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A REGULARIZING PARAMETER FOR SOME FREDHOLM INTEGRAL EQUATIONS

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Abstract — The regularizing parameter appearing in some Fredholm integral equations of the second kind is discussed. Theoretical estimates and the results of numerical tests confirming the theoretical expectations are given.

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1. Introduction

In [5], the author introduced a particular procedure to regularize the following Fredholm integral equation of the second kind:

$$f(y) - \frac{\mu}{G_2(y)} \int_0^\infty k(x,y) f(x) w_\alpha(x) dx = \frac{g(y)}{G_1(y)}, \tag{1.1}$$

where $w_\alpha(x) = x^\alpha e^{-x^\beta}$, $\alpha > -1$, $\beta > \frac{1}{2}$ is a generalized Laguerre weight, f is the unknown, $\mu \in \mathbb{R}$, g and k are given smooth functions, and G_1 and G_2 are functions with zeros at the origin of the type y^λ with $0 < \lambda < 1$. The suggested approach consists in "moving" the singularities into the kernel and then regularizing the equation by applying a smoothing transformation depending on the parameter $q \in \mathbb{N}$. Hence, the Nyström method is used to approximate the solution of the equation in a suitable Banach weighted space C_v equipped with a uniform norm.

In this paper, we discuss the choice of the parameter q. Indeed, the approximate solution F_m^* tends to the exact solution F^* with an error of the type

$$\|F^* - F_m^*\|_{C_v} = \mathcal{O}\left(\frac{1}{m}\right)^\sigma,$$

where σ depends on q and increases with increasing q. Consequently, it would appear natural to take q very large to have a good order of convergence. But when the parameter q increases, the speed of convergence slows down compromising the numerical results. Then, the aim of this paper is to propose a suitable choice of the parameter q in order to approximate the solution of the considered equation with a satisfactory theoretical order of convergence and with positive numerical results.

The paper is organized as follows. In Section 2, the regularizing procedure proposed in [5] is described. Section 3 presents the main results including some numerical tests. Section 4 gives proofs to conclude the paper.

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2. Preliminaries: a regularizing procedure

Let us consider Eq. (1.1) in the weighted space C_u, $u(x) = (1+x)^\rho x^\gamma e^{-x^\beta/2}$, $x, \rho, \gamma \geqslant 0$, defined as

$$C_u = \left\{ f \in C((0,\infty)) : \lim_{\substack{x\to\infty \\ x\to 0}} (fu)(x) = 0 \right\}, \tag{2.1}$$

where $C(J)$ denotes the collection of all continuous functions on $J \subseteq [0,\infty)$. If $\gamma = 0$, then the space C_u consists of all continuous functions on $[0,\infty)$ such that $\lim_{x\to\infty}(fu)(x) = 0$. This space equipped with the following norm

$$\|f\|_{C_u} = \|fu\|_\infty = \sup_{x\geqslant 0} |(fu)(x)|$$

is a Banach space.

In order to approximate the solution of Eq. (1.1) in C_u (if it exists), we could apply the Nyström method or the projection method based on orthogonal polynomials with respect to the weight w_α appearing in the integral (see, for instance, [12]). Nevertheless, it is possible to see (see, for instance, [5], [7], [6]) that, in virtue of the low smoothness of the given functions, these methods lead to very poor numerical results.

Hence the necessity arises to introduce a regularizing procedure that would allow us to improve the smoothness properties of the given functions in order to approximate the solution of (1.1) with a satisfactory order of convergence. In [5], an alternative numerical approach was proposed in this direction. The suggested procedure consists mainly of three steps which we now summarize.

The aim of the first step is to reduce the given equation to a *regularized equation.* To this end we consider (1.1) and for the sake of simplicity, but without loss of generality, we assume

$$G_1(y) = y^\delta, \quad G_2(y) = y^\epsilon, \quad 0 < \epsilon < \delta < 1.$$

We multiply both sides of (1.1) by y^δ and setting $\lambda = \delta - \epsilon$ we get

$$(y^\delta f)(y) - \mu y^\lambda \int_0^\infty k(x,y) x^\delta f(x) x^{\alpha-\delta} e^{-x^\beta} dx = g(y), \quad \delta < \alpha + 1. \tag{2.2}$$

Now, in order to improve the smoothness of the kernel, we introduce the following one-to-one map $\gamma_q : [0,\infty) \to [0,\infty)$ defined as

$$\gamma_q(t) = t^{q/\lambda}, \quad 1 \leqslant q \in \mathbb{N} \tag{2.3}$$

and we change the variables $x = \gamma_q(t)$ and $y = \gamma_q(s)$ in (2.2).

In this way we obtain

$$F(s) - \mu \int_0^\infty h(t,s)\ F(t)\ w_\eta(t) dt = G(s), \tag{2.4}$$

where $F(s) = f(\gamma_q(s)) s^{\frac{q\delta}{\lambda}}$ is the new unknown,

$$G(s) = g(\gamma_q(s)) \quad \text{and} \quad h(t,s) = \frac{q}{\lambda}\ k(\gamma_q(t), \gamma_q(s))\ t^{[\eta]}\ s^q, \tag{2.5}$$

are the new given functions, and $w_\eta(t) = t^{\eta-[\eta]}e^{-t^{q\beta/\lambda}}$ is the new Laguerre weight with $\eta = \frac{q}{\lambda}(1+\alpha-\delta)-1$, $\delta < \alpha+1$, $\beta > \frac{1}{2}$ and $[\eta]$ is its integer part.

We immediately note that the new equation, which we will call the *regularized equation*, has smooth given functions.

Now we have to fix the space in which we will study the *regularized equation* to assure its being unisolvent. This is the second step.

To this end, we introduce the weighted space C_v, with

$$v(s) = u(\gamma_q(s))s^{-\frac{q\delta}{\lambda}} = (1+s^{\frac{q}{\lambda}})^\rho s^{\frac{q}{\lambda}(\gamma-\delta)} e^{-\frac{s^{\frac{q\beta}{\lambda}}}{2}} \tag{2.6}$$

and study the smoothness properties of the new functions and the characteristics of the integral operator associated with the regularized equation

$$(\mathcal{K}F)(s) = \mu \int_0^\infty h(t,s)\ F(t)\ w_\eta(t)dt. \tag{2.7}$$

We denote by k_x (respectively by k_y) the function $k(x,y)$ as a function of the only variable y (respectively x).

