# Max-plus polyhedra-based state characterization for uMPL systems

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Guilherme Espindola-Winck[1], Laurent Hardouin[1] and Mehdi Lhommeau[1]

***Abstract*— This paper presents a mathematical tool for stochastic filter design based on reach sets for general Uncertain Max-Plus Linear (uMPL) systems. The reach sets are defined as the computation of the set of all states that can be reached from a known previous state vector (forward) and from an available source of measurement (backward). The existing approaches in [13, 10] have exponential complexity, which is an important drawback in higher-dimensional systems. In this work, we propose a max-plus polyhedra-based procedure with complexity that is in practice polynomially-bounded.**

## I. INTRODUCTION

Discrete Event Dynamic Systems (DEDS) are systems whose dynamics are event-driven, *i.e.* the state evolution depends entirely on the occurrence of asynchronous discrete events over time. Manufacturing systems, telecommunication networks, transportation networks, are example of DEDS [4]. To describe the behaviour of these systems the ordinary or partial differential equations are not suitable, hence more relevant theoretical setting are considered, among them the following can be cited: languages and automata, Markov chain and Petri nets, the reader is invited to consult [11] for an overview.

Among the DEDS, a particular class involving *synchronization* and *delay* phenomena has been the subject of a dedicated algebraic development, generally called max-plus linear algebra. This class of DEDS can be represented graphically, depicted by Timed Event Graphs (TEG). A TEG is a timed Petri net in which each place admits only one upstream transition and one downstream transition. In analogy to classical linear system theory, the Max-Plus Linear (MPL) systems arise to characterize the behaviour of systems governed by the occurrence of those phenomena.

The knowledge of the system states is a key point in control design and fault detection. The filtering problem consists in the estimation of these states, subject or not to noisy dynamics and measurements. For this reason, the development of methods for observing/estimating the system states are of great interest. In [21] a dynamic observer for MPL systems is proposed, in [17, 15] a Luenberger-inspired observer is developed. In both cases, uncertainties w.r.t. the model parameters are considered, however, the probabilistic aspects are not taken into account. Though, these probabilistic aspects are of great interest in the filtering problems where the model parameters are influenced by random processes [22, 9].

In this work, although the probabilistic aspects of the uncertainties are not considered, we are interested in systems where the uncertain parameters can vary over a known interval. Formally, we are interested in the Uncertain Max-Plus Linear (uMPL) systems, where at each event step, the entries of the system matrices can take an arbitrary value within a real interval.

In previous works [13, 10], reach sets are introduced as the sets of all states that can be reached from a known previous state vector (forward) and from an available source of measurement (backward), in a procedure that resembles the two-fold bayesian filtering scheme. The latter has a more efficient performance, however both approaches have exponential complexity.

As already remarked in [10], the filter design can take advantage of the reach sets calculations. Indeed, these sets can be applied to compute and represent the support of the posterior probability density function (p.d.f) of the states of the uMPL system and a solution to the estimation problem can be obtained using Monte Carlo method called *Importance Sampling*.

In this paper, we make the following contributions:

i We propose a max-plus polyhedra-based state characterization with a probabilistic consistency guarantee for uMPL systems.

ii The proposed calculation procedure, based on the resolution of a two-sided max-plus equation, drastically reduces the calculation time compared to other approaches [13, 10].

This paper is organized as follows. In Section II we present some background on MPL systems and interval analysis. Section III introduces previous results concerning reach sets and presents the main contribution of this work as the solution of a two-sided max-plus equation. In Section IV, some simulations are given in order to compare the execution time of the proposed approach with the one in [10]. Finally, Section V concludes the work and give hints about the filtering design with the aid of reach sets.

## II. MATHEMATICAL BACKGROUND

### A. Algebraic framework

Define $\varepsilon = -\infty$, $e = 0$ and the *max-plus semiring* $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ endowed with two internal operations: *sum* ($\oplus$) and *product* ($\otimes$) are defined as $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$ for any $x, y \in \mathbb{R}_{\max}$. Moreover, $\varepsilon$ is the absorbing element, *i.e.* for any $a \in \mathbb{R}_{\max}$, $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$.

*This work was supported by the RFI Atlanstic 2020.

[1]G. Espindola-Winck, L. Hardouin and M. Lhommeau are with LARIS/Polytech Angers/University of Angers, FRANCE `guilherme.espindolawinck@univ-angers.fr`, `laurent.hardouin@univ-angers.fr` and `mehdi.lhommeau@univ-angers.fr`, respectively.

The $\oplus$ and $\otimes$ operations can be extended to matrices as follows: if $A, B \in \mathbb{R}_{\max}^{n\times p}$ and $C \in \mathbb{R}_{\max}^{p\times q}$, then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes C)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}$. As in classical algebra, the operator $\otimes$ will be usually omitted in expressions for the sake of readability.

In this algebraic structure, operation $\oplus$ induces a partial order relation

$$a \succeq b \iff a = a \oplus b. \tag{1}$$

It coincides with the usual order $\geq$ on $\mathbb{R} \cup \{-\infty\}$.

For a more exhaustive presentation, the reader is invited to refer to [4, 18].

