# Performance of a third-order Volterra MVDR beamformer in the presence of non-Gaussian and/or non-circular interference

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# PERFORMANCE OF A THIRD-ORDER VOLTERRA MVDR BEAMFORMER IN THE PRESENCE OF NON-GAUSSIAN AND/OR NON-CIRCULAR INTERFERENCE

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**ABSTRACT**

Linear beamformers are optimal, in a mean square (MS) sense, when the signal of interest (SOI) and observations are jointly Gaussian and circular. When the SOI and observations are zero-mean, jointly Gaussian and non-circular, optimal beamformers become widely linear (WL). They become non-linear with a structure depending on the unknown joint probability distribution of the SOI and observations when the latter are jointly non-Gaussian, assumption which is very common in radiocommunications. In this context, a third-order Volterra minimum variance distortionless response (MVDR) beamformer has been introduced recently for the reception of a SOI, whose waveform is unknown, but whose steering vector is known, corrupted by non-Gaussian and potentially non-circular interference, omnipresent in practical situations. However its statistical performance has not yet been analyzed. The aim of this paper is twofold. We first introduce an equivalent generalized sidelobe canceller (GSC) structure of this beamformer and then, we present an analytical performance analysis of the latter in the presence of one interference. This allows us to quantify the improvement of the performance with respect to the linear and WL MVDR beamformers.

***Index Terms*—** Beamformer, non-circular, non-Gaussian, higher order, interferences, MVDR, Volterra, widely non linear, widely linear, third order, fourth order, sixth order.

## 1. INTRODUCTION

The conventional time invariant linear beamformers, such as the Capon beamformer [1], are only optimal for stationary Gaussian observations whose complex envelope is necessarily second-order (SO) circular. However in many applications such as in radiocommunications, signals are non-Gaussian and non-circular either at the SO and/or at a higher order (HO) [2]. For this reason, several non-linear beamformers have been proposed in the literature. A widely linear [3] MVDR beamformer has been introduced in [4] to improve the performance of the Capon beamformer in SO noncircular contexts. A third-order Volterra GSC [5] structure has been proposed in [6] to improve the performance of the Capon beamformer in non-Gaussian contexts, without taking into account the potential non-circularity of interference. Then a family of third-order Volterra MVDR beamformers [7] which takes into account both the potential non-Gaussian character and the potential SO, fourth-order (FO) and sixth-order (SIO) non-circularity of interference has been presented. But these beamformers have been implemented by an intricate GSC structure and their performance have been presented only by numerical illustrations.

In this context, the first purpose of this paper is to introduce an alternative GSC structure of these beamformers allowing much simpler implementations. The second purpose is to present an analytical performance analysis of these beamformers in the presence of interference. It has been shown in particular that, depending on the SO, FO and SIO interference statistics, some of these Volterra MVDR beamformers outperform the Capon beamformer for a circular non-Gaussian interference, while some others, outperform the WL MVDR beamformer for a non-circular non-Gaussian interference.

## 2. HYPOTHESES AND PROBLEM FORMULATION

### 2.1. Hypotheses

We consider an array of $N$ narrowband sensors and we denote by $\mathbf{x}(t)$ the vector of the complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of an SOI corrupted by interference and a background noise. Under these assumptions, the observation vector $\mathbf{x}(t)$ can be written as follows

$$\mathbf{x}(t) = s(t)\mathbf{s} + \mathbf{n}(t) \in \mathbb{C}^N, \quad (1)$$

where $s(t)$ and $\mathbf{s}$ correspond to the complex envelope, assumed zero-mean, and the steering vector, assumed perfectly known, of the SOI respectively. The vector $\mathbf{n}(t)$ is the total noise vector, containing the background noise and the interference, and assumed to be zero-mean, potentially non-Gaussian and non-circular, and independent of $s(t)$.