Moreover, we define the Zygmund type space

$$Z_{s,r}(v) = \left\{ f \in C_v : \sup_{\tau>0} \frac{\Omega^r_\varphi(f,\tau)_v}{\tau^s} < \infty,\ r > s > 0 \right\}, \tag{2.8}$$

equipped with the norm

$$\|f\|_{Z_{s,r}(v)} = \|fv\|_\infty + \sup_{\tau>0} \frac{\Omega^r_\varphi(f,\tau)_v}{\tau^s},$$

where [17]

$$\Omega^r_\varphi(f,\tau)_v = \sup_{0<h\leqslant\tau} \|(\Delta^r_{h\varphi}f)v\|_{I_{rh}}$$

denotes the main part of modulus of smoothness with $r \geqslant 1$, $\varphi(x) = \sqrt{x}$, $\|\cdot\|_{I_{rh}}$ is the uniform norm on the interval $I_{rh} = [8r^2h^2, \mathcal{C}h^*]$, $h^* = h^{-2/(2\beta-1)}$, and $\mathcal{C}$ is a fixed constant and

$$\Delta^r_{h\varphi}f(x) = \sum_{i=0}^r (-1)^i \binom{r}{i} f\left(x + \left(\frac{r}{2}-i\right)h\varphi(x)\right).$$

For the sake of brevity, we will set $Z_{s,r}(v) := Z_s(v)$.

The following two propositions hold true.

Proposition 2.1. *Let $u(x) = (1+x)^\rho x^\gamma e^{-x^\beta/2}$ and v as in (2.6) with $\beta > \frac{1}{2}$, $\rho > 0$ and $\gamma > \delta$. If the known functions of the original equation are such that*

$$g \in Z_r(u), \tag{2.9}$$

$$\sup_{x\geqslant 0} u(x)\|k_x\|_{Z_r(u)} < \infty, \tag{2.10}$$

$$\sup_{y\geqslant 0} u(y)\|k_y\|_{Z_r(u)} < \infty, \tag{2.11}$$

with $r > 2\delta$, then the known functions of the regularized equation are such that

$$G \in Z_{\sigma_1}(v), \tag{2.12}$$

$$\sup_{t\geqslant 0} v(t)\|h_t\|_{Z_{\sigma_2}(v)} < \mathcal{C}\ \mathcal{D}, \tag{2.13}$$

$$\sup_{s\geqslant 0} v(s)\|h_s\|_{Z_{\sigma_3}(v)} < \mathcal{C}\ \mathcal{A}\ , \tag{2.14}$$

where $\sigma_1 = \frac{q}{\lambda}(r-2\delta)$, $\sigma_2 = \frac{q}{\lambda}(r-2\delta)+2q$, $\sigma_3 = \frac{q}{\lambda}(r-2\delta)+2[\eta]$, q and λ are the parameters appearing in the transformation γ_q defined in (2.3)*, $\mathcal{C}$ is a positive constant independent of the given functions and of q and λ, while $\mathcal{A}$ and $\mathcal{D}$ are constants depending on q and λ.*

Remark 2.1. We note that the choice of q as a natural number is closely related to the smoothness properties of the given functions (on which the order of convergence depends, as we will see in Section 3). Indeed, if q is not necessarily a natural number, the kernel h_t has a worse smoothness because of the factor s^q. In fact, in this case we have $h_t \in Z_{2\frac{q}{\lambda}(\gamma-\delta)+2q}(v)$. As an example, consider $k(x,y) = (x^{2/3} + y^{7/2})$, $\gamma_q(t) = t^{\frac{3}{2}q}$, $\lambda = \delta = 2/3$, $\gamma = 0.7$. Then, if $q \in \mathbb{N}$, we have $h_t(s) \in Z_{12.6q}(v)$, otherwise $h_t(s) \in Z_{2.1q}(v)$.

Proposition A [5]. *Let $u(s) = (1+s)^\rho s^\gamma e^{-\frac{s^\beta}{2}}$, $\beta > \frac{1}{2}$, v as in* (2.6) *with ρ and γ such that*

$$\rho > \frac{1}{2}, \qquad \max\left\{\delta, \frac{\delta+\alpha}{2}\right\} < \gamma < \frac{\alpha+1}{2}. \tag{2.15}$$

Then, if the kernel k_x satisfies (2.10)*, the operator $\mathcal{K} : C_v \to C_v$ defined in* (2.7) *is compact and for* (2.4) *the Fredholm Alternative Theorem holds true in C_v.*

We remark that if $\alpha < \delta$, find the value of γ, it is essential that $0 < \delta < \frac{\alpha+1}{2}$.

Now by means of the previous propositions it is possible to determine the conditions under which the *regularized equation* (2.4) is unisovent in C_v.

Proposition B [5]. *Let u and v be as in Proposition A and let* (2.10) *be satisfied. Then the original equation* (1.1) *has a unique solution $f^* \in C_u$ for each given right-hand side in C_u if and only if the regularized equation* (2.4) *has a unique solution $F^* \in C_v$ for each given right-hand side $G \in C_v$. Moreover the following relation holds true:*

$$(f^*u)(t) = (F^*v)(\gamma_q^{-1}(t)) \tag{2.16}$$

for each point $t \in [0,\infty)$.

The last step consists in applying the Nyström method to the *regularized equation* (2.4) in order to approximate its solution.

To this end, we first approximate the integral $\mathcal{K}F$ by using the following truncated Gaussian rule (see, e.g., [14],[15],[4]):

$$\int_0^\infty h(t,s)F(t)w_\eta(t)dt = \sum_{k=1}^{j} \lambda_k(w_\eta)h(x_k,s)F(x_k) + e_m^*(h_yF), \tag{2.17}$$

where $x_k = x_{m,k}(w_\eta)$, $k = 1, ..., j$, are the zeros of the polynomial $p_m(w_\eta)$ which is orthonormal with respect to the weight w_η, λ_k, $k = 1, ..., j$ are the Christoffel numbers corresponding to w_η,

$$x_j = \min_{1 \leqslant k \leqslant m} \{x_k : \ x_k \geqslant \theta a_m\}, \quad 0 < \theta < 1, \tag{2.18}$$

$$a_m = 4 \frac{\Gamma(\frac{q}{\lambda}\beta)^{\frac{2\lambda}{q\beta}}}{\Gamma(2\frac{q}{\lambda}\beta)^{\frac{\lambda}{q\beta}}} m^{\frac{\lambda}{q\beta}}, \tag{2.19}$$

denotes the Mhaskar-Rahmanov-Saff number (see, e.g., [11]) and $e_m^*(h_y F)$ is the remainder term.