### B. Interval arithmetic in $\mathbb{R}_{\max}^{n}$

Interval arithmetic is presented in [23]. An interval in $\mathbb{R}_{\max}$ is defined as $[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_{\max} : \underline{x} \preceq x \preceq \overline{x}\}$. The max-plus operations can be, therefore, extended to intervals as follows [5, 16, 20, 19]:

$$[x] \oplus [y] = \{x \oplus y : x \in [x], y \in [y]\} = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \ , \tag{2}$$

$$[x] \otimes [y] = \{x \otimes y : x \in [x], y \in [y]\} = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}] \ . \tag{3}$$

The intersection of the intervals $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$ can be defined as:

$$[x] \cap [y] = [\max\{\underline{x}, \underline{y}\}, \min\{\overline{x}, \overline{y}\}], \tag{4}$$

If $[x] \cap [y] \neq \emptyset$ then the union of $[x]$ and $[y]$ is:

$$[x] \cup [y] = [\min\{\underline{x}, \underline{y}\}, \max\{\overline{x}, \overline{y}\}], \tag{5}$$

otherwise the union cannot be represented as a single interval, but by a collection of intervals.

The intersection and union operations of two interval vectors can be computed as the element-wise operation of the corresponding entries.

The $\oplus$ and $\otimes$ are extended to interval matrices as follows: if $[A]$, $[B]$ and $[C]$ are, respectively, $(n \times p)$, $(n \times p)$ and $(p \times q)$-dimensional interval matrices, then $([A] \oplus [B])_{ij} = [a_{ij}] \oplus [b_{ij}]$ and $([A] \otimes [C])_{ij} = \bigoplus_{k=1}^{p}([a_{ik}] \otimes [c_{kj}])$.

If we consider the max-plus equation $[\mathbf{z}] = [C] \otimes \mathbf{x}$, with, $[\mathbf{z}]$ an $q$-dimensional interval vector, $[C]$ an $(q \times n)$-dimensional interval matrix and $\mathbf{x}$ an $n$-dimensional vector, then we can write the $i$-th component of $[\mathbf{z}]$ as follows:

$$[z]_i = \left[ \bigoplus_{j=1}^{n} \underline{c}_{ij} \otimes x_j, \bigoplus_{j=1}^{n} \overline{c}_{ij} \otimes x_j \right], \text{ for all } i \in \{1, \ldots, q\}. \tag{6}$$

### C. MPL Systems

A general MPL model that describes the TEGs class is defined as:

$$\tilde{\mathbf{x}}(k) = \bigoplus_{l=0}^{M} \tilde{A}_l \tilde{\mathbf{x}}(k-l) \oplus \bigoplus_{l=0}^{N} \tilde{B}_l \tilde{\mathbf{u}}(k-l), \tag{7a}$$

$$\tilde{\mathbf{z}}(k) = \tilde{C}\tilde{\mathbf{x}}(k), \tag{7b}$$

where $\tilde{\mathbf{x}} \in \mathbb{R}_{\max}^{n}$, $\tilde{A}_l \ (l = 0, \ldots, M) \in \mathbb{R}_{\max}^{n\times n}$, $\tilde{\mathbf{u}} \in \mathbb{R}_{\max}^{p}$, $\tilde{B}_l \ (l = 0, \ldots, N) \in \mathbb{R}_{\max}^{n\times p}$, $\tilde{\mathbf{z}} \in \mathbb{R}_{\max}^{q}$ and $\tilde{C} \in \mathbb{R}_{\max}^{q\times n}$.

After some modifications[1], it is possible to obtain the following explicit form:

$$\mathbf{x}(k) = A_0\mathbf{x}(k) \oplus A_1\mathbf{x}(k-1) \oplus B_0\mathbf{u}(k), \tag{8a}$$

$$\mathbf{z}(k) = C\mathbf{x}(k). \tag{8b}$$

[1] Each place of the TEG is assumed to be with initially one or zero token since it is sufficient to add extra places, *i.e.* to increase $n$ in a suitable way.

In practice, the token-free matrix $A_0$ is assumed to be written in strictly lower triangular form ($a_0^{ij} = \varepsilon$ for all $i \leq j$) since the corresponding subgraph of $A_0$ is without circuit, *i.e.* without frozen transitions. Thus,

$$x_i(k) = \bigoplus_{j=1}^{i-1} a_0^{ij} x_j(k) \oplus y_i, \text{ for all } i \in \{1, \ldots, n\}, \tag{9}$$

where $\mathbf{y} = A_1\mathbf{x}(k-1) \oplus B_0\mathbf{u}(k)$.

The matrix entries of the equations above are considered to be *bounded noisy*, *i.e.* it is assumed that at each event $k$ these entries[2] can take an arbitrary value within a real interval. Hence, it is possible to model uMPL systems as defined in [13, 10] from Eq.(8) considering that $A_0 \doteq A_0(k) \in [\underline{A}_0, \overline{A}_0]$, $A_1 \doteq A_1(k) \in [\underline{A}_1, \overline{A}_1]$, $B_0 \doteq B_0(k) \in [\underline{B}_0, \overline{B}_0]$ and $C \doteq C(k) \in [\underline{C}, \overline{C}]$ are matrices of independent random variables with finite support and whose entries are mutually independent[3]. For instance, matrices $\underline{A}_0$ and $\overline{A}_0$ are respectively the lower and upper bounds of $[A_0]$, such that $a_0^{ij} \in [\underline{a}_0^{ij}, \overline{a}_0^{ij}]$. The same reasoning is applied to the lower and upper bounds of $[A_1]$, $[B_0]$ and $[C]$.

