Exploring Quantum Supremacy in Access Structures of Secret Sharing by Coding Theory

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Abstract—We consider secret sharing schemes with a classical secret and quantum shares. One example of such schemes was recently reported whose access structure cannot be realized by any secret sharing schemes with classical shares. In this paper, we report further quantum secret sharing schemes whose access structures cannot be realized by any classical secret sharing schemes.

I. Introduction

Secret sharing [13] is a cryptographic scheme to encode a secret into multiple pieces of information (called shares) so that only qualified sets of shares can reconstruct the original secret. Secret sharing has become even more important as its application to the cloud storage is spreading [1]. The security criterion of secret sharing is usually information theoretic one and thus cannot be broken even by quantum computers [14].

Quantum supremacy [12] is the potential ability of quantum computing devices to solve problems that classical computers practically cannot. Discovery of new quantum supremacy is important in research of quantum information processing. Since majority of secret sharing schemes are secure against both classical and quantum computers, quantum supremacy cannot be found in that respect. On the other hand, the author recently reported new quantum supremacy in the access structure of secret sharing [8]. An access structure of a secret sharing schemes is a set of qualified share sets and forbidden share sets, where a share set is said to be forbidden (resp. qualified) if the set has no information about the secret (resp. can reconstruct the secret) [11].

Specifically, when we use the famous [[5, 1, 3]] binary quantum stabilizer error-correcting code to encode a 1-bit classical secret into five quantum shares, its access structure cannot be realized by any secret sharing schemes with classical shares.

However, it was not clarified whether or not there exists another secret sharing schemes with quantum shares whose access structures cannot be realized by classical shares. In this paper, we use different necessary conditions on the existence of access structures realized by secret sharing schemes with classical shares, and report 9 new quantum secret sharing schemes whose access structures cannot be realized by secret sharing schemes with classical shares.

II. QUANTUM ERROR-CORRECTING CODES AND SECRET SHARING

Quantum error-correcting codes have been used for constructing secret sharing schemes for quantum secrets [5], [7], [9]. Since classical information can be regarded as a special case of quantum information [10], it is easy to construct a secret sharing scheme for a classical secret from a quantum error-correcting code. Suppose that we have a k-bit string \vec{s} as a classical secret and we want to encode \vec{s} into n shares. For this goal, we select a binary [[n, k, d]] quantum error-correcting code Q, where [[n, k, d]] means that the code encodes k qubits into n qubits and has the minimum distance d. We prepare a k-qubit quantum state $|\vec{s}\rangle$ and encode $|\vec{s}\rangle$ into n qubits $|\vec{x}\rangle$ by Q. Then each qubit in the quantum codeword $|\vec{x}\rangle$ is distributed to each of n participants.

We say that a secret sharing scheme has t-privacy if any set of t shares has absolutely no information about the secret, and has r-reconstruction if any set of t shares uniquely reconstruct the secret [3]. For simplicity, t is assumed to be smallest possible and t to be largest possible. For a secret sharing scheme to be useful, we must know t and t. We will relate t and t in order to demonstrate the quantum supremacy.

III. QUANTUM SUPREMACY IN ACCESS STRUCTURES

Suppose that one has n-d+1 or more shares. Then the number of missing shares is d-1 or less. By setting the quantum state of missing shares to any state (e.g., the completely mixed state) and treating them as erasures, the quantum erasure correction procedure reconstructs the n shares $|\vec{x}\rangle$ from available shares [8], and the secret \vec{s} can be reconstructed from $|\vec{x}\rangle$. This means that $r \le n - d + 1$.

On the other hand, when we have a secret sharing scheme with a classical secret and quantum shares and a set of shares can reconstruct the secret, then the complementary set of shares has absolutely no information about the secret [11]. This implies that $t \ge d - 1$.

The difference r-t is called the threshold gap. When we construct a secret sharing scheme from a binary [[n, k, d]] quantum error-correcting codes, we have

$$r - t \le n + 2 - 2d. \tag{1}$$

On the other hand, when we have a secret sharing scheme in which each classical share has $\log_2 q$ bits and the classical secret has $k \log_2 q$ bits, we must have [2]

$$r - t \ge \frac{r + 1}{q}.\tag{2}$$

A secret sharing scheme with classical shares is said to be *linear* if the reconstruction from shares to secrets is a linear map [4]. Most of studied secret sharing schemes with classical shares are linear, as they enable efficient encoding and reconstruction by linear algebraic algorithms. When a scheme is linear, we must have [3]

$$r - t \ge \frac{q^m - 1}{q^{m+1} - 1}(n+2) + \frac{q^{m+1} - q^m}{q^{m+1} - 1}(k - 2m)$$
(for all $0 \le m \le k - 1$). (3)

We consider the case that each share is one bit or one qubit, and search for an access structure that can be realized by quantum shares but cannot be realized by classical shares. If we have a binary [[n, k, d]] quantum code and we also have

$$n + 2 - 2d < \frac{n + 2 - d}{2},\tag{4}$$

then by Eqs. (1) and (2) the binary [[n, k, d]] quantum code realizes an access structure that cannot

TABLE I

Parameters of binary [[n,k,d]] quantum error-correcting codes that exhibit quantum supremacy in the access structures of associated secret sharing schemes

n	k	d	Eq. (4)	Eq. (5)
6	1	3	true	false
11	1	5	true	false
12	1	5	true	false
17	1	7	true	false
18	1	7	true	false
27	3	9	false	true with $m = 2$
28	3	9	false	true with $m = 2$
29	1	11	true	false
30	1	11	true	false

be realized by secret sharing schemes with classical 1-bit shares, thus it exhibits quantum supremacy in the access structure.

In addition, if we have a binary [[n, k, d]] quantum code and we also have

$$n+2-d < \frac{q^m-1}{q^{m+1}-1}(n+2) + \frac{q^{m+1}-q^m}{q^{m+1}-1}(k-2m)$$
(for some $0 \le m \le k-1$), (5)

then by Eqs. (1) and (3) the binary [[n, k, d]] quantum code realizes an access structure that cannot be realized by *linear* secret sharing schemes with classical 1-bit shares, thus it also exhibits quantum supremacy in the access structure.

Grassl [6] maintains the table of best binary quantum error-correcting codes. We searched for codes with properties (4) or (5), and found the codes in Table I.

IV. Conclusion

As a continuation of the author's recent paper [8], we searched quantum error-correcting codes that give secret sharing schemes whose access structures cannot be realized by classical information processing. We reported 9 new codes having access structures impossible by classical information processing in Table I. However, it remains unknown whether or not there exist infinitely many quantum error-correcting codes having access structures impossible by classical information processing. It is a further research agenda.

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