Thus, setting

$$(\mathcal{K}_m F)(s) = \mu \sum_{k=1}^{j} \lambda_k(w_\eta) h(x_k, s) F(x_k),$$

we go to consider the operator equation

$$(I - \mathcal{K}_m) F_m = G,$$

where F_m is unknown.

Then, multiplying this equation by the weight v chosen as in Proposition A and collocating on the zeros x_i, $i = 1, ..., j$, we obtain the following linear system:

$$\sum_{k=1}^{j} \left[\delta_{i,k} - \mu \lambda_k(w_\eta) \frac{v(x_i)}{v(x_k)} h(x_k, x_i) \right] b_k = (Gv)(x_i), \quad i = 1, \dots, j, \tag{2.20}$$

where $\delta_{i,k}$ is a Kronecker symbol and $b_k = F_m(x_k)v(x_k)$, $k = 1, \dots, j$ are the unknowns. Now, if the above system has a unique solution $[b_1^*, \dots, b_j^*]^T$, then we can construct the following weighted Nyström interpolant:

$$F_m^*(s)v(s) = \mu \sum_{k=1}^{j} \lambda_k(w_\eta) \frac{v(s)}{v(x_k)} h(x_k, s) b_k^* + G(s)v(s). \tag{2.21}$$

Hence, in order to obtain an approximate solution of (2.4), we have to solve a linear system of j equations in j unknowns rather than a system of m equations in m unknowns and this implies a significant economy in computations. Moreover, we remark that system (2.20) can easily be constructed because it only requires the computation of the zeros x_k, $k = 1, ..., j$ and of the Christoffel Numbers $\lambda_k(w_\eta)$, $k = 1, ..., j$. To this end, one can use, in the Laguerre case, the routine *gaussq* (see [8]) or routines *recur* and *gauss* (see [9] and [10]), and in the general case, the Mathematica Package "OrthogonalPolynomials" (see [3]).

The stability and the convergence of the proposed method is stated in the following theorem proved in [5].

Theorem A [5]. *Assume that Eq.* (1.1) *has a unique solution f^* in C_u and that the hypotheses of Proposition* 2.1 *are satisfied. Then for m sufficiently large, system* (2.20) *is unisolvent and its matrix $\mathbf{B}_j$ is well conditioned holding*

$$cond(\mathbf{B}_j) \leqslant \mathcal{C}, \tag{2.22}$$

where $\mathcal{C}$ does not depend on m and $cond(\mathbf{B}_j) = \|\mathbf{B}_j\|_\infty \|\mathbf{B}_j^{-1}\|_\infty$.

3. Main Results

3.1. Why the choice of q? The error estimate.

The regularizing procedure and the Nyström method summarized in the previous section do not impose any restriction on the parameter q. Indeed, until now we have seen that, for each value of q, the given functions of Eq. (2.4) are smooth , the *regularized equation* is unisolvent in the space C_v, and system (2.20) has a unique solution and is well conditioned. Nevertheless, we need an optimal choice of q. In order to understand the reason of this necessity, let us estimate the error.

To this end, we denote by F^* the unique solution of (2.4) in C_v and by F^*_m the Nyström interpolant defined in (2.21).

By the well-known argument (see, e.g., [1])

$$\begin{aligned} \|[F^* - F^*_m]v\|_\infty &\sim \|[\mathcal{K}F^* - \mathcal{K}_m F^*]v\|_\infty \\ &= \sup_{s\geqslant 0} v(s) \left| \int_0^\infty h(x,y)F^*(x)w_\eta(x)dx - \sum_{k=1}^{j} \lambda_k(w_\eta)h(x_k,s)F^*(x_k) \right| \\ &= \sup_{s\geqslant 0} v(s)|e^*_M(h_sF^*)|, \end{aligned} \tag{3.1}$$

where $e^*_M(h_sF^*)$ is the remainder term of the Gaussian rule (2.17).

Now, since in virtue of the assumptions about the parameters of the weight v it results in $\displaystyle\int_0^\infty \frac{w_\eta(t)}{v^2(t)}dt < \infty$, we have [12]

$$|e^*_m(h_yF)| \leqslant \mathcal{C}[E_M(h_yF)_{v^2} + e^{-Am}\|h_yFv^2\|_\infty], \tag{3.2}$$

where the constants $\mathcal{C}$ and A are independent of m and F, $M = [(\frac{\theta}{1+\theta})^\beta m]$ and $E_n(f)_v = \inf_{P_n\in\mathbb{P}_n} \|(f-P_n)v\|_\infty$ denotes the error of the best approximation of $f \in C_v$ by polynomials of degree n at most ($P_n \in \mathbb{P}_n$).

Hence, choosing $M = am$, $0 < a < 1$ and taking into account that for all $f, g \in C_v$, we get

$$E_m(fg)_{v^2} \leqslant \mathcal{C}[\ \|fv\|E_m(g)_v + 2\|gv\|_\infty E_m(f)_v\ ], \tag{3.3}$$

by (3.1) we have

$$\|[F^* - F^*_m]v\|_\infty \leqslant \mathcal{C}\left[\|F^*v\|_\infty \sup_{s\geqslant 0} v(s)E_{[\frac{M}{2}]}(h_s)_v + \sup_{s\geqslant 0} v(s)\|h_sv\|_\infty E_{[\frac{M}{2}]}(F^*)_v\right].$$

By Proposition 2.1 we deduce that $h, g \in Z_\sigma(v)$, with $\sigma = \frac{q}{\lambda}(r-2\delta)$ and then $F^* \in Z_\sigma(v)$, too. Moreover, since $\forall f \in Z_s(v)$ (see, e.g., [17])

$$E_m(f)_v \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_s(v)}, \quad m > s \quad \mathcal{C} \neq \mathcal{C}(m,f), \tag{3.4}$$

we have

$$\|[F^* - F^*_m]v\|_\infty \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^\sigma \|F^*\|_{Z_{\sigma(v)}} \sup_{s\geqslant 0} v(s)\|h_s\|_{Z_{\sigma 3}(v)}. \tag{3.5}$$

We have proved the following result.

Theorem 3.1. *Assume that the assumptions of Theorem A are satisfied. Then if F^* denotes the unique solution of Eq.* (2.4) *and F^*_m is the Nyström interpolant defined in* (2.21), *then*

$$\|[F^* - F^*_m]v\|_\infty \leqslant \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^\sigma \|F^*\|_{Z_{\sigma(v)}} \sup_{s\geqslant 0} v(s)\|h_s\|_{Z_{\sigma_3}(v)}, \tag{3.6}$$

where $\mathcal{C} \neq \mathcal{C}(m, F^*)$ and $\sigma = \frac{q}{\lambda}(r - 2\delta)$.