### D. Residuation

The max-plus inequality $A\mathbf{x} \preceq \mathbf{y}$ with matrix $A \in \mathbb{R}_{\max}^{n\times p}$ and vectors $\mathbf{x} \in \mathbb{R}_{\max}^{p}$ and $\mathbf{y} \in \mathbb{R}_{\max}^{n}$ admits a greatest solution in terms of the completion of $\mathbb{R}_{\max}$, *i.e.* the set $\mathbb{R}_{\max} \cup \{+\infty\}$ (see [4]). This solution is given by $\hat{\mathbf{X}} = A \diamond\!\!\!\backslash \mathbf{y}$ such that $A\hat{\mathbf{X}} \preceq \mathbf{y}$. Below each entry of $\hat{\mathbf{X}}$ is computed with the convention $-\infty + \infty = +\infty$:

$$\hat{X}_i = \min_{k=1}^{n}\{a_{ki} \diamond\!\!\!\backslash y_k\}, \text{ for all } i \in \{1, \ldots, p\}, \tag{10}$$

where $a_{ki} \diamond\!\!\!\backslash y_k$ is the greatest solution of the inequality $a_{ki} \otimes x \preceq y_k$. It is worth to mention that an equivalent calculation of $\hat{\mathbf{X}}$ can be obtained thanks to the fact that $A \diamond\!\!\!\backslash \mathbf{y} = -A^{\mathrm{t}} \odot \mathbf{y}$ with $(\bullet)^{\mathrm{t}}$ being the matrix transposition operator and with $\odot$ being the matrix multiplication of $-A^{\mathrm{t}}$ by $\mathbf{y}$ but by replacing the usual $\oplus$ with min.

### E. The two-sided equation $E\mathbf{z} = F\mathbf{z}$ in $\mathbb{R}_{\max}^{n+1}$

In this section we recall some notions we shall use in the following (for more details see [2, 3, 7, 8]).

*Definition 1:* A max-plus polyhedron $\mathscr{P}$ of $\mathbb{R}_{\max}^{n}$ is the intersection of finitely many inequality constraints of form: $\mathbf{a}^{\mathrm{t}}\mathbf{x} \oplus b \succeq \mathbf{c}^{\mathrm{t}}\mathbf{x} \oplus d$, where $\mathbf{a}, \mathbf{c}, \mathbf{x} \in \mathbb{R}_{\max}^{n}$ and $b, c \in \mathbb{R}_{\max}$.

In the max-plus semiring, a system of constraints represented by inequalities can also be represented by equations with the same expressiveness (see Eq.(1)). Therefore, the constraints in this work are represented by equations and max-plus polyhedra can also be defined as the intersection of finitely many equality constraints.

The equation $A\mathbf{x} \oplus \mathbf{b} = C\mathbf{x} \oplus \mathbf{d}$ with $A, C \in \mathbb{R}_{\max}^{s\times n}$ and $\mathbf{b}, \mathbf{d} \in \mathbb{R}_{\max}^{s}$ is considered to be non-homogeneous and to

[2] For the sake of brevity, we drop $k$ everywhere whenever it is clear the recursive relation.

[3] This assumption of statistical independence between the matrix entries means that the minimum task duration or transportation time are independent of each other. This assumption is reasonable for practical problems, *e.g.*, in the field of transport systems, a failure of one train does not affect the potential efficiency of the others, even if they are blocked due to precedence constraint.

properly obtain a solution in $\mathbf{x}$ it must be associated to the homogeneous equation $E\mathbf{z} = F\mathbf{z}$ where $E = (A\ \mathbf{b})$, $F = (C\ \mathbf{d})$ and $\mathbf{z} = (\mathbf{x}^t, e)^t$ with solution in $\mathbf{z}$ for the first $n$ coordinates and $z_{n+1} = e$.

The solutions of $E\mathbf{z} = F\mathbf{z}$ in $\mathbb{R}^{n+1}_{\max}$ are obtained using existing algorithms:

i *the elimination method* of double exponential complexity [7],

ii implementation by induction on $s$ of the *the elimination method* in the Max-Plus toolbox [12] of `Scilab` and `ScicosLab` of complexity $\mathcal{O}(c_n^4 sn)$ (refined in [2]),

iii *the double description method* of complexity $\mathcal{O}(c_n^2 s^2 n)$ [3].

The term $c_n$ is related to the maximal number of solutions of the $s$ equations represented by $E\mathbf{z} = F\mathbf{z}$ in $\mathbb{R}^{n+1}_{\max}$, and its calculation falls back on a combinatorial problem that is hard to be determined beforehand. In worst-case scenario, $c_n$ is exponential (see [1] for more details). Nevertheless, it is observed, in practice, that $c_n$ appears to be polynomially-bounded for the solution of the two-sided max-plus equations that will be used in this work, *i.e.* the maximal number of solutions remains polynomial in $n$. Future work could focus on the mathematical proof of this claim.

*Definition 2:* Given a collection of vectors $V = \{\mathbf{v}^i\}_{i=1}^k$, we define $\mathtt{span}(V) = \{k \in \mathbb{N}, \mathbf{v}^i \in V, \lambda_i \in \mathbb{R}_{\max} : \bigoplus_{i=1}^k \lambda_i \mathbf{v}^i\}$ as the set of all possible combinations of vectors $\mathbf{v}^i$.