## 2.2. Problem formulation

The problem addressed in this paper is to estimate the unknown signal $s(t)$ from the observation $\mathbf{x}(t)$. It is well-known [3] that the optimal estimate $\widehat{s}(t)$ of $s(t)$, in a MS sense, from the observation $\mathbf{x}(t)$, is the conditional expectation $\mathrm{E}[s(t)|\mathbf{x}(t)]$. Note, that for respectively circular or non-circular mutually Gaussian distributions of $(s(t), \mathbf{x}(t))$, this conditional expectation becomes linear or widely linear [3]. But for non-Gaussian distribution of $(s(t), \mathbf{x}(t))$, the derivation of this conditional expectation needs this distribution which is unknown in practice.

To approximate this conditional expectation, a particular class of non-linear beamformers, the complex Volterra beamformers, has been introduced for the first time in [8,9]. Then a particular third-order Volterra beamformer, whose output is defined by (2) has been proposed and briefly analyzed in [7].

$$\begin{aligned} y(t) \quad &= \quad \mathbf{w}_{1,0}^H\mathbf{x}(t) + \mathbf{w}_{1,1}^H\mathbf{x}^*(t) \\ &+\mathbf{w}_{3,0}^H[\mathbf{x}(t)\otimes\mathbf{x}(t)\otimes\mathbf{x}(t)]+\mathbf{w}_{3,1}^H[\mathbf{x}(t)\otimes\mathbf{x}(t)\otimes\mathbf{x}^*(t)] \\ &+\mathbf{w}_{3,2}^H[\mathbf{x}(t)\otimes\mathbf{x}^*(t)\otimes\mathbf{x}^*(t)]+\mathbf{w}_{3,3}^H[\mathbf{x}^*(t)\otimes\mathbf{x}^*(t)\otimes\mathbf{x}^*(t)] \\ &\stackrel{\text{def}}{=} \quad \widetilde{\mathbf{w}}^H\widetilde{\mathbf{x}}(t), \end{aligned} \tag{2}$$

where $\otimes$ denotes the Kronecker product. In (2), the third-order terms $\mathbf{x}_{3,q}(t) \stackrel{\text{def}}{=} [\mathbf{x}(t)^{\otimes(3-q)} \otimes \mathbf{x}^*(t)^{\otimes q}]$, $q = 0,1,2,3$ are called cubic (C) terms and $q$ are their indexes. The beamformers containing the linear term and the cubic terms $q_1, q_2, .., q_r$ are called L-C$(q_1, q_2, .., q_r)$ and those containing the WL terms and the cubic terms $q_1, q_2, .., q_r$ are called WL-C$(q_1, q_2, .., q_r)$. Note that only the L-C(1) beamformer have been considered in [6].

# 3. THIRD-ORDER VOLTERRA MVDR BEAMFORMER

To impose no distortion on $s(t)$ at the output $y(t)$, the spatial filters of the first order terms of (2) must verify the constraints:

$$\mathbf{w}_{1,0}^H\mathbf{s} = 1 \ \text{ and } \ \mathbf{w}_{1,1}^H\mathbf{s}^* = 0.$$

To obtain the constraints on the cubic terms, we must decompose the random variable $\mathbf{n}(t)$ on a fixed orthogonal basis $(\mathbf{s}, \mathbf{u}_1, ...\mathbf{u}_{N-1})$ of $\mathbb{C}^N$: $\mathbf{n}(t) = n_0(t)\mathbf{s} + \sum_{i=1}^{N-1} n_i(t)\mathbf{u}_i$. To cancel the SOI contributions in the cubic term $\mathbf{w}_{3,q}^H\mathbf{x}_{3,q}(t)$, it is equivalent to cancel all the terms of $\mathbf{w}_{3,q}^H\mathbf{x}_{3,q}(t)$ excluding the terms containing the $\mathbf{u}_i$'s only. For example, for $q = 1$, we must impose the $1+3(N-1)+3(N-1)^2 = N^3-(N-1)^3$ constraints:

$$\begin{aligned} \mathbf{w}_{3,1}^H(\mathbf{s}\otimes\mathbf{s}\otimes\mathbf{s}^*) \quad &= \quad 0 \\ \mathbf{w}_{3,1}^H(\mathbf{u}_i\otimes\mathbf{s}\otimes\mathbf{s}^*) \quad &= \quad 0, \quad \mathbf{w}_{3,1}^H(\mathbf{s}\otimes\mathbf{u}_i\otimes\mathbf{s}^*) = 0, \\ \mathbf{w}_{3,1}^H(\mathbf{s}\otimes\mathbf{s}\otimes\mathbf{u}_i^*) \quad &= \quad 0, \qquad\quad 1 \le i \le N-1 \\ \mathbf{w}_{3,1}^H(\mathbf{u}_i\otimes\mathbf{u}_j\otimes\mathbf{s}^*) \quad &= \quad 0, \quad \mathbf{w}_{3,1}^H(\mathbf{u}_i\otimes\mathbf{s}\otimes\mathbf{u}_j^*) = 0, \\ \mathbf{w}_{3,1}^H(\mathbf{s}\otimes\mathbf{u}_i\otimes\mathbf{u}_j^*) \quad &= \quad 0, \qquad\quad 1 \le i,j \le N-1, \end{aligned} \tag{3}$$

or equivalently $\mathbf{C}_1^H\mathbf{w}_{3,1} = \mathbf{0}_{N^3-(N-1)^3}$. For any other index $q = 0,2,3$, the constraints can be written as $\mathbf{C}_q^H\mathbf{w}_{3,q} = \mathbf{0}_{N^3-(N-1)^3}$ and the global set of constraints takes the form:

$$\mathbf{C}^H\widetilde{\mathbf{w}} = \mathbf{f}, \tag{4}$$

where $\mathbf{C} = \mathrm{Diag}(\mathbf{s}, \mathbf{s}^*, \mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3)$ and $\mathbf{f}$ is the vector $(1, \mathbf{0}_{1+4[N^3-(N-1)^3]}^T)^T$.

Thus the best SO estimate $s(t)$ exploiting the noise statistics only, corresponds to the output of the third-order Volterra beamformer $\widetilde{\mathbf{w}}_{\mathrm{MVDR}}$ which minimizes the time-averaged output power, $\widetilde{\mathbf{w}}^H\mathbf{R}_{\widetilde{x}}\widetilde{\mathbf{w}}$ under the previous constraint.

$$\widetilde{\mathbf{w}}_{\mathrm{MVDR}} = \arg\{\min_{\mathbf{C}^H\widetilde{\mathbf{w}}=\mathbf{f}} \widetilde{\mathbf{w}}^H\mathbf{R}_{\widetilde{x}}\widetilde{\mathbf{w}}\}, \tag{5}$$

where $\mathbf{R}_{\widetilde{x}} \stackrel{\text{def}}{=} < \mathrm{E}[\widetilde{\mathbf{x}}(t)\widetilde{\mathbf{x}}^H(t)] >$ is the time-averaged correlation matrix of $\widetilde{\mathbf{x}}(t)$. As the extended observation $\widetilde{\mathbf{x}}(t)$ has redundant components for $N > 1$, $\mathbf{R}_{\widetilde{x}}$ is singular and consequently the solutions of the constrained optimization problem (5) are difficult to derive (see e.g., [10, sec.19.3c]). To solve this difficulty, an intricate equivalent GSC structure, which transforms a constrained least MS problem to an unconstrained least MS one, has been proposed in [7].

# 4. EQUIVALENT THIRD-ORDER VOLTERRA GSC STRUCTURE

We present here a simpler equivalent GSC structure. It is based on the existence of a full column rank matrix $\mathbf{B}$:

$$\mathbf{B} \stackrel{\text{def}}{=} \mathrm{Diag}(\mathbf{B}_{1,0}, \mathbf{B}_{1,0}^*, \mathbf{B}_{3,0}, \mathbf{B}_{3,1}, \mathbf{B}_{3,2}, \mathbf{B}_{3,3}), \tag{6}$$