Hence, the theoretical order of convergence depends on the smoothness properties of the given functions. Consequently, we emphasize again the importance of choosing q as a natural number. And if it is not natural, then, by Remark 2.1, we obtain that the theoretical order of convergence is worse $\mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{q}{\lambda}(\gamma-\delta)+2q}\right)$.

From estimate (3.6) it follows that for any constant $\mathcal{C}$ independent of m the error tends to zero as $\left(\frac{\sqrt{a_m}}{m}\right)^\sigma$, since theoretically we can choose m sufficiently large. Moreover, the rate of convergence increases with increasing q. Consequently, we tempt to take q very large to have a good order of convergence. But now we linger over the Zygmund norms appearing on the right hand side of (3.6). By Proposition 2.1 it follows that

$$\sup_{s\geqslant 0} v(s)\|h_s\|_{Z_{\sigma_3}(v)} \leqslant \mathcal{C}\ \mathcal{A},$$

where $\mathcal{A}$ is a constant depending on q. Moreover, using the same argument we can see that $\|F^*\|_{Z_{\sigma(v)}}$ also has the same behavior.

The error estimate is of the following type:

$$\|[F^* - F^*_m]v\|_\infty = \mathcal{C}\mathcal{A}^2\mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^\sigma\right), \tag{3.7}$$

i.e., a constant $\mathcal{A}$ depending on a parameter (which can be changed) appears. Consequently, it is necessary to analyze the behavior of this constant when the parameter varies. Indeed, if it becomes large as q does, the numerical convergence can be compromised even if the theoretical one is ensured.

In the following subsection we will make an evaluation of this constant and study its behavior. Here we only observe that in the approximation theory an error estimate in which a parameter-dependent constant frequently appears. For instance, in $[-1, 1]$, if we consider the function $f(x) = \log(1 + x)$, it is possible to prove that there exists a polynomial P (see, e.g., [16]) such that

$$\|[f - P]v^{\gamma,\delta}\|_\infty \leqslant \mathcal{C}(r-1)!\frac{\log m}{m^r}, \quad v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta.$$

Then also in this case a constant $\mathcal{A} = (r-1)!$ appears. Moreover, here the numerical problem we have is evident: if r increases, then the order of convergence becomes large but the speed of convergence slows down because the constant $\mathcal{A}$ increases. This will be our problem.

3.2. Constant $\mathcal{A}$ and the crucial problem of choosing the regularizing parameter

In [5], to give an idea of the constant $\mathcal{A}$, the following estimate was proved in the case where the parameter η appearing in (2.5) is equal to zero.

Proposition C [5]. *Let $q \geqslant 1$ and $0 < \lambda < 1$. Then*

$$\mathcal{A} \leqslant \left(\left[\frac{q}{\lambda}\right]+1\right)^{\left[\frac{q}{\lambda}\right]} \mathcal{W}\left(\left[\frac{q}{\lambda}\right]\right),$$

where $\mathcal{W}\left(\left[\frac{q}{\lambda}\right]\right)$ denotes the $\left[\frac{q}{\lambda}\right]$th Bell number.

Now we will make an evaluation for $\mathcal{A}$.

By the proof of Proposition 2.1, setting $\ell = \min\{[\frac{q}{\lambda}], [r]\}$ it follows that

$$\mathcal{A} = \begin{cases} \displaystyle\sum_{i=0}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} \\ \qquad \displaystyle\sum_{m=0}^{i} \mathcal{B}_{i,m}\left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right), ..., \frac{q}{\lambda}\cdot ...\cdot\left(\frac{q}{\lambda}-i+m\right)\right), & \ell \leqslant [\eta]; \\ \displaystyle\sum_{i=\ell-[\eta]}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} \\ \qquad \displaystyle\sum_{m=0}^{i} \mathcal{B}_{i,m}\left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right), ..., \frac{q}{\lambda}\cdot ...\cdot\left(\frac{q}{\lambda}-i+m\right)\right), & \ell > [\eta] \end{cases} \tag{3.8}$$

where $\mathcal{B}_{i,m}$ denotes the partial Bell polynomials defined in (4.1) with $\mathcal{B}_{0,m} = 1$ for all $m = 0, \cdots, i$ and $\mathcal{B}_{i,0} = 0$ for all $i = 1, \cdots, \ell$.

Now, from the theory of Bell's polynomials it is known that

$$\sum_{k=1}^{n} B_{n,k}(x_1, x_2, ..., x_{n-k+1}) = B_n(x_1, x_2, ..., x_n),$$

where $B_n(x_1, x_2, ..., x_n)$ are the so-called complete Bell polynomials which satisfy the following property:

$$B_n(x_1, x_2, ..., x_n)$$
$$:= det \begin{pmatrix} x_1 & \binom{n-1}{1} x_2 & \binom{n-1}{2} x_3 & \dots & \binom{n-1}{n-2} x_{n-1} & x_n \\ -1 & x_1 & \binom{n-2}{1} x_2 & \dots & \binom{n-2}{n-3} x_{n-2} & x_{n-1} \\ 0 & -1 & x_1 & \dots & \binom{n-3}{n-4} x_{n-3} & x_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_1 & x_2 \\ 0 & 0 & 0 & \dots & -1 & x_1 \end{pmatrix}. \tag{3.9}$$

Then, in virtue of this relation, in order to compute the constant $\mathcal{A}$ we have only to compute special sums of determinants of a particular matrix. Indeed, since $\mathcal{B}_{0,m} = 1$ for all $m = 0, \cdots, i$ and $\mathcal{B}_{i,0} = 0$ for all $i = 1, \cdots, \ell$, by (3.8) and (3.9), the constant $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = \begin{cases} \displaystyle\frac{[\eta]!}{([\eta]-\ell)!} + \sum_{i=1}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} det(\mathbf{A}_i), & \ell \leqslant [\eta]; \\ \displaystyle\sum_{i=\ell-[\eta]}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} det(\mathbf{A}_i), & \ell > [\eta] \end{cases} \tag{3.10}$$