*Definition 3:* The *convex hull* of $V = \{\mathbf{v}^i\}_{i=1}^k$ is denoted as $\mathtt{hull}(V) = \{k \in \mathbb{N}, \mathbf{v}^i \in V, \bigoplus_{i=1}^k \lambda_i = e : \bigoplus_{i=1}^k \lambda_i \mathbf{v}^i\}$, with no positivity constraints on $\lambda_i$.

*Definition 4:* Given $\mathtt{span}(G) \subseteq \mathbb{R}^n_{\max}$ and a vector $\mathbf{v} \in \mathbb{R}^n_{\max}$, we say that $\mathbf{v}$ is *redundant* or linearly dependent on $\mathtt{span}(G)$ if $\mathbf{v} \in \mathtt{span}(G)$.

In [8], it is proven that if $\mathbf{v} = G(G \backslash \mathbf{v})$ then $\mathbf{v}$ is redundant on $\mathtt{span}(G)$. However, we will rather use [6] to compute

$$\bigcup_{i=1}^{k} \arg\min_{j=1}^{n_G}\{-g_j^i + v_j\} = \mathscr{N}, \tag{11}$$

such that if $\mathscr{N} = \{1, \dots, n_G\}$ then $\mathbf{v} \in \mathtt{span}(G)$. Additionally, this test is twice faster than checking if $\mathbf{v} = G(G \backslash \mathbf{v})$.

Let $\mathscr{P} = \bigcap_{i=1}^s \{\mathbf{z} \in \mathbb{R}^{n+1}_{\max} : E_i\mathbf{z} = F_i\mathbf{z}\}$ where $E_i$ and $F_i$ are respectively the $i$-th row of $E$ and $F$. The solution of $E\mathbf{z} = F\mathbf{z}$ in $\mathbb{R}^{n+1}_{\max}$ is defined as the *minimal system of generators* represented by $\mathcal{G} = \mathtt{span}(G)$ where the columns of $G$ satisfy $E\mathbf{z} = F\mathbf{z}$, *i.e.* $\mathcal{G} = \{\mathbf{g} \in G : E\mathbf{g} = F\mathbf{g}\}$ is the *minimal generating set* that belongs to the max-plus polyhedron $\mathscr{P}$, precisely $\mathcal{G} \subseteq \mathscr{P}$ is called the inner region of $\mathscr{P}$.

*Lemma 1:* If the last row of $G \in \mathbb{R}^{n_G \times k}_{\max}$ that determines $\mathtt{span}(G) \subseteq \mathbb{R}^n_{\max}$ is $(e, \dots, e)$ then $\mathtt{span}(G) = \mathtt{hull}(G)$.

*Proof:* For any $\mathbf{v} \in \mathtt{span}(G)$ we first check if $v_{n_G} = e$, if it is not the case then we $\otimes$-multiply $\mathbf{v}$ by a non-$\varepsilon$ scalar. After this phase, we have the following that holds: $\bigoplus_{i=1}^k \lambda_i g^i_{n_G} = e \Rightarrow \bigoplus_{i=1}^k \lambda_i e = e$ which is the span of $G$ constrained to $\bigoplus_{i=1}^k \lambda_i = e$. Hence, $\mathtt{span}(G) = \mathtt{hull}(G)$. ■

## III. Reach sets computation using two-sided max-plus equations

This section presents the computation of the intersection of the direct image of a previous state vector $\mathbf{x}(0)$ and the inverse image of a measurement $\mathbf{z}$ w.r.t. the uMPL system derived from Eq.(9) and Eq.(8b) where $\mathbf{y} = A_1\mathbf{x}(0) \oplus B_0\mathbf{u}$ will simply be represented by $\mathbf{y} = A_1\mathbf{x}(0)$ without loss of generality[4].

### A. The direct image of $\mathbf{x}(0)$ - forward reach set

Let Eq.(9) be an uMPL system. If the point $\mathbf{x}(0)$ is given, then $x_i \in [x_i]$ for all $i \in \{1, \dots, n\}$ where:

$$[x_i] = \bigoplus_{j=1}^{i-1} [a_0^{ij}][x_j] \oplus [y_i], \tag{12}$$

and $[\mathbf{y}] = [A_1]\mathbf{x}(0)$.

From Eq.(6) we compute the direct image of $\mathbf{x}(0)$ as follows[5]:

$$\mathcal{I}_{[A_0],[A_1]}\{\mathbf{x}(0)\} = \\ = \bigcap_{i=1}^{n} \left( \left\{ x_i \succeq \underbrace{\bigoplus_{j=1}^{i-1} \underline{a}_0^{ij} \underline{x}_j \oplus \underline{y}_i}_{\underline{X}_i} \right\} \cap \left\{ x_i \preceq \underbrace{\bigoplus_{j=1}^{i-1} \overline{a}_0^{ij} \overline{x}_j \oplus \overline{y}_i}_{\overline{X}_i} \right\} \right), \tag{13}$$

where $[y_i] = [\bigoplus_{j=1}^n \underline{a}_1^{ij} \otimes x_j(0), \bigoplus_{j=1}^n \overline{a}_1^{ij} \otimes x_j(0)]$. Therefore,

$$\mathcal{I}_{[A_0],[A_1]}\{\mathbf{x}(0)\} = \{\mathbf{x} \succeq \underline{\mathbf{X}}\} \cap \{\mathbf{x} \preceq \overline{\mathbf{X}}\}, \tag{14}$$
$$\mathbf{x} \in \mathcal{I}_{[A_0],[A_1]}\{\mathbf{x}(0)\} \iff \underline{\mathbf{X}} \preceq \mathbf{x} \preceq \overline{\mathbf{X}},$$

an $n$-dimensional interval vector.