with $\mathbf{B}_{1,0} \stackrel{\text{def}}{=} [\mathbf{u}_1, ..., \mathbf{u}_{N-1}]$ and $\mathbf{B}_{3,q} = [\mathbf{B}_{1,0}^{\otimes(3-q)} \otimes \mathbf{B}_{1,0}^{*\otimes q}]$, $q = 0, .., 3$, whose columns span $\mathrm{span}(\mathbf{C})^\perp$, which implies $\mathbf{B}^H\mathbf{C} = \mathbf{O}_{[2(N-1)+4(N-1)^3]\times[2+4(N^3-(N-1)^3)]}$. Consequently, any filter $\widetilde{\mathbf{w}}$ that may be decomposed into two components: $\widetilde{\mathbf{w}} = \widetilde{\mathbf{w}}_f - \widetilde{\mathbf{v}}$ where $\widetilde{\mathbf{w}}_f \stackrel{\text{def}}{=} [\mathbf{w}_f^T, \mathbf{0}_{N+4N^3}^T]^T$, such that $\mathbf{w}_f$ is an $N \times 1$ filter satisfying $\mathbf{w}_f^H\mathbf{s} = 1$, satisfies the constraint (4) if and only if $\widetilde{\mathbf{v}} = \widetilde{\mathbf{w}}_f - \mathbf{B}\widetilde{\mathbf{w}}_a$, where $\widetilde{\mathbf{w}}_a$ is an unconstrained $[2(N-1) + 4(N-1)^3] \times 1$ vector. Consequently, the constrained minimization problem (5) is equivalent to the unconstrained one:

$$\min_{\widetilde{\mathbf{w}}_a} < \mathrm{E}|(\widetilde{\mathbf{w}}_f - \mathbf{B}\widetilde{\mathbf{w}}_a)^H\widetilde{\mathbf{x}}(t)|^2 > . \tag{7}$$

Using $\widetilde{\mathbf{z}}(t) \stackrel{\text{def}}{=} [\mathbf{z}^T(t), \mathbf{z}^H(t), [\mathbf{z}(t)\otimes\mathbf{z}(t)\otimes\mathbf{z}(t)]^T, [\mathbf{z}(t)\otimes\mathbf{z}(t)\otimes\mathbf{z}^*(t)]^T, [\mathbf{z}(t)\otimes\mathbf{z}^*(t)\otimes\mathbf{z}^*(t)]^T, [\mathbf{z}^*(t)\otimes\mathbf{z}^*(t)\otimes\mathbf{z}^*(t)]^T$, where $\mathbf{z}(t) \stackrel{\text{def}}{=} \mathbf{B}_{1,0}^H\mathbf{x}(t)$, it is straightforward to prove that (7) is equivalent to the unconstrained minimization:

$$\min_{\widetilde{\mathbf{w}}_a} < \mathrm{E}|\mathbf{w}_f^H\mathbf{x}(t) - \widetilde{\mathbf{w}}_a^H\widetilde{\mathbf{z}}(t)|^2 > . \tag{8}$$

To solve easily this minimization (8), we have to withdraw the redundancies in the 4 terms $\mathbf{z}_{3,q}(t) \stackrel{\text{def}}{=} [\mathbf{z}(t)^{\otimes(3-q)} \otimes$

$\mathbf{z}^*(t)^{\otimes q}]$ of $\widetilde{\mathbf{z}}(t)$. This can be obtained by using selection matrices $\mathbf{K}_q$ such that $\mathbf{z}'_{3,q}(t) = \mathbf{K}_q\mathbf{z}_{3,q}(t)$. Consequently $\widetilde{\mathbf{z}}(t)$ can be replaced by $\widetilde{\mathbf{z}}'(t) = \mathbf{K}\widetilde{\mathbf{z}}(t)$ with $\mathbf{K} = \mathrm{Diag}(\mathbf{I}_{N-1}, \mathbf{I}_{N-1}, \mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3)$ and the minimization (8) is equivalent to the minimization:

$$\min_{\widetilde{\mathbf{w}}'_a} < \mathrm{E}|(\mathbf{w}_f^H\mathbf{x}(t) - \widetilde{\mathbf{w}}_a^{\prime H}\tilde{\mathbf{z}}^{\prime}(t)|^2 >, \tag{9}$$