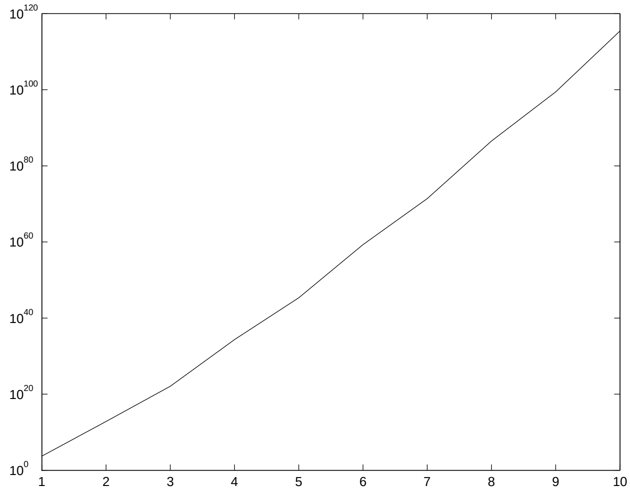

Fig. 3.1. $\mathcal{A}$

where $\det(\mathbf{A}_i)$ denotes the determinant of the matrix defined in (3.9) with $n = i$ and with

$$(x_1, x_2, ..., x_n) = \left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda} - 1\right), ..., \frac{q}{\lambda} \cdot ... \cdot \left(\frac{q}{\lambda} - i + 1\right)\right).$$

Note that in the simple case where $\eta = 0$ we have in $\mathcal{A} = det(\mathbf{A}_\ell)$.

We also underline that in order to compute the determinant of the matrix $\mathbf{A}_i$, one can use the following formula:

$$det(\mathbf{A}_i) = \sum_{k=0}^{i-1} \binom{i-1}{k} x_{k+1}\; det(\mathbf{A}_{i-k-1}), \quad i \geqslant 2,$$

with $det(\mathbf{A}_0) = 1$ and $det(\mathbf{A}_1) = x_1$.

Now by (3.10) the behavior of the constant is evident: as q increases, it becomes very large. Moreover, we note that since the constant depends on the ratio $\frac{q}{\lambda}$, when λ is close to zero, this constant becomes large even when q is small (see Fig. 3.1). On the contrary, if λ is close to one, it becomes large when q is large (see Table 3.4). Figure 3.1 shows the trend of the constant when q changes in the case where $\ell = [\frac{q}{\lambda}]$, $\lambda = 2/9$, $\delta = 2/9$ and $\alpha = -1/3$.

The problem announced in the previous subsection is confirmed: when q becomes large the numerical convergence is compromised even if the theoretical one is assured. Indeed, by the error estimate (3.7)

$$\|[F^* - F_m^*]v\|_\infty = \mathcal{C}\mathcal{A}^2\mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{q}{\lambda}(r-2\delta)}\right),$$

we deduce that if q becomes large, then the order of convergence increases but the speed of convergence slows down because of the presence of the constant $\mathcal{A}$. Consequently, we need a very large number of points m to obtain the required convergence. For instance, assume $\lambda = \delta = 2/9$, $r = 2$. According to (3.10), if $q = 8$, then $\mathcal{A} = 3.351200611656362e + 086$. Therefore, to have the approximate solution with, e.g., 7 correct digits, we need a number of points $m > 1899$. But this is not realistic. In fact, in order to construct the Nyström interpolant F_m^* defined in (2.21), we have to solve system (2.20). Thus, we have to compute the zeros x_k and the Christoffel numbers λ_k of a polynomial of degree $m > 1899$. And this requires a computational effort.

Then, for this reason, an optimal choice of the parameter q is necessary. To this end, we suggest to proceed in the following way:

1. Regularize the given equation as shown in Section 2.

2. Compute the order of convergence according to (3.6).

3. Compute the constant $\mathcal{A}$ according to (3.10) for different values of q. Now, as mentioned above, the constant $\mathcal{A}$ becomes very large as q increases. Consequently, after a certain value q_0 of q, the constant $\mathcal{A}^2$ (we need it later to compute the optimal parameter q) cannot be computed numerically. Among the highest values $\mathcal{A}^2 \sim 10^{292}$. After this value it is impossible to know $\mathcal{A}^2$. Because of this, in this phase we fix the range $[1, q_0]$ in which we can choose our optimal parameter q.

4. Fix the correct digits we want to be exact in the approximate solution and then compute the number of points m we need to obtain it. For instance, if we want to have an approximate solution with a correct digits, taking into account (3.7) and (2.19), it has to be

$$m \geqslant \left(\mathcal{A}^2 2^\sigma \left(\frac{\Gamma(\frac{q}{\lambda}\beta)}{\sqrt{\Gamma(2\frac{q}{\lambda}\beta)}} \right)^{\frac{\lambda\sigma}{q\beta}} 10^{a+1} \right)^{\frac{1}{(1-\frac{\lambda}{2q\beta})\sigma}}, \quad 1 \leqslant q \leqslant q_0. \tag{3.11}$$

5. Choose the optimal parameter $q \in [1, q_0]$, that is the natural number which minimize the right-hand side of (3.11).

6. Solve system (2.20) and construct the Nyström interpolant (2.21).

7. Compute the solution of the original equation according to (2.16).

Proceeding in this way, we will approximate the solution of the considered equations with a satisfactory theoretical order of convergence and with positive numerical results.

We note that theoretically the parameter q_0 can be large (it depends on the other parameters involved in the computation of $\mathcal{A}$). Consequently, the optimal parameter q can be large. On the other hand, it is very difficult to give an analytical expression of the minimal point of the right-hand side of (3.11). In any case, if $q \in [1, q_0]$ is large, then we have no numerical problem: system (2.20) is well conditioned for each value of q. However, we underline that in all the examples tested the optimal parameter q has always been small.

In the following subsection we will carry out some numerical tests confirming our theoretical expectations.

3.3. Numerical Tests

In this subsection, we will give the numerical results obtained for some Fredholm integral equations.

To this end, we will follow the procedure suggested in the previous subsection. Thus, first of all, we will regularize the given equation as shown in Section 2. Subsequently, we will choose the optimal parameter q to avoid a compromise of the numerical convergence.

Then with fixed q, we will construct the Nyström interpolant (2.21) of Eq. (2.4) and finally we will compute the approximate solution f_m^* of the original equation according to (2.16).

In each numerical test, we take, as a reference solution, the approximated solution obtained at $m = 256$ and in all the tables we will give $e_m = \max_i |(f_{256}u)(y_i) - (f_m u)(y_i)|$, where $\{y_i\}_{i=1}^{20}$ denotes 20 equispaced points on the interval $(0, \infty)$ and the condition number in the infinity norm of system (2.20). All computations were performed in 16-digit arithmetics.