*Proposition 1:* Let $\mathscr{D}$ be a max-plus polyhedron. The following properties are equivalent:

*i)* $\mathtt{span}(D) \subseteq \mathscr{D}$, where $\mathtt{span}(D)$ is given by the solution set of the following two-sided max-plus equation:

$$\begin{pmatrix} I_n & \mathcal{E}_{n\times 1} \\ \mathcal{E}_{n\times n} & \overline{\mathbf{X}} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix} = \begin{pmatrix} I_n & \underline{\mathbf{X}} \\ I_n & \overline{\mathbf{X}} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix}, \tag{15}$$

(representing $2n$ single equations) where $I$ and $\mathcal{E}$ are the identity matrix (square matrix with $e$ on the main diagonal and $\varepsilon$ elsewhere) and the zero matrix (rectangular matrix whose entries are $\varepsilon$) respectively. In addition, the pair $(\underline{\mathbf{X}}, \overline{\mathbf{X}})$ is calculated beforehand by Eq.(14),

*ii)* $\mathscr{D}$ is the smallest max-plus polyhedron such that $\mathcal{I}_{[A_0],[A_1]}\{\mathbf{x}(0)\} \subseteq \mathscr{D}$.

*Proof:* For each $i \in \{1, \dots, n\}$, the Eq.(14) is depicted, using the partial order induced by $\oplus$ (see Eq.(1)), as follows:

$$\begin{cases} x_i \succeq \underline{X}_i \iff x_i = x_i \oplus \underline{X}_i, \\ x_i \preceq \overline{X}_i \iff \overline{X}_i = x_i \oplus \overline{X}_i, \end{cases}$$

[4]The equation $\mathbf{y} = A_1\mathbf{x}(0) \oplus B_0\mathbf{u}$ can be transformed in $\mathbf{y} = \mathcal{M}\mathbf{x}'(0)$ by considering $\mathcal{M} = (A_1\, B_0)$ and $\mathbf{x}'(0) = (\mathbf{x}^t(0)\ \mathbf{u}^t)^t$ of appropriate dimensions and does not change the nature of the calculation.

[5]For brevity, the sets $\{\mathbf{x} \in \mathbb{R}^n_{\max} : x_j \bowtie \mathit{constant}\}$ with $\bowtie \in \{\preceq, \succeq\}$ can rather be represented in short notation by $\{x_j \bowtie \mathit{constant}\}$.

which corresponds to

$$(\varepsilon,\dots,e,\dots,\varepsilon)\mathbf{x}\oplus(\varepsilon)=(\varepsilon,\dots,e,\dots,\varepsilon)\mathbf{x}\oplus(\underline{X}_i), \tag{16}$$
$$(\varepsilon,\dots,\varepsilon)\mathbf{x}\oplus(\overline{X}_i)=(\varepsilon,\dots,e,\dots,\varepsilon)\mathbf{x}\oplus(\overline{X}_i). \tag{17}$$

Hence, it is straightforward to see that Eq.(15) is the vertical concatenation of Eq.(16) and Eq.(17) for all $i\in\{1,\dots,n\}$. ■

The solution of Eq.(14) is simply defined by its bounds $[\underline{\mathbf{X}},\overline{\mathbf{X}}]$, *i.e.* a hyperrectangle which is a non-empty compact convex subset of $\mathbb{R}^n_{\max}$ and also of $\mathbb{R}^n$. A hyperrectangle can be seen as a subset $S\subseteq\mathbb{R}^{n\times m}_{\max}$, where $m=2^n$ represents its corners/vertices, and according to the Max-Plus Minkowski Theorem [14, Theorem 3.2] every element $\mathbf{s}\in[\underline{\mathbf{X}},\overline{\mathbf{X}}]\Leftrightarrow\mathbf{s}\in S$, is therefore the convex hull of $n+1$ generators $\tilde{\mathbf{d}}$. Furthermore, Eq.(15) and Eq.(14) are similar, then $(\tilde{\mathbf{d}}^{\mathrm{t}},e)^{\mathrm{t}}\in\mathtt{hull}(D)=\mathtt{span}(D)$ (see Lemma 1) where $D\in\mathbb{R}^{(n+1)\times(n+1)}_{\max}$ is given by the following clause form:

$$D=\begin{pmatrix}\underline{X}_1&\overline{X}_1&\underline{X}_1&\dots&\underline{X}_1\\ \underline{X}_2&\underline{X}_2&\overline{X}_2&\dots&\underline{X}_2\\ \vdots&\vdots&\vdots&\dots&\vdots\\ \underline{X}_n&\underline{X}_n&\underline{X}_n&\dots&\overline{X}_n\\ e&e&e&\dots&e\end{pmatrix}. \tag{18}$$

*Example 1:* Let $[\mathbf{x}]=([0,2],[1,3])^{\mathrm{t}}$. Thus, if $\mathbf{x}\in[\mathbf{x}]$ then $(\mathbf{x}^{\mathrm{t}},e)^{\mathrm{t}}\in\mathtt{span}(D)$ with $D=\begin{pmatrix}0&2&0\\1&1&3\\e&e&e\end{pmatrix}$ calculated thanks to Eq.(18). This is consistent $\forall\mathbf{x}\in[\mathbf{x}]$, as it is shown in Figure 1.

