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PARALEL PREDICTOR-CORRECTOR SCHEMES FOR PARABOLIC PROBLEMS ON GRAPHS

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Abstract — We consider a predictor-corrector type finite difference scheme for solving one-dimensional parabolic problems. This algorithm decouples computations on different subdomains and thus can be efficiently implemented on parallel computers and used to solve problems on graph structures. The stability and convergence of the discrete solution is proved in the special energy and maximum norms. The results of computational experiments are presented.

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1. Introduction

Many applied problems are described by reaction - diffusion equations on branched structures. The well-known examples are given by neuron simulation models based on the Hodgkin–Huxley (HH) reaction–diffusion system [4]. In order to describe the functioning of the brain we must take into account the complex architecture of the synaptic connections between neurons, the spatial distribution of the membrane channels within the branched structure of neurons, and the highly nonlinear electrical properties of the neurons. The amount of computations needed to make a mathematical model fit experimental data by exploring exhaustively the parameter space grows exponentially with the number of parameters.

Numerical algorithms for solving HH type problems were proposed and investigated in [1, 6, 7, 11]. They use the finite difference method for the approximation of space derivatives and apply different techniques for numerical integration in time. In constructing numerical approximations of reaction-diffusion systems on branched structures, very important problem arises due to the approximation of flux conservation equations at branch points.

Recently, the predictor-corrector splitting and domain decomposition methods have been widely used to solve elliptic and parabolic PDEs in multidimensional domains [3, 10, 16]. Important results for domain decomposition methods are described in [5]. A general framework for the construction and analysis of various classes of splitting and domain decomposition algorithms is presented in [13]. Very interesting developments of these ideas are presented in [15], where new domain decomposition methods with overlapping subdomains are investigated. The main idea is to choose a simple generating scheme for the classical approximation of the continuous problem, then to increase its stability by adding the regularization operator at the second stage of developing an efficient discrete scheme. General operator stability conditions [12] can be used to prove the stability of the new discrete schemes that are obtained.

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Such algorithms are well suited for parallel implementation. Similar techniques can also be used for problems on graph structures (see [11, 14] for related discussions). Equations on different edges of the graph are decoupled and can be solved in parallel. Parallelization of the algorithm is very important in solving real-world problems when the amount of computations increases, and the application of parallel algorithms enables us to reduce substantially the CPU time. In this paper, we consider predictor and predictor–corrector type finite difference schemes for soluting parabolic problems. These results can be generalized to problems on graphs quite straightforwardly. The main goal is to investigate the stability of the obtained discrete algorithms with respect to the approximation errors introduced at the predictor step of the algorithm. Stability analysis in the maximum norm was performed in [2], but only the conditional stability can be proved by this method and no influence of the corrector step on the stability is obtained.

In the present paper, we have investigated the stability of the proposed discrete scheme in the energy and maximum norms. For ease of presentation, we restrict ourselves to a detailed analysis of one-dimensional linear parabolic problems. The main goal is to investigate the stability of the obtained discrete algorithms with respect to the approximation errors introduced at the predictor step of the algorithm.

The rest of the paper is organized as follows. The mathematical model of a linear parabolic problem and the predictor-corrector algorithm are presented in Section 2. The convergence of the discrete solution in the energy and maximum norms is investigated in Section 3. Some numerical results are presented in Section 4, they confirm the obtained theoretical results. Finally, some conclusions are given in Section 5.

2. Problem formulation

We consider a one-dimensional linear parabolic problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) - q(x)u(x,t) + f(x,t), \quad 0 < x < 1, \ \ t > 0, \qquad (2.1)\\
&0 < a_0 \leqslant a(x) \leqslant a_M, \quad q(x) \geqslant 0,\\
&u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1,\\
&u(0,t) = \mu_L(t), \quad u(1,t) = \mu_R(t), \quad 0 < t \leqslant T.
\end{aligned}$$

2.1. Predictor – corrector discrete scheme

We construct uniform grids for the x and t variables, respectively,

$$\begin{aligned}
&\omega_h = \{x_j : \ \ x_j = jh,\ j = 1,\dots,N-1\},\ h = 1/N, \quad \bar\omega_h = \omega_h \cup \{x_0, x_N\},\\
&\omega_\tau = \{t^n : t^n = n\tau,\ n = 0,\dots,N,\ N\tau = T\}.
\end{aligned}$$