whose solution is

$$\begin{aligned}\widetilde{\mathbf{w}}^{\prime}_{a,\mathrm{opt}} &\stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{z}'}^{-1}\mathbf{R}_{\tilde{z}',x}\mathbf{w}_f = [\mathbf{K}\mathbf{B}^H\mathbf{R}_{\tilde{x}}\mathbf{B}\mathbf{K}^H]^{-1}\mathbf{K}\mathbf{B}^H\mathbf{R}_{\tilde{x}}\widetilde{\mathbf{w}}_f \\ &= [\mathbf{K}\mathbf{B}^H\mathbf{R}_{\tilde{n}}\mathbf{B}\mathbf{K}^H]^{-1}\mathbf{K}\mathbf{B}^H\mathbf{R}_{\tilde{n}}\widetilde{\mathbf{w}}_f,\end{aligned} \tag{10}$$

where $\mathbf{R}_{\tilde{z}'} \stackrel{\mathrm{def}}{=} < \mathrm{E}[\tilde{\mathbf{z}}^{\prime}(t)\tilde{\mathbf{z}}^{\prime H}(t)] >$, $\mathbf{R}_{\tilde{z}',x} \stackrel{\mathrm{def}}{=} < \mathrm{E}[\tilde{\mathbf{z}}^{\prime}(t)\mathbf{x}^H(t)] >$. The output $y(t)$ of the GSC structure is then given by:

$$\begin{aligned} y(t) &= \mathbf{w}_f^H\mathbf{x}(t) - \widetilde{\mathbf{w}}^{\prime H}_{a,\mathrm{opt}}\tilde{\mathbf{z}}^{\prime}(t) \\ &= s(t) + \mathbf{w}_f^H\mathbf{n}(t) - \widetilde{\mathbf{w}}^{\prime H}_{a,\mathrm{opt}}\mathbf{K}\mathbf{B}^H\widetilde{\mathbf{x}}(t) \\ &= s(t) + (\widetilde{\mathbf{w}}_f - \mathbf{B}\mathbf{K}^H\widetilde{\mathbf{w}}^{\prime}_{a,\mathrm{opt}})^H\widetilde{\mathbf{n}}(t). \end{aligned} \tag{11}$$

$\mathbf{w}_f^H$

$$G_{\mathrm{LC}(0)} = 1+\frac{\alpha^2\beta^6\epsilon_j^4|\kappa_{j,nc,2}-3\gamma_j|^2}{(1+\epsilon_j)A_0-\alpha^2\beta^6\epsilon_j^4|\kappa_{j,nc,2}-3\gamma_j|^2} \quad (21)$$

$$G_{\mathrm{LC}(1)} = 1+\frac{\alpha^2\beta^6\epsilon_j^4(\kappa_{j,c}-2)^2}{(1+\epsilon_j)A_1-\alpha^2\beta^6\epsilon_j^4(\kappa_{j,c}-2)^2}, \quad (22)$$

$$G_{\mathrm{LC}(2)} = 1+\frac{\alpha^2\beta^6\epsilon_j^4|(\kappa_{j,nc,2}-\gamma_j)+2\gamma_j/(\beta^2\epsilon_j)|^2}{(1+\epsilon_j)A_2-\alpha^2\beta^6\epsilon_j^4|(\kappa_{j,nc,2}-\gamma_j)+2\gamma_j/(\beta^2\epsilon_j)|^2} \quad (23)$$

$$G_{\mathrm{LC}(3)} = 1+\frac{\alpha^2\beta^6\epsilon_j^4|\kappa_{j,nc,1}^2|}{(1+\epsilon_j)A_3-\alpha^2\beta^6\epsilon_j^4|\kappa_{j,nc,1}^2|}. \quad (24)$$

Expressions (14) to (17) and (21) to (24) of the SINR and gain $G$ of the considered L-C($q$) MVDR beamformers depend on the statistical properties of the interference and more precisely on the coefficients $\gamma_j$, $\kappa_{j,c}$, $\kappa_{j,nc,1}$, $\kappa_{j,nc,2}$ and $\chi_{j,c}$. Consequently many insights can be deduced from these expressions. In the following, we concentrate on the behavior of these beamformers for a strong interference (i.e., $\epsilon_j \gg 1$):

