqs. (15) and (16) one gets Eq. (22).

XI. APPENDIX B: PROOF OF LEMMAS 3 AND 4

A. Proof of Lemma 3

As in [13], let us define $Q(q^{-1}, \hat{\theta})$ such that $Q(q^{-1}, \hat{\theta})\frac{\partial \varepsilon_{CL}}{\partial \hat{\theta}} = -\varphi(t, \hat{\theta})$. Notice that we have $\hat{\beta}(t+1) = \frac{P_0}{S}\hat{y}(t+1, \hat{\theta}) - B_0 r(t)$, and $\hat{\beta}(t+1) = \hat{\theta}^T \varphi(t)$. Therefore $\frac{\partial \hat{y}(t+1, \partial \hat{\theta})}{\hat{\theta}} = \frac{S}{P_0}\left[\varphi(t) + \hat{\theta}^T \frac{\partial \varphi(t)}{\partial q\varepsilon}\frac{\partial q\varepsilon}{\partial \hat{\theta}}\right]$, and $\left[\frac{P_0}{S} + \hat{\theta}^T \frac{\partial \varphi}{\partial q\varepsilon}\right]\frac{\partial \varepsilon(t+1, \hat{\theta})}{\partial \hat{\theta}} = -\varphi(t)$. Therefore $Q(q^{-1}, \hat{\theta}) = \frac{P_0}{S} + \hat{\theta}^T \frac{\partial \varphi}{q\varepsilon} = \frac{P_0 \hat{\Gamma}}{S}$. Now, Theorem 1 of [13] can be used with the present expression of $Q(q^{-1}, \hat{\theta}) = \frac{P_0 \hat{\Gamma}}{S}$. Since from [8], p.308, $\varepsilon_{CL}(t) = \frac{\hat{A}S}{P_0 \hat{\Gamma}}\left[G - \hat{G}\right]\frac{AS}{\Gamma P_0}r(t) + \frac{AS}{\Gamma P_0}w(t)$. One obtains, the equivalent prediction error $\varepsilon_E(t) = \hat{A}\left[G - \hat{G}\right]\frac{AS}{\Gamma P_0}r(t) + \frac{AS}{\Gamma P_0}w(t)$, and one has $\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \Psi_{\varepsilon_E}(\omega)d\omega$, where $\Psi_{\varepsilon_E}(\omega)$ is the PSD of ε_E, hence the limit model in the frequency domain.

B. Proof of Lemma 4

Like in case 1 of $Dual - CLOE$ one has $Q(q^{-1}, \hat{\theta}) = \frac{P_0 \hat{\Gamma}}{S}$. On the other hand, one has $\varphi_f(t) = \frac{S}{P_0 \Gamma_0}\varphi(t) = \frac{1}{Q_f(q^{-1})}\varphi(t)$, where $Q_f(q^{-1}) = \frac{P_0 \Gamma_0}{S}$. Therefore one can apply Theorem 2 in [13], with the present expressions of $Q(q^{-1}, \hat{\theta})$ and $Q_f(q^{-1})$, and the equivalent prediction error is $\varepsilon_E(t) = Q(q^{-1}, \hat{\theta})\nu_{CL}(t) + (1 - Q(q^{-1}, \theta))\frac{AS}{\Gamma_0 P_0}w(t+1)$, and one gets $\varepsilon_E(t) = \frac{\hat{A}S}{\Gamma_0 P_0}\left[G - \hat{G}\right]\frac{AS}{\Gamma P_0}r(t) + \frac{AS}{\Gamma P_0}w(t)$, hence the limit model in the frequency domain.