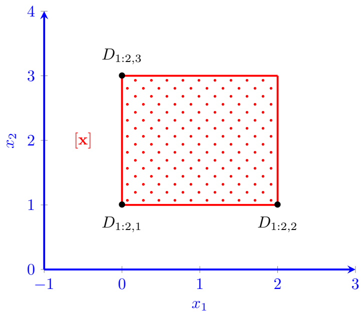


Fig. 1. $[\mathbf{x}]$ and $\mathtt{span}(D)$ of Example 1.

## B. The inverse image of $\mathbf{z}$ - backward reach set

The computation of the inverse image of a given measurement, depicted by vector $\mathbf{z}$ w.r.t. Eq.(8b), was properly discussed in [10] and it is defined as the solution in $\mathbf{x}$ of the following problem:

$$\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}=\{\mathbf{x}\in\mathbb{R}^n_{\max}:\exists C\in[C]:C\mathbf{x}=\mathbf{z}\}, \tag{19}$$
$$\mathbf{x}\in\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}\iff\underline{C}\mathbf{x}\preceq\mathbf{z}\preceq\overline{C}\mathbf{x}.$$

Therefore, we recall the solution in $\mathbf{x}$ obtained in [10] and we invite the reader to refer to this work for more details[6]. Let

$$\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}=L\cap U, \tag{20}$$

where

$$L=\bigcap_{i=1}^{q}L_i,\quad L_i=\{z_i\preceq(\overline{C}\mathbf{x})_i\},\quad U=\{\mathbf{x}\preceq\underline{C}\diamond\mathbf{z}\}, \tag{21}$$

and

$$L_i\equiv\bigcup_{j=1}^{n}\{x_j\succeq\overline{c}_{ij}\diamond z_i\}. \tag{22}$$

Thus $L\cap U=(\bigcap_{i=1}^{q}L_i)\cap U=\bigcap_{i=1}^{q}L_i\cap U$ and then

$$\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}=\bigcap_{i=1}^{q}\left(\bigcup_{j=1}^{n}\{x_j\succeq\overline{c}_{ij}\diamond z_i\}\cap\{\mathbf{x}\preceq\underline{C}\diamond\mathbf{z}\}\right), \tag{23}$$

is a set of cardinality bounded by $n^q$

In the following, we will show that it is possible to reinterpret the previous results with the objectives of improving the computation times and representing $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ as the span of a single matrix rather than a set with exponential cardinality.

*Lemma 2:* The following two-sided max-plus equation directly encodes $L_i$ of Eq.(22) with the same expressiveness:

$$(\overline{c}_{i1},\dots,\overline{c}_{in})\mathbf{x}\oplus(z_i)=(\overline{c}_{i1},\dots,\overline{c}_{in})\mathbf{x}\oplus(\varepsilon). \tag{24}$$

*Proof:* The proof is straightforward by considering the partial order induced by $\oplus$ (see Eq.(1)):

$$z_i\preceq(\overline{C}\mathbf{x})_i\Leftrightarrow(\overline{C}\mathbf{x})_i\oplus z_i=(\overline{C}\mathbf{x})_i,$$
$$\Leftrightarrow(\overline{c}_{i1},\dots,\overline{c}_{in})\mathbf{x}\oplus(z_i)=(\overline{c}_{i1},\dots,\overline{c}_{in})\mathbf{x}\oplus(\varepsilon).$$

■

*Proposition 2:* Let $\mathscr{H}$ be a max-plus polyhedron. The following properties are equivalent:

i $\mathtt{span}(H)\subseteq\mathscr{H}$, where $\mathtt{span}(H)$ is given by the solution set of the following two-sided max-plus equation:

$$\begin{pmatrix}\overline{C}&\mathbf{z}\\ \mathcal{E}_{n\times n}&\overline{\mathbf{X}}^U\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ e\end{pmatrix}=\begin{pmatrix}\overline{C}&\mathcal{E}_{q\times1}\\ I_n&\overline{\mathbf{X}}^U\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ e\end{pmatrix}, \tag{25}$$

(representing $q+n$ single equations) where $\overline{\mathbf{X}}^U=\underline{C}\diamond\mathbf{z}$,

ii $\mathscr{H}$ is the smallest max-plus polyhedron such that $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}\subseteq\mathscr{H}$.

*Proof:* The proof is a direct consequence of Lemma 2 with the intersections along $i\in\{1,\dots,q\}$ as presented in Eq.(24) and along $j\in\{1,\dots,n\}$ as presented in Eq.(17) but by replacing $\overline{X}_j$ by $\overline{X}^U_j$. Summing-up, the inverse image is seen as a vertical concatenation of $q+n$ constraints, as presented in Subsection II-E. ■

Eq.(25) is proper to be solved using any algorithm of Subsection II-E.