For $\chi_{j,c}=\kappa_{j,c}^2$ (which means that $|j(t)|$ takes at most two values corresponding to zero and a non-zero constant value) and $\beta\neq 0$, (15) and (22) become:

$$\mathrm{SINR}_{\mathrm{LC}(1)} \approx \epsilon_s\left(1-\frac{\alpha^2(5\kappa_{j,c}-4)}{\kappa_{j,c}(\kappa_{j,c}+1)-\alpha^2(\kappa_{j,c}-2)^2}\right) \quad (25)$$

$$G_{\mathrm{LC}(1)} \approx 1+\frac{\alpha^2(\kappa_{j,c}-2)^2}{\kappa_{j,c}(\kappa_{j,c}+1)-\alpha^2(\kappa_{j,c}-2)^2}, \quad (26)$$

In particular for CPM, FSK and non-filtered PSK interference: $\mathrm{SINR}_{\mathrm{LC}(1)} \approx \epsilon_s[1-\alpha^2/(2-\alpha^2)]$ and $G_{\mathrm{LC}(1)} \approx 1+\alpha^2/(2-\alpha^2)$. For impulsive interference, such that $|j(t)|$ is Bernoulli distributed and noting $p \overset{\mathrm{def}}{=} P(|j(t)|\neq 0)$, we can verify that $\chi_{j,c}=\kappa_{j,c}^2=1/p^2$ and we obtain as $p$ decreases to zero, i.e., for very impulsive interference, $\mathrm{SINR}_{\mathrm{LC}(1)} \approx \epsilon_s$ and $G_{\mathrm{LC}(1)} \approx 1+\alpha^2/(1-\alpha^2)$. In this case, the SINR is the one obtained in the absence of interference.

For $\chi_{j,c}=|\kappa_{j,nc,2}|^2$ (which means that $j^2(t)$ takes at most two values corresponding to zero and a non-zero constant value, and consequently that $j(t)$ is necessarily rectilinear) and $\beta\neq 0$, (14), (16), (21) and (23) become:

$$\mathrm{SINR}_{\mathrm{LC}(0)} \approx \epsilon_s \left(1-\frac{9\alpha^2(\kappa_{j,c}-|\gamma_j^2|)}{|\kappa_{j,nc,2}|^2+9\kappa_{j,c}-6\mathrm{Re}(\gamma_j\kappa_{j,nc,2}^*)-\alpha^2|\kappa_{j,nc,2}-3\gamma_j|^2}\right) \quad (27)$$

$$\mathrm{SINR}_{\mathrm{LC}(2)} \approx \epsilon_s \left(1-\frac{\alpha^2(9\kappa_{j,c}-|\gamma_j^2|-4\mathrm{Re}(\gamma_j\kappa_{j,nc,2}^*))}{|\kappa_{j,nc,2}|^2+9\kappa_{j,c}-6\mathrm{Re}(\gamma_j\kappa_{j,nc,2}^*)-\alpha^2|\kappa_{j,nc,2}-\gamma_j|^2}\right) \quad (28)$$

$$G_{\mathrm{LC}(0)} \approx 1+\frac{\alpha^2|\kappa_{j,nc,2}-3\gamma_j|^2}{|\kappa_{j,nc,2}|^2+9\kappa_{j,c}-6\mathrm{Re}(\gamma_j\kappa_{j,nc,2}^*)-\alpha^2|\kappa_{j,nc,2}-3\gamma_j|^2} \quad (29)$$

$$G_{\mathrm{LC}(2)} \approx 1+\frac{\alpha^2|\kappa_{j,nc,2}-\gamma_j|^2}{|\kappa_{j,nc,2}|^2+9\kappa_{j,c}-6\mathrm{Re}(\gamma_j\kappa_{j,nc,2}^*)-\alpha^2|\kappa_{j,nc,2}-\gamma_j|^2} \quad (30)$$

In particular for a non-filtered BPSK interference:

$$\mathrm{SINR}_{\mathrm{LC}(0)} \approx \epsilon_s, \qquad G_{\mathrm{LC}(0)} \approx 1+\frac{\alpha^2}{1-\alpha^2} \quad (31)$$

$$\mathrm{SINR}_{\mathrm{LC}(2)} \approx \epsilon_s(1-\alpha^2), \quad G_{\mathrm{LC}(2)} \approx 1. \quad (32)$$

In this case, (32) shows that the L-C(2) MVDR beamformer does not improve the Capon beamformer, whereas (31) shows that the L-C(0) MVDR beamformer outperforms the Capon beamformer by completely canceling the interference whatever $\alpha$ .