On $\bar\omega_h \times \omega_\tau$ we define the discrete function $U_j^n = U(x_j, t^n)$ approximating the solution $u(x_j, t^n)$ at time t^n. The backward time, the forward and backward space difference quotients with respect to t and x are defined by

$$U_{\bar t}^n = \frac{U^n - U^{n-1}}{\tau}, \quad \partial_x U_j^n = \frac{U_{j+1}^n - U_j^n}{h}, \quad \partial_{\bar x} U_j^n = \frac{U_j^n - U_{j-1}^n}{h}.$$

Let U, V be grid functions, then we define the discrete inner product (U, V), the discrete L^2 norm $\|U\|$, and the discrete H^1 seminorm $|U|_b$ by

$$(U,V) = \sum_{j=1}^{J-1} U_j V_j h, \quad \|U\| = \sqrt{(U,U)}, \quad |U|_b^2 = \sum_{j=1}^{J} b_{j-1/2} \big(\partial_{\bar{x}} U_j\big)^2 h,$$

where $b(x)$ is a given positive function $b(x) > 0$, $x \in [0,1]$.

Recently, the predictor-corrector splitting and domain decomposition methods have been widely used to solve elliptic and parabolic PDEs in multidimensional regions and on graph structures (see [11, 14] for related discussions). The presented algorithm is based on the standard decomposition of the problem into simple 1D problems on each edge. New values of the solution at branch points are predicted by using an explicit algorithm. In order to improve the stability of the domain decomposition algorithms, a new approach was proposed in [9]. The main idea is to drop the values of the solution at the interface of two domains computed by the predictor algorithm and compute new values by using the basic implicit discrete algorithm (*correction* step). We note that predictor-corrector splitting fits well with the general framework from [15], since the predictor step — implemented as the explicit Euler algorithm — can be taken as a generating scheme, and the implicit Euler scheme at the corrector step can be regarded as a regularized algorithm in this framework.

Problem (2.1) is approximated by the following scheme. First, the domain $\bar{\Omega} = [0,1]$ is divided into K subdomains

$$\bar{\Omega} = \bigcup_{k=1}^{K} \bar{\Omega}_k, \quad \bar{\Omega}_k = [l_{k-1}, l_k], \quad l_0 = 0, \;\; l_K = 1.$$

In accordance with this domain decomposition, the space grid is decomposed into subgrids $\bar{\omega}_h = \bigcup\limits_{k=1}^{K} \bar{\omega}_h^k$, where

$$\bar{\omega}_h^k = \{x_j : \;\; x_j = jh, \, j = j_{k-1}, \dots, j_k\}, \;\; x_{j_k} = l_k.$$

- **Predictor step.** First, we compute in parallel the new values of the solution at the boundary points of the subdomains. The explicit-implicit Euler approximation is used to discretize the differential equation (2.1):

$$\frac{\tilde{U}_{j_k}^n - U_{j_k}^{n-1}}{\tau} = \partial_{\bar{x}} \big(a_{j_k + \frac{1}{2}} \partial_x U_{j_k}^{n-1} \big) - q_{j_k} \tilde{U}_{j_k}^n + f_{j_k}^{n-1}, \quad k = 1, \dots, K-1. \tag{2.2}$$

- **Domain decomposition step.** Second, the solutions on each subdomain are computed in parallel using the implicit finite difference scheme (2.3). The predicted values $\tilde{U}_{j_k}^n$ are used as the interface boundary conditions.

$$U_{\bar{t}}^n = \partial_{\bar{x}} \big(a_{j+\frac{1}{2}} \partial_x U_j^n \big) - q_j U_j^n + f_j^n, \quad \forall x_j \in \omega_h^k, \;\; k = 1, \dots, K-1, \tag{2.3}$$

$$U_{j_{k-1}}^n = \tilde{U}_{j_{k-1}}^n, \quad U_{j_k}^n = \tilde{U}_{j_k}^n,$$

$$U_0^n = \mu_L(t^n), \quad U_J^n = \mu_R(t^n).$$

- **Corrector step.** Third, using the implicit finite difference scheme (2.3) and taking the solution U^n computed at the second step, we update in parallel the values of the solution at the boundary points of the subdomains

$$U^n_{j_k,\bar t} = \partial_{\bar x}\big(a_{j_k+\frac12}\partial_x U^n_{j_k}\big) - q_{j_k}U^n_{j_k} + f^n_{j_k}, \quad k = 1,\dots,K-1. \tag{2.4}$$

Our goal is to investigate the stability and the accuracy of the discrete algorithm.