*Example 2:* Let $\mathbf{z}=(5,4)^{\mathrm{t}}$ and $[C]=\begin{pmatrix}[1,4]&[2,3]\\ [1,2]&[e,4]\end{pmatrix}$. Thus, if $\mathbf{x}\in\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ (see Eq.(19)) then $(\mathbf{x}^{\mathrm{t}},e)^{\mathrm{t}}\in\mathtt{span}(H)$ with $H=\begin{pmatrix}\varepsilon&3&1&\varepsilon&2\\3&\varepsilon&0&2&\varepsilon\\e&e&e&e&e\end{pmatrix}$ calculated thanks to Eq.(25) with $\begin{pmatrix}4&3&5\\2&4&4\\ \varepsilon&\varepsilon&3\\ \varepsilon&\varepsilon&3\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ e\end{pmatrix}=\begin{pmatrix}4&3&\varepsilon\\2&4&\varepsilon\\e&\varepsilon&3\\ \varepsilon&e&3\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ e\end{pmatrix}$. This is consistent $\forall\mathbf{x}\in\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$, as it is shown in Figure 2.

[6]The max-plus mapping is generally residuated but not dually residuated, *i.e.* given $\mathbf{z}$, there is a unique greatest $\mathbf{x}$ given by Eq.(10) such that $\underline{C}\mathbf{x}\preceq\mathbf{z}$, but not a unique least $\mathbf{x}$ such that $\overline{C}\mathbf{x}\succeq\mathbf{z}$. Hence, the task of finding $\mathbf{x}$ in $\underline{C}\mathbf{x}\preceq\mathbf{z}\preceq\overline{C}\mathbf{x}$ is not as straightforward as the direct image computation.

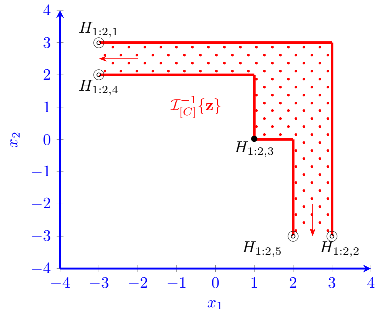


Fig. 2. $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ and $\mathtt{span}(H)$ of Example 2.

### C. Merging the direct and inverse images

Let $\chi$ be the set that merges information obtained from Eq.(14) and Eq.(19), *i.e.* a set that considers the *a priori* information computed thanks to the dynamic equation over $\mathbf{x}(0)$ and the *a posteriori* information obtained thanks to the observation equation over $\mathbf{x}$, as follows:

$$\begin{aligned}\chi &= \mathcal{I}_{[A_0],[A_1]}\{\mathbf{x}(0)\} \cap \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}, \\ &= \{\mathbf{x} \succeq \underline{\mathbf{X}}\} \cap \{\mathbf{x} \preceq \overline{\mathbf{X}}\} \cap L \cap \left\{\mathbf{x} \preceq \overline{\mathbf{X}}^U\right\}, \\ &= L \cap \{\mathbf{x} \succeq \underline{\mathbf{X}}\} \cap \left\{\mathbf{x} \preceq \min\{\overline{\mathbf{X}}, \overline{\mathbf{X}}^U\}\right\}.\end{aligned} \tag{26}$$

*Proposition 3:* Let $\mathscr{S}$ be a max-plus polyhedron. The following properties are equivalent:

i $\mathtt{span}(S) \subseteq \mathscr{S}$, where $\mathtt{span}(S)$ is given by the solution set of the following two-sided max-plus equation:

$$\begin{pmatrix} I_n & \mathcal{E}_{n\times 1} \\ \mathcal{E}_{n\times n} & \overline{\mathbf{X}}' \\ \overline{C} & \mathbf{z} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix} = \begin{pmatrix} I_n & \underline{\mathbf{X}} \\ I_n & \overline{\mathbf{X}}' \\ \underline{C} & \mathcal{E}_{q\times 1} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix}, \tag{27}$$

(representing $2n+q$ single equations) with the necessary calculations of $\underline{\mathbf{X}}$ and $\overline{\mathbf{X}}' = \min\{\overline{\mathbf{X}}, \overline{\mathbf{X}}^U\}$ in a previous step,

ii $\mathscr{S}$ is the smallest max-plus polyhedron such that $\chi \subseteq \mathscr{S}$.

*Proof:* The proof is straightforward, mixing Propositions 1 and 2 and does not require further details. ■

Eq.(27) is proper to be solved using any algorithm of Subsection II-E.

*Example 3:* Consider the uMPL system given below:

$$\mathbf{x} = A\mathbf{x}(0), \; A \in \begin{pmatrix} [1,3] & [e,4] \\ [2,4] & 2.5 \end{pmatrix}, \tag{28a}$$

$$\mathbf{z} = C\mathbf{x}, \; C \in \begin{pmatrix} 0.5 & [e,1] \end{pmatrix}, \tag{28b}$$

with $\mathbf{x}(0) = (e,e)^{\mathrm{t}}$, $\mathbf{x}' = (3.3, 3.3)^{\mathrm{t}}$, $\mathbf{z} = (0.5\ 0.5)\mathbf{x}' = 3.8$. From [10], the solution of $\chi = \mathcal{I}_{A\in[A]}\{\mathbf{x}(0)\} \cap \mathcal{I}^{-1}_{C\in[C]}\{\mathbf{z}\}$ is given by the union of pairwise disjoint hyperrectangles (interval vectors with strictness sign) as shown in the following:

$$\begin{aligned}\chi &= \mathtt{hyper}_1 \cup \mathtt{hyper}_2, \\ &= \{x_1 = 3.3,\ 2.5 \preceq x_2 \preceq 3.8\} \\ &\cup \{1 \preceq x_1 < 3.3,\ 2.8 \preceq x_2 \preceq 3.8\}.\end{aligned} \tag{29}$$