For $\chi_{j,c}=|\kappa_{j,nc,1}|^2$ (which means that $j^4(t)$ takes at most two values corresponding to zero and a non-zero constant value) and $\beta\neq 0$, (17) and (24) become:

$$\mathrm{SINR}_{\mathrm{LC}(3)} \approx \epsilon_s\left(1-\frac{9\alpha^2\kappa_{j,c}}{9\kappa_{j,c}+(1-\alpha^2)|\kappa_{j,nc,1}|^2}\right) \quad (33)$$

$$G_{\mathrm{LC}(3)} \approx 1+\frac{\alpha^2|\kappa_{j,nc,1}|^2}{9\kappa_{j,c}+(1-\alpha^2)|\kappa_{j,nc,1}|^2}. \quad (34)$$

In particular, for non-filtered BPSK and non-filtered QPSK interference, we obtain:

$$\mathrm{SINR}_{\mathrm{LC}(3)} \approx \epsilon_s\left(1-\frac{9\alpha^2}{10-\alpha^2}\right) \quad (35)$$

$$G_{\mathrm{LC}(3)} \approx 1+\frac{\alpha^2}{10-\alpha^2}, \quad (36)$$

which proves that the L-C(3) MVDR beamformer improves only slightly the Capon beamformer and remains less powerful than the WL beamformer for a rectilinear interference.

### 5.3. Performance of WL-C($q$) and L-C($q_1,q_2$) MVDR beamformers

Closed-form expressions of $G_{\mathrm{WL–C}(q_1)}/G_{\mathrm{WL}}$ and $G_{\mathrm{L–C}(q_1,q_2)}$ are too intricate to derive. However, using symbolic math toolboxes and the results of Section 5.2, it is possible to prove in particular, that for a strong BPSK interference:

$$G_{\mathrm{WL-C}(0)}\approx G_{\mathrm{WL-C}(1)}\approx G_{\mathrm{WL-C}(3)}\approx G_{\mathrm{L-C}(0,1)}\approx G_{\mathrm{L-C}(0)} \approx 1+\frac{\alpha^2}{1-\alpha^2} > G_{\mathrm{WL}} \approx 1+\frac{\alpha^2}{2-\alpha^2} > G_{\mathrm{L-C}(3)}\approx 1, \quad (37)$$

whereas for a strong QPSK interference:

$$G_{\mathrm{L-C}(1,3)} \approx 1+\frac{\alpha^2}{1-\alpha^2} > G_{\mathrm{L-C}(1)} \approx 1+\frac{\alpha^2}{2-\alpha^2} > G_{\mathrm{L-C}(3)} > G_{\mathrm{WL}} = G_{\mathrm{L-C}(0)} = G_{\mathrm{L-C}(2)} = 1. \quad (38)$$

This result shows in particular that in this latter case, $\mathrm{SINR}_{\mathrm{L-C}(1,3)} \approx \epsilon_s$, which proves the quasi-optimality (among the beamformers which use the total noise statistics only) of the L-C(1,3) MVDR beamformer for a strong QPSK interference. Finally, let us note that in all cases, the WL-C(0,1,2,3) MVDR beamformer reaches at least the performance of the best WL-C($q_1$) and L-C($q_1,q_2$) MVDR beamformer and is thus quasi-optimal not only for strong BPSK and QPSK interference but also for very impulsive interference, circular or not.