Example 3.1. Consider the equation

$$f(y) - \frac{1}{5y^{2/3}} \int_0^\infty (x^2 + y^2 + 8)\ f(x)\ x^{4/5} e^{-x^{3/4}} dx = \frac{(y^{3/2} + 2)}{y^{7/9}}.$$

It has a unique solution in the weighted space C_u with $u(x) = (1+x)^{0.6} x^{0.85} e^{-\frac{x^{3/4}}{2}}$. Note that the function $k(x, y) = x^2 + y^2 + 8$ is an analytical function while $g(y) = (y^{3/2} + 2) \in Z_{4.7}(u)$. Now, applying the regularizing procedure shown in Section 2, we obtain

$$F(s) - \frac{9}{5} q s^q \int_0^\infty (t^{18q} + s^{18q} + 8)\ t^{[\eta]}\ F(t)\ w_\eta(t) dt = (s^{\frac{27q}{2}} + 2),$$

with $\eta = \frac{46}{5} q - 1$ and $w_\eta(t) = t^{\eta - [\eta]} e^{-t^{\frac{27q}{4}}}$. The new equation has a unique solution in C_v with v as in (2.6), according to Proposition B. We note that the new kernel is an analytical function while the right-hand side pertains to $Z_{28.30q}(v)$. Consequently, the order of convergence is $\mathcal{O}\left((\frac{\sqrt{a_m}}{m})^\sigma \right)$ with $\sigma = 28.30q$ according to (3.6). Now, we choose the optimal parameter q. Then, first of all, we compute the constant $\mathcal{A}$ according to (3.10) for different values of q. Thus we fix the greatest value of q, namely q_0, for which we can compute numerically $\mathcal{A}^2$. In this case, we have $q_0 = 5$. Now, we would like to know the approximate solution with 6 correct digits and we compute the optimal parameter q, that is the value of $q \in [0, 5]$ which minimize the right hand side of (3.11). The following graph shows the behavior of (3.11) when q changes.

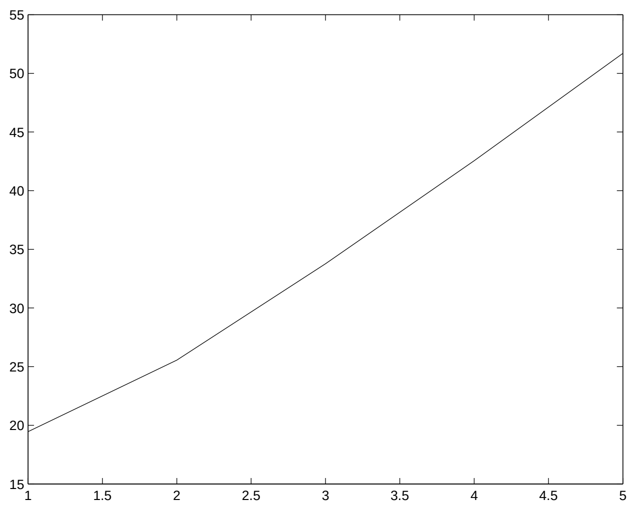

Fig. 3.2. m

Hence we deduce that the optimal parameter is $q = 1$ in accordance with which we have the required convergence with a number of points $m \geqslant 19$. Table 3.3 shows the weighted approximate solution obtained with this optimal parameter ($\theta = 0.9$).

Table 3.1. **q=1**

m	j	e_m	$cond(\mathbf{B}_j)$
16	16	4.31192e-007	24.63689043040089
32	31	1.88633e-007	29.56259271555317
64	60	1.04805e-013	32.56276574794698
128	120	7.10542e-015	34.42444895667656

If the parameter q increases, for instance, $q = 4$, then the numerical results are poor. Indeed, as shown in Table 3.2, in order to have 6 correct digits we have to solve a system of order 63 rather than 16 as done in the case $q = 1$. From the last table we can also see that if the parameter q increases, the condition number in the infinity norm of system (2.20) is still bounded.

Table 3.2. **q=4**

m	j	e_m	$cond(\mathbf{B}_j)$
32	32	4.05913e-006	25.04999125397745
64	63	7.50975e-007	29.87180783730457
128	126	2.27320e-011	32.77334652130129

Example 3.2. We consider the following Fredholm integral equation:

$$f(y) - \frac{1}{2}\int_0^\infty (x^2 + y + 3) f(x) x^{4/3} e^{-x} dx = \frac{y+1}{y^{1/3}} - \frac{3}{2}(13 + y),$$

whose exact solution is $f(y) = \frac{1+y}{y^{1/3}}$.
The considered equation has a unique solution in the weighted space C_u with $u(x) = (1 + x)^{0.7} x^{9/8} e^{-x/2}$. Using the regularizing procedure shown in Section 2, we get

$$F(s) - \frac{3q}{2} s^q \int_0^\infty (t^{6q} + s^{3q} + 3) t^{6q-1} F(t) e^{-t^{3q}} dt = s^{3q} + 1 - \frac{3}{2}(13 + s^{3q}) s^q,$$

which has a unique solution in C_v according to Proposition B. We immediately notice that all given functions are polynomials for each q and the convergence is very fast. Table 3.3 shows the numerical results obtained at $q = 1$ ($\theta = 0.7$). Note that in this case $\mathcal{A} = 132$. If the parameter q increases the given functions are still polynomials and we expect the same numerical results but they are poor. Indeed, since the constant $\mathcal{A}$ increases, the speed of convergence slows down compromising the numerical results. Table 3.4 shows what happens in the case where $q = 8$ ($\theta = 0.96$). Note that in this case $\mathcal{A} = 8.321415742355469e + 050$.

Table 3.3. **q=1**

m	j	e_m	$cond(\mathbf{B}_j)$
8	8	5.68434e-014	21.78681961756113

Table 3.4. **q=8**

m	j	e_m	$cond(\mathbf{B}_j)$
16	16	1.16761e-001	16.75283784307782
32	31	2.57434e-002	25.56426459953316
64	61	6.02116e-005	31.11032195144892
128	122	9.02389e-013	33.82723852547484
256	242	4.26325e-014	37.34898825173112

Example 3.3. Consider the equation

$$f(y) - \frac{1}{7}\int_0^\infty (x^{7/2} + y^{2/3} + 7)\ f(x)\ \sqrt{x}e^{-x}dx = \frac{2}{y^{2/3}e^{(1+y^{4/3})}}.$$

It has a unique solution in the weighted space C_u with $u(x) = (1+x)^{0.6}x^{0.7}e^{-x/2}$. Applying the procedure shown in Section 2, the given equation is equivalent to

$$F(s) - \frac{3q}{14}s^q\int_0^\infty (t^{\frac{21q}{4}} + s^q + 7)t^{[\eta]}\ F(t)\ w_\eta(t)dt = \frac{2}{e^{(1+s^{2q})}}$$

with $\eta = \frac{5}{4}q - 1$ and $w_\eta(t) = t^{\eta-[\eta]}e^{-t^{\frac{3}{2}q}}$. The new equation has a unique solution in C_v with v as in (2.6), in virtue of Proposition B.