3. Convergence analysis

Let us denote the error functions of the discrete solution as $Z^n_j = U^n_j - u(x_j,t^n)$, $\tilde Z^n_j = \tilde U^n_j - u(x_j,t^n)$, $x_j \in \bar\omega_h$. By putting them into the finite-difference scheme we get a discrete problem for the error functions

$$Z^n_{j,\bar t} = \partial_{\bar x}\big(a_{j+\frac12}\partial_x Z^n_j\big) - q_j Z^n_j + \psi^n_j, \quad x_j \in \omega_h \setminus \{x_{j_k\pm1}, \ \ k = 1,\dots,K-1\}, \tag{3.1}$$

$$Z^n_{j_k\pm1,\bar t} = \partial_{\bar x}\big(a_{j_k\pm1+\frac12}\partial_x Z^n_{j_k\pm1}\big) - q_{j_k\pm1}Z^n_{j_k\pm1} + a_{j_k\pm\frac12}\frac{\tilde Z^n_{j_k} - Z^n_{j_k}}{h^2} + \psi^n_{j_k\pm1}, \tag{3.2}$$

$$\frac{\tilde Z^n_{j_k} - Z^{n-1}_{j_k}}{\tau} = \partial_{\bar x}\big(a_{j_k+\frac12}\partial_x Z^{n-1}_{j_k}\big) - q_{j_k}\tilde Z^n_{j_k} + +\varphi^n_{j_k}, \quad k = 1,\dots,K-1, \tag{3.3}$$

$$Z^n_0 = 0, \quad Z^n_J = 0, \quad Z^0_j = 0, \quad j = 0,\dots,J, \tag{3.4}$$

where ψ^n and φ^n are the truncation errors of the implicit and explicit-implicit Euler schemes

$$\psi^n_j = -u^n_{j,\bar t} + \partial_{\bar x}\big(a_{j+\frac12}\partial_x u^n_j\big) - q_j u^n_j + f^n_j, \ \ \varphi^n_j = -u^n_{j,\bar t} + \partial_{\bar x}\big(a_{j+\frac12}\partial_x u^{n-1}_j\big) - q_j u^n_j + f^{n-1}_j.$$

Let us assume that the initial data, the coefficients, and the exact solution of problem (2.1) are smooth enough. Using the Taylor formula, we can eatimate the truncation errors of the finite difference scheme (2.2)–(2.4) as

$$|\psi^n_j| \leqslant C(\tau + h^2), \quad 1 \leqslant j \leqslant J, \quad |\varphi^n_{j_k} - \psi^n_{j_k}| \leqslant C\tau, \quad 1 \leqslant k < K. \tag{3.5}$$

Next, we investigate the stability of the discrete solution of the finite difference scheme (2.2) – (2.4).

3.1. Stability in H^1

Our stability analysis uses the results of [14]. Let $\bar a_{j_k} = (a_{j_k-1/2}+a_{j_k+1/2})/2$, $k = 1,\dots,K-1$.

Lemma 3.1. *Let Z^n satisfy problem (3.1) – (3.4). Then*

$$E^n \leqslant E^0 + \tau\sum_{l=1}^{n}\Big\{\frac12\|\psi^l\|^2 + \sum_{k=1}^{K-1}\Big[\frac{2\tau}{h}\bar a_{j_k}(\psi^l_{j_k})^2 + \Big(\frac{a^2_M(K-1)\tau}{a_0h^2} + \frac h2\Big)\big(\varphi^l_{j_k} - \psi^l_{j_k}\big)^2\Big]\Big\}, \tag{3.6}$$

holds, where the functional E^n is defined as

$$E^n = |Z^n|^2_a + \|\sqrt{q}\,Z^n\|^2 + \frac{2\tau^2}{h}\sum_{k=1}^{K-1}\frac{\bar a_{j_k}}{1+\tau q_{j_k}}\big(\partial_x(a_{j_k-1/2}\partial_{\bar x}Z^n_{j_k})\big)^2.$$