### 5.4. Performance illustrations

We consider throughout this section a two-element array with omnidirectional sensors and we assume that the SOI has a signal to noise ratio (SNR) $\pi_s/\eta_2$, equal to 10dB. This SOI is assumed to be corrupted by a single interference whose interference to noise ratio (INR) $\pi_j/\eta_2$, is equal to 30dB. Under these assumptions, Fig.2 displays, for a non-filtered BPSK interference, the variations of $\mathrm{SINR_B}$ at the output of different MVDR beamformers. Note that this figure confirms the results (37), i.e., the equivalent performance of the L-C(0) and WL-C(0) MVDR beamformers and the better performance of the L-C(0) MVDR beamformer with respect to the WL MVDR beamformer, itself better than the L-C(3) MVDR beamformer, the latter being equivalent to the Capon beamformer. Moreover, Fig.2 shows the very weak information brought by the WL-C(0,1), WL-C(0,1,3) and WL-C(0,1,2,3) MVDR beamformers with respect to the L-C(0) or L-C(1,3) MVDR beamformers which are quasi-optimal.

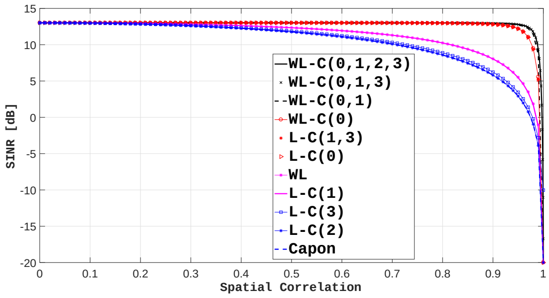


Fig.2 $\mathrm{SINR_B}$ as a function of $\alpha$, non-filtered BPSK interference.

Fig.3 displays the same variations as Fig.2, but for a QPSK interference. Again, this figure confirms the results (38), i.e., the better performance of the L-C(1,3) MVDR beamformer with respect to the L-C(1) MVDR beamformer, itself better than the L-C(3) MVDR beamformer, itself better than the Capon beamformer.

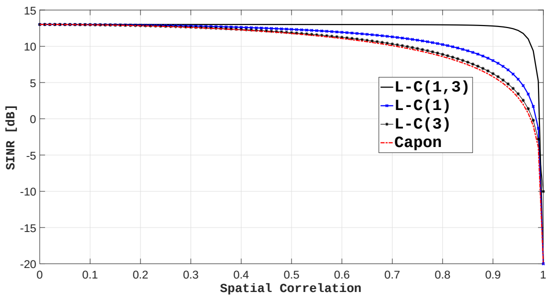


Fig.3 $\mathrm{SINR_B}$ as a function of $\alpha$, non-filtered QPSK interference.

Fig.4 displays the variations of $G_\mathrm{B}$ at the output of the different MVDR beamformers as a function of $\alpha$, for an impulsive circular interference, such that $|j(t)|$ follows a Bernoulli distribution with $P(|j(t)| \neq 0) = 0.001$ associated with $\kappa_{j,c} = 1000$. This figure confirms that in this case, $G_{\mathrm{LC}(1)} \approx 1 + \alpha^2/(1-\alpha^2)$ is quasi-optimal for very high value of $\kappa_{j,c}$. As a consequence, the beamformers L-C(1,3) and L-C(0,1,2,3) bring no further gains with respect to L-C(1).

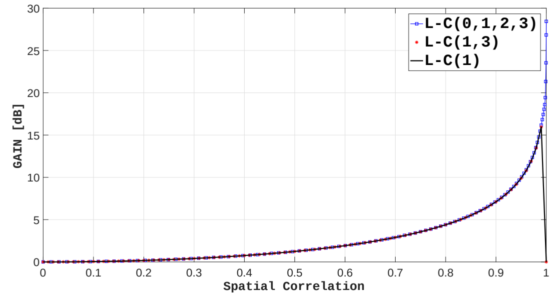


Fig.4 $G_\mathrm{B}$ as a function of $\alpha$, circular Bernoulli impulsive interference.

## 6. CONCLUSION

An alternate equivalent GSC structure of a previously introduced third-order Volterra MVDR beamformer has been proposed. It permits both simpler implementations of this beamformer and analytical SINR computations at its output, which has been done in the paper. This analytical performance analysis allows us to specify how this beamformer outperforms the Capon and the WL beamformers for respectively, circular or not circular non-Gaussian interference, depending on the SO, FO and SIO interference statistics.

## 7. REFERENCES


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