Note that the right-hand side and the kernel with respect to the variable s of the new equation are analytical functions while the kernel with respect to the variable t pertains to $Z_{6.45q}(v)$. Consequently, the order of convergence is $\mathcal{O}(\frac{1}{m^{5.37q}})$, according to (3.6) and (2.19).

Now we choose the optimal parameter q to obtain an approximate solution with 7 correct digits. Computing expression (3.10), we can see that we can determine numerically $\mathcal{A}^2$ if $q \in [1, q_0]$ with $q_0 = 32$. Then, taking into account (3.11), we can construct Table 3.5.

Hence we deduce that the optimal parameter is $q = 5$. Table 3.6 shows the obtained numerical results ($\theta = 0.9$).

Example 3.4. Consider the equation

$$f(y) - \frac{1}{12}\int_0^\infty \sin{(xy)}e^{-xy}\ f(x)\ x^{-1/5}e^{-x^{3/2}}dx = \frac{\log{(1+y)}}{\sqrt{y}(y^2+4)}.$$

Table 3.5

q	$\mathcal{A}$	$m \geqslant$
1	1.5	381
2	132	27
3	7012.6875	13
4	13610520	14
5	1.769577199804688e+009	11
6	1.190512570851900e+013	13
$\vdots$	$\vdots$	$\vdots$
25	2.134477512167769e+090	26
$\vdots$	$\vdots$	$\vdots$
32	7.851241025230311e+125	33

Table 3.6. **q=5**

m	j	e_m	$cond(\mathbf{B}_j)$
16	16	3.41507e-006	28.78177615020002
32	31	2.94802e-010	34.21321407988217
64	62	2.02615e-015	37.37669660028023

It is unisovent in the weighted space C_u with $u(x) = (1+x)^{0.8}x^{0.35}e^{-x^{3/2}/2}$. Using the regularizing procedure described in Section 2, we find that it is equivalent to

$$F(s) - \frac{q}{6}s^q \int_0^\infty \sin{(ts)^{2q}}\ e^{-(st)^{2q}}\ F(t)\ t^{\frac{3}{5}q-1}e^{-t^{3q}}dt = \frac{\log{(1+s^{2q})}}{(s^{4q}+4)},$$

which has a unique solution in C_v with v as in (2.6) according to Proposition B.

We immediately notice that all given functions are analytical for each value of q. Consequently, the convergence is very fast as shown in Table 3.7 in which the results were obtained with $q = 1$ and $\theta = 0.7$. Note that in this case the constant $\mathcal{A} = 6$.

Table 3.7. **q=1**

m	j	e_m	$cond(\mathbf{B}_j)$
8	8	8.87958e-007	1.045102472850763
16	14	1.03388e-010	1.046930825888115
32	27	6.92534e-018	1.048257667797889

If the parameter q increases, the order of convergence remains the same because the functions are still analytic but the speed of convergence slows down because the constant increases. Indeed, if, for instance, $q = 7$, we have $\mathcal{A} = 1.257542760359232e + 024$. Table 3.8

Table 3.8. **q=7**

m	j	e_m	$cond(\mathbf{B}_j)$
16	15	2.16707e-005	1.044237627930557
32	28	1.00269e-005	1.046535459802606
64	56	3.66678e-008	1.047805484514525
128	110	4.82443e-011	1.048941221806483

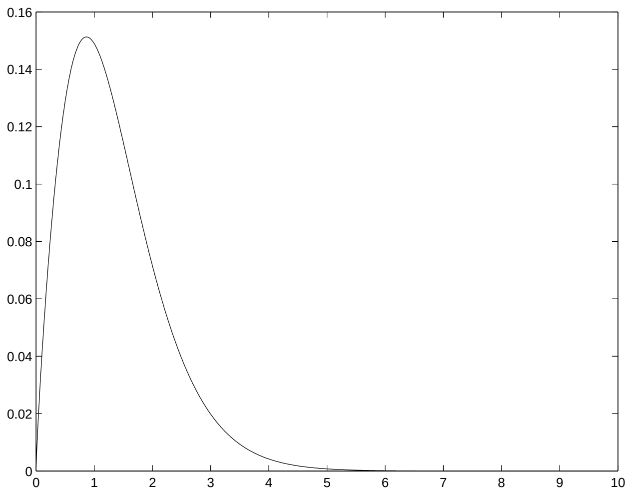

Fig. 3.3. $(f_{32}^* u)(y)$

shows the results obtained at $q = 7$ $(\theta = 0.9)$. Note that the condition number of system (2.20) is also small in the case where the parameter q increases.

Figure 3.3 shows the graph of the weighted approximate solution $f_{32}^* u$.

4. Proofs

Proof of Proposition 2.1.
We begin by proving (2.12). Let $\ell = \min\{[\frac{q}{\lambda}], [r]\}$. By the Faá di Bruno Formula we have

$$G^{(\ell)}(s) = \sum_{k=1}^{\ell} g^{(k)}(\gamma_q(s))\ \mathcal{B}_{\ell,k}(\gamma_q^{(1)}(s), \gamma_q^{(2)}(s), ..., \gamma_q^{(\ell-k+1)}(s)),$$

where $\mathcal{B}_{\ell,k}$ denotes the partial Bell polynomials defined as (see, e.g., [2, p. 134])

$$\mathcal{B}_{\ell,k}(x_1, x_2, ..., x_{\ell-k+1}) = \sum \frac{\ell!}{k_1!k_2!...k_{\ell-k+1}!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdot ... \cdot \left(\frac{x_{\ell-k+1}}{(\ell-k+1)!}\right)^{k_{\ell-k+1}}, \quad (4.1)$$

where the sum is extended to all positive integers $k_1, k_2, ..., k_{\ell-k+1}$ such that $k = k_1 + k_2 + ... + k_{\ell-k+1}$ and $k_1 + 2k_2 + ... + (\ell - k + 1)k_{\ell-k+1} = \ell$.
Developing $\mathcal{B}_{\ell,k}(\gamma_q^{(1)}(s), \gamma_q^{(2)}(s), ..., \gamma_q^{(\ell-k+1)}(s))$ leads to

$$G^{(\ell)}(s) = \sum_{k=1}^{\ell} g^{(k)}(\gamma_q(s)) s^{\frac{q}{\lambda}k-\ell} \mathcal{B}_{\ell,k}\left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right), ..., \frac{q}{\lambda} \cdot ... \cdot \left(\frac{q}{\lambda} - \ell + k\right)\right).$$