Proof. Multiplying equations (3.1) – (3.2), respectively, by $2h\tau\partial_{\bar t}Z_j^n$, $j=1,\dots,J-1$ and adding the resulting equalities, we get

$$2\tau\|\partial_{\bar t}Z^n\|^2 = 2\tau\big(\partial_{\bar t}Z^n,\,\partial_x\big(a_{j-\frac12}\partial_{\bar x}Z_j^n\big)\big) + \frac{2\tau}{h}\sum_{k=1}^{K-1}(\tilde Z_{j_k}^n - Z_{j_k}^n)$$
$$\times\,(a_{j_k-\frac12}\partial_{\bar t}Z_{j_k-1}^n + a_{j_k+\frac12}\partial_{\bar t}Z_{j_k+1}^n) - 2\tau\big(qZ^n,\,\partial_{\bar t}Z^n\big) + 2\tau\big(\psi^n,\,\partial_{\bar t}Z^n\big). \qquad (3.7)$$

Applying equality $2Z^n = Z^n + Z^{n-1} + \tau\partial_{\bar t}Z^n$ and the well-known formula of summation by parts, we obtain

$$2\tau\big(\partial_{\bar t}Z^n,\,\partial_x\big(a_{j-\frac12}\partial_{\bar x}Z_j^n\big)\big) = -|Z^n|_a^2 + |Z^{n-1}|_a^2 - \tau^2\,|\partial_{\bar t}Z^n|_a^2, \qquad (3.8)$$

$$2\tau\big(qZ^n,\,\partial_{\bar t}Z^n\big) = \|\sqrt{q}\,Z^n\|^2 - \|\sqrt{q}\,Z^{n-1}\|^2 + \tau^2\|\sqrt{q}\,\partial_{\bar t}Z^n\|^2. \qquad (3.9)$$

Using the Schwarz inequality and the ε inequality, we have

$$2\tau\big(\psi^n,\,\partial_{\bar t}Z^n\big) \leqslant 2\tau\,\|\partial_{\bar t}Z^n\|^2 + \frac{\tau}{2}\,\|\psi^n\|^2. \qquad (3.10)$$

It remains to estimate the error introduced at the predictor step. From (3.1), (3.3) it follows that

$$(1+\tau q_{j_k})(\tilde Z_{j_k}^n - Z_{j_k}^n) = -\tau^2\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big) + \tau(\varphi_{j_k}^n - \psi_{j_k}^n).$$

We note that

$$a_{j_k-\frac12}\partial_{\bar t}Z_{j_k-1}^n + a_{j_k+\frac12}\partial_{\bar t}Z_{j_k+1}^n = h^2\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big) + 2\bar a_{j_k}\,\partial_{\bar t}Z_{j_k}^n$$
$$= h^2\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big) + 2\bar a_{j_k}\Big(\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big) + \psi_{j_k}^n\Big).$$

Applying the Schwarz inequality, the ε inequality, and the a priori estimate $q_j \geqslant 0$, we get the following estimates:

$$T_1 = -\frac{2\tau^3 h}{1+\tau q_{j_k}}\Big(\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\Big)^2,$$

$$T_2 = -\frac{4\tau^3\bar a_{j_k}}{h(1+\tau q_{j_k})}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)$$
$$= -\frac{2\tau^2\bar a_{j_k}}{h(1+\tau q_{j_k})}\Big(\big(\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\big)^2 - \big(\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^{n-1}\big)\big)^2 + \tau^2\big(\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\big)^2\Big),$$

$$T_3 = \frac{4\tau^3\bar a_{j_k}}{h(1+\tau q_{j_k})}\,\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\,\psi_{j_k}^n \leqslant \frac{2\tau^4\bar a_{j_k}}{h(1+\tau q_{j_k})}\big[\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\big]^2 + \frac{2\tau^2\bar a_{j_k}}{h}\,\big(\psi_{j_k}^n\big)^2,$$

$$T_4 = \frac{2\tau^2 h}{1+\tau q_{j_k}}\,\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)(\varphi_{j_k}^n - \psi_{j_k}^n) \leqslant \frac{2\tau^3 h}{1+\tau q_{j_k}}\Big(\partial_{\bar t}\partial_x\big(a_{j_k-\frac12}\partial_{\bar x}Z_{j_k}^n\big)\Big)^2 + \frac{\tau h}{2}\big(\varphi_{j_k}^n - \psi_{j_k}^n\big)^2.$$