Alternatively, solving $\chi$ can be done by considering Eq.(27) as follows:

$$\begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & 3.3 \\ \varepsilon & \varepsilon & 3.8 \\ 0.5 & 1 & 3.8 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix} = \begin{pmatrix} e & \varepsilon & 1 \\ \varepsilon & e & 2.5 \\ e & \varepsilon & 3.3 \\ \varepsilon & e & 3.8 \\ 0.5 & 1 & \varepsilon \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix}, \tag{30}$$

such that its solution is given by $\mathtt{span}(S)$ where

$$S = \begin{pmatrix} 1 & 3.3 & 1 \\ 3.8 & 2.5 & 2.8 \\ e & e & e \end{pmatrix}. \tag{31}$$

Clearly, checking if the unknown target $\mathbf{x}'$ belongs to $\chi$ is easily done either using Eq.(11) with $S$ and $(\mathbf{x}'^{\mathrm{t}}, e)^{\mathrm{t}}$, *i.e.* if $(\mathbf{x}'^{\mathrm{t}}, e)^{\mathrm{t}} \in \mathtt{span}(S)$, or if $\mathbf{x} \in \mathtt{hyper}_i$ for any $i \in \{1,2\}$. The results using both approaches are shown in Figure 3.

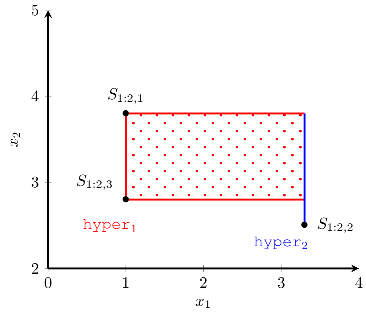


Fig. 3. $\chi$ as a union of 2 hyperrectangles and as the span of 3 generators of Example 3.

## IV. Numerical Simulations

In this section we aim to compare the time and memory consumption of the herein proposed method to compute the set $\chi$ given in Eq.(26) and the one proposed in [10] with overall complexity that amounts to $\mathcal{O}(qn^{q+1})$ in the worst-case scenario and with data-storage of at most $n^q$ hyperrectangles.

For the sake of simplicity and without loss of generality, we therefore drop the matrix $A_0$ in Eq.(14) which makes the direct image to be only related to $[\mathbf{y}]$.

For simulation purposes, we define each bound of each entry of the interval matrix $[A_1]$ to be respectively: a pseudorandom value $\underline{a}_1^{ij}$ drawn from the standard uniform distribution on the range between 0 and 10 and $\overline{a}_1^{ij} = \underline{a}_1^{ij} + \Delta^{ij}$ with $\Delta^{ij}$ drawn from the same range. The definition of the bounds of $[C]$ follows the same procedure with the same ranges. Let the vector $\mathbf{x}(0)$ be equal to $\mathbf{e}$, the vector $\mathbf{x}$ is obtained from the $\otimes$-multiplication between a random matrix $A_1 \in [A_1]$ and the vector $\mathbf{x}(0)$ and the measurement $\mathbf{z}$ is obtained from the $\otimes$-multiplication between a random matrix $C \in [C]$ and the vector $\mathbf{x}$.

The Table I shows the results for different values of $n$ and $q$. The simulation scenario is defined with the mean values of time (execution time[7] $T(s)$ in the solution of Eq.(27) using [12] and $T'(s)$ using the codes available in [10]) and memory (number of generators $N_{\mathtt{gen}}$ and number of hyperrectangles $N_{\mathtt{hyper}}$[8]) consumption of $10^2$ simulations for each pair $(n,q)$.

[7]The simulations were done running `ScicosLab-4.4.2` on a Dell Precision 5530 - 2.6 GHz Intel(R) Core(TM) i7 processor.

[8]Each hyperrectangle is represented by two matrices of dimension $(n\times 2)$ each, one for the upper and lower bounds and the other for the strictness sign of these bounds. Hence, each hyperrectangle needs 4 units of storage in $n$.

| $n$ | $q$ | $T(s)$ | $T'(s)$ | $N_{\text{gen}}$ | $4N_{\text{hyper}}$ |
|---|---|---|---|---|---|
| 2 | 1 | 0.0035 | 0.0170 | 3.3737 | 5.8180 |
| 3 | 2 | 0.0036 | 0.044 | 5.9393 | 12.4444 |
| 4 | 3 | 0.0053 | 0.1887 | 11.6969 | 31.4340 |
| 5 | 4 | 0.0171 | 1.2358 | 19.9393 | 67.4744 |
| 6 | 5 | 0.2064 | 5.7839 | 36.6969 | 179.0300 |
| 7 | 5 | 1.3217 | 12.3333 | 58.1616 | 268.3232 |

TABLE I

COMPARISON OF TIME AND MEMORY CONSUMPTION.

## V. CONCLUSIONS

This paper presents a faster way, if compared to [10], to compute the intersection of the forward and backward reach sets for general uMPL systems that is defined as the solution of a two-sided max-plus equation taking into account the *a priori* and *a posteriori* information that are available. In future work, we aim to develop an analytic solution of Eq.(27) using a similar procedure to that developed in Eq.(18) in order to avoid the use of a computational procedure. Stochastic filter design can take advantage of this characterization of the dynamic evolution in a recursive way using either the approach presented in [10, Section IV] or the tools developed in [22] with initial interval vector defined as the smallest envelope that enclosures $\chi$.

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