Then denoting by $\varphi(s)=\sqrt{s}$, $u(s)=(1+s)^\rho s^\gamma e^{-s^\beta/2}$ and $v(s)=u(\gamma_q(s))s^{-\frac{q}{\lambda}\delta}$, we deduce

$$\begin{aligned}&|(G^{(\ell)}\varphi^\ell v)(s)|\\ &\leqslant \sum_{k=1}^{\ell}|(g^{(k)}\varphi^k u)(\gamma_q(s))|s^{\frac{q}{\lambda}(\frac{k}{2}-\delta)-\frac{\ell}{2}}\mathcal{B}_{\ell,k}\left(\frac{q}{\lambda},\frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right),...,\frac{q}{\lambda}\cdot...\cdot\left(\frac{q}{\lambda}-\ell+k\right)\right).\end{aligned} \tag{4.2}$$

Now, by the assumption $g\in Z_r(u)$. Therefore, taking into account that

$$\Omega_\varphi^k(g,t)_u\leqslant \mathcal{C}\sup_{0<h\leqslant t}h^k\|g^{(k)}\varphi^k u\|_{I_{hk}}, \tag{4.3}$$

with $I_{hk}=[8k^2h^2,\mathcal{C}h^{-2}]$ by some computations we have

$$|(g^{(k)}\varphi^k u)(\gamma_q(s))|<\mathcal{C}s^{\frac{q}{\lambda}(\frac{r}{2}-\frac{k}{2})}(1+s^{q/\lambda})^\rho\ e^{-s^{q/\lambda}/2}M(s),$$

where M is a smooth function.

Thus, by (4.2) taking the supremum on $I_{h\ell}$, we have

$$\|G^{(\ell)}\varphi^\ell v\|_{I_{h\ell}}\leqslant \mathcal{C}h^{\frac{q}{\lambda}(r-2\delta)-\ell}\sum_{k=1}^{\ell}\mathcal{B}_{\ell,k}\left(\frac{q}{\lambda},\frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right),...,\frac{q}{\lambda}\cdot...\cdot\left(\frac{q}{\lambda}-\ell+k\right)\right),$$

from which by using (4.3) and some properties of the main part of modulus of smoothness (see, e.g., [13]) we deduce

$$\sup_{\tau>0}\frac{\Omega_\varphi^n(g,\tau)_u}{\tau^{\frac{q}{\lambda}(r-2\delta)}}\leqslant \mathcal{C}\sup_{\tau>0}\frac{\Omega_\varphi^\ell(g,\tau)_u}{\tau^{\frac{q}{\lambda}(r-2\delta)}}<\infty,\quad n>\frac{q}{\lambda}(r-2\delta).$$

Now we prove (2.13). As for the uniform norm, it is easy to see that

$$\begin{aligned}\sup_t v(t)\|h_t v\|_\infty &=\sup_t v(t)\sup_{s\geqslant 0}|k(\gamma_q(t),k(\gamma_q(s))s^q v(s)|\\ &\leqslant \sup_{x\geqslant 0}u(x)\|k_x u\|_\infty<\sup_{x\geqslant 0}u(x)\|k_x\|_{Z_r(u)},\end{aligned} \tag{4.4}$$

which is bounded by the assumptions. Moreover, by applying the Leibnitz formula we have

$$h_t^{(\ell)}(s)=\begin{cases}\displaystyle\sum_{i=0}^{\ell}\binom{\ell}{i}\frac{q!}{(q-\ell+i)!}s^{q+i-\ell}[k(\gamma_q(t),\gamma_q(s))]^{(i)}, & \ell\leqslant q;\\ \displaystyle\sum_{i=\ell-q}^{\ell}\binom{\ell}{i}\frac{q!}{(q-\ell+i)!}s^{q+i-\ell}[k(\gamma_q(t),\gamma_q(s))]^{(i)}, & \ell>q.\end{cases}$$

Hence, by using the Bruno di Fáa formula for computing $[k(\gamma_q(t),\gamma_q(s))]^{(i)}$ according to which we have

$$[k(\gamma_q(t),\gamma_q(s))]^{(i)}=\sum_{m=0}^{i}s^{\frac{q}{\lambda}m-i}\mathcal{B}_{i,m}\left(\frac{q}{\lambda},\frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right),...,\frac{q}{\lambda}\cdot...\cdot\left(\frac{q}{\lambda}-i+m\right)\right)k^{(m)}(\gamma_q(t),\gamma_q(s)),$$

with $\mathcal{B}_{0,m}=1$ for all $m=0,...,i$ and $\mathcal{B}_{i,0}=0$ for all $i=1,...,\ell$ and proceeding as already done for the function G, we get

$$\sup_t v(t)\sup_{\tau>0}\frac{\Omega_\varphi^j(h_t,\tau)_v}{\tau^{\frac{q}{\lambda}(r-2\delta)+2q}}<\ \mathcal{C}\ \mathcal{D},$$

where $\mathcal{D}$ is a constant depending on q and λ. Then (2.13) is proved. Proceeding in the same way, it is possible to prove (2.14), i.e.,

$$\sup_s v(s) \sup_{\tau>0} \|h_s\|_{Z_{\frac{q}{\lambda}(r-2\delta)+2[\eta]}(v)} < \mathcal{C} \ \mathcal{A},$$

where

$$\mathcal{A} = \begin{cases} \displaystyle\sum_{i=0}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} \\ \qquad \displaystyle\sum_{m=0}^{i} \mathcal{B}_{i,m} \left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right), ..., \frac{q}{\lambda} \cdot ... \cdot \left(\frac{q}{\lambda}-i+m\right)\right), \quad \ell \leqslant [\eta]; \\ \displaystyle\sum_{i=\ell-[\eta]}^{\ell} \binom{\ell}{i} \frac{[\eta]!}{([\eta]-\ell+i)!} \\ \qquad \displaystyle\sum_{m=0}^{i} \mathcal{B}_{i,m} \left(\frac{q}{\lambda}, \frac{q}{\lambda}\left(\frac{q}{\lambda}-1\right), ..., \frac{q}{\lambda} \cdot ... \cdot \left(\frac{q}{\lambda}-i+m\right)\right), \quad \ell > [\eta]. \end{cases}$$

□

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