Now we will estimate the last term in a different way than it was done in [14]. The following imbedding theorem $|Z_j^n| \leqslant \dfrac{1}{2\sqrt{a_0}}\,|Z^n|_a$ is used. Then we obtain

$$T_5 = \frac{4\tau^2\bar a_{j_k}}{h(1+\tau q_{j_k})}\,(\varphi_{j_k}^n - \psi_{j_k}^n)\,\partial_{\bar t}Z_{j_k}^n \leqslant \frac{4\tau^2 a_M}{h}\,|\varphi_{j_k}^n - \psi_{j_k}^n|\,|\partial_{\bar t}Z_{j_k}^n|$$
$$\leqslant \frac{\tau^2}{K-1}|\partial_{\bar t}Z^n|_a^2 + \frac{a_M^2(K-1)\tau^2}{a_0h^2}\big(\varphi_{j_k}^n - \psi_{j_k}^n\big)^2.$$

Substituting (3.8) – (3.10) into (3.7) and using the resulting inequalities T_m, $m = 1, \dots, 5$, we obtain

$$E^n \leqslant E^{n-1} + \tau \Big\{ \frac{1}{2} \|\psi^n\|^2 + \sum_{k=1}^{K-1} \Big[\frac{2\tau}{h} \bar{a}_{j_k} (\psi^n_{j_k})^2 + \Big(\frac{a_M^2 (K-1)\tau}{a_0 h^2} + \frac{h}{2} \Big) \big(\varphi^n_{j_k} - \psi^n_{j_k} \big)^2 \Big] \Big\}.$$

Repeated application yields the desired result (3.6). □

Remark 3.1. It follows from Lemma 3.1 that the finite difference scheme (2.2)–(2.4) is unconditionally A-stable. Note that in [14] only the ρ-stability with $\rho = 1 + c\tau$ was proved.

Using the stability estimate given in Lemma 3.1 and the estimates of the truncation errors (3.5), we get the following convergence result in the H^1 norm.

Theorem 1. *Let U be the solution of the finite difference scheme (2.2)–(2.4) and $u(x,t)$ be the solution of the differential problem (2.1). Then*

$$|U^n - u^n|_a \leqslant C t^n \Big(\tau + h^2 + \frac{\tau\sqrt{\tau}}{h} \Big). \tag{3.11}$$

3.2. Stability in L_∞

In this section, we investigate the asymptotic optimality of the error estimate (3.11). For simplicity of notations, we restrict ourselves to the case $K = 1$ and assume that $a(x) \equiv 1$, $q(x) \equiv= 0$. Let us denote $m = j_1$ and define

$$g = \max_{1 \leqslant n \leqslant N} \max_{x_j \in \omega_h} |\psi^n_j|, \quad \tilde{g} = \max_{1 \leqslant n \leqslant N} |\varphi^n_m - \psi^n_m|, \quad G = \tilde{g} + g.$$

Then we consider the stationary analogue of the predictor-corrector algorithm

$$\begin{cases} -\partial_x \partial_{\bar{x}} W_j = g, & 1 \leqslant j < m, \\ W_0 = 0, \quad W^p_m = \tilde{W}_m, \end{cases} \qquad \begin{cases} -\partial_x \partial_{\bar{x}} W_j = g, & m < j < J, \\ W^p_m = \tilde{W}_m, \quad W_J = 0, \end{cases} \tag{3.12}$$

$$\frac{\tilde{W}_m - W_m}{\tau} = \partial_x \partial_{\bar{x}} W_m + G,$$

$$-\partial_x \partial_{\bar{x}} W_m = g, \quad j = m.$$

The solution of (3.12) can be written in explicit form

$$W_j = \begin{cases} A \dfrac{x_j}{x_m} + \dfrac{g}{2} x_j (1 - x_j), & 0 \leqslant j \leqslant m, \\ A \dfrac{1 - x_j}{1 - x_m} + \dfrac{g}{2} x_j (1 - x_j), & m < j \leqslant J, \end{cases} \qquad A = \frac{2\tau (x_m - x_m^2 - h/2)}{h} \tilde{g}.$$

We see that the function W is bounded by

$$|W_j| \leqslant C \Big(\tau + h^2 + \frac{\tau^2}{h} \Big),$$

and this estimate predicts a better convergence rate than it follows from the a priori error estimate (3.11).

4. Numerical experiments

We have applied the developed finite difference scheme (2.2)–(2.4) to the model problem (2.1), where we take $T = 1$ and the following coefficients $a(x) \equiv 1$, $q(x) \equiv= 0$. The source term f is chosen such that the exact solution is given by $u = \exp(t)\exp(2x)$. The computational domain is decomposed into four subdomains at points $l_1 = 0.25$, $l_2 = 0.5$, $l_3 = 0.8$.

In the numerical experiments we have tested the convergence of the algorithm for different values of h and τ. The error of the solution is presented in the uniform norm

$$\|Z^N\|_\infty = \max_{1 \leqslant j < J} |U_j^N - u(x_j, T)|.$$

The results are given in Table 4.1.

Table 4.1. **Error of the solution of the predictor–corrector scheme at $T = 1$**

N	$\tau = 0.02$	$\tau = 0.01$	$\tau = 0.005$	$\tau = 0.0025$	$\tau = 0.00125$
50	0.11833	0.02791	0.05522	0.00045	0.00058
100	0.24904	0.06204	0.01444	0.02963	0.00036
200	0.57610	0.12636	0.03142	0.00731	0.00152
400	1.13165	0.25529	0.06508	0.01582	0.00241

The results show that the global error in most cases converges as $O(\tau^2/h)$, i.e., the error introduced at the prediction step dominates the total error.

It follows from Theorem 1 that the predictor–corrector scheme is unconditionally stable with respect to the initial condition in the energy norm, i.e., $E_h^n \leqslant E_h^{n-1}$ for $n > 0$, where for the test problem

$$E^n = |Z^n|_a^2 + + \frac{2\tau^2}{h} \sum_{k=1}^{3} \left(\partial_x \partial_{\bar{x}} Z_{j_k}^n\right)^2.$$

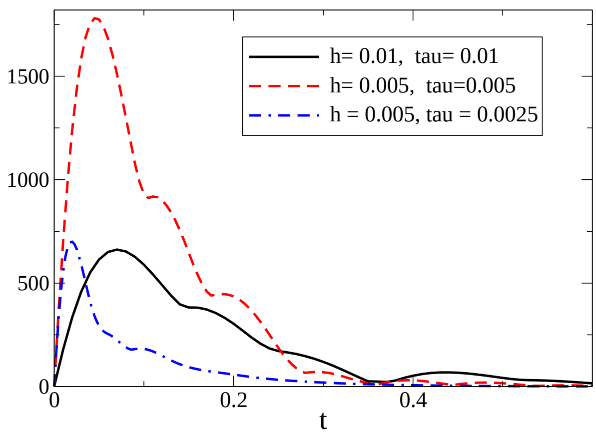

Fig. 4.1. Dynamics of the error $\|Z^n\|_\infty$ in time

But in the case of nonlinear problems we are interested to get error estimates also in the uniform norm. In Fig. 4.1, we plot the dynamics of the error $\|Z^n\|_\infty$ in time, when zero initial conditions are disturbed only at one grid point $Z^0_{J/2} = 1$ and the source term $f \equiv 0$.

5. Conclusions

The predictor – corrector algorithm splits the computations for each time step into a set of discrete one-dimensional parabolic problems which can be solved efficiently by the standard factorization algorithm. This leads to a good parallelism of the algorithm.

It is wellknown that the simple prediction strategy loses the unconditional stability of the backward Euler algorithm. By the energy estimates method we have proved that the predictor-corrector algorithm is unconditionally stable. The relation between the space and time steps is still required in order to get the convergence of the discrete solution, since the truncation error introduced at the prediction step is of order $O(\tau^2/h)$.

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