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NUMERICAL SOLUTION OF SYSTEMS OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

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Abstract — A survey is given of current research into the numerical solution of time-independent systems of second-order differential equations whose diffusion coefficients are small parameters. Such problems are in general singularly perturbed. The equations in these systems may be coupled through their reaction and/or convection terms. Only numerical methods whose accuracy is guaranteed for all values of the diffusion parameters are considered here. Some new unifying results are also presented.

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1. Introduction

Systems of linear reaction — convection — diffusion equations (where reaction or convection terms, but not both, may be absent) appear in various applications. For example, the Oseen equations, which arise when using a fixed-point iteration to solve the Navier-Stokes equations written in velocity-pressure form, are a convection — diffusion system; while a linearization of the Navier-Stokes equations written in rotation form will yield a reaction — diffusion system. For large Reynolds number (i.e., small viscosity coefficient) these systems are singularly perturbed.

A search of the research literature shows that since 2003 there has been a surge of interest in numerical methods for reaction — convection — diffusion systems. Several papers prove convergence results that are uniform in the singular perturbation parameter(s). Convergence results of this type are the focus of this survey. Only stationary problems are considered here. As well as summarizing the current state of knowledge in this area, we also prove some new results that unify previously published analyses.

Let $\ell \geqslant 2$ be an integer. Let Ω be a domain in $\mathbb{R}$ or $\mathbb{R}^2$. Let ε_i, for $i = 1, \dots, \ell$, be a set of parameters that satisfy $0 < \varepsilon_i \ll 1$ for all i. (In some cases below we shall take $\varepsilon_i = \varepsilon$ for all i.) Consider the system of ℓ reaction — convection — diffusion problems

$$\boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\Delta \boldsymbol{u} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}. \tag{1.1}$$

where $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$ with matrix-valued functions $\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2 : \bar{\Omega} \to \mathbb{R}^{\ell,\ell}$, and vector-valued $\boldsymbol{f}, \boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^{\ell}$. Here $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f}$ and $\boldsymbol{g}$ are given while $\boldsymbol{u}$ is unknown.

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We divide (1.1) into three subclasses:

(i) reaction — diffusion: $-\mathrm{diag}\,(\boldsymbol{\varepsilon})\Delta\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$;

(ii) weakly coupled convection — reaction — diffusion: $-\mathrm{diag}\,(\boldsymbol{\varepsilon})\Delta\boldsymbol{u} - \mathrm{diag}\,(\boldsymbol{b}) \cdot \nabla\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$;

(iii) strongly coupled convection — diffusion: $-\mathrm{diag}\,(\boldsymbol{\varepsilon})\Delta\boldsymbol{u} - \boldsymbol{B} \cdot \nabla\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$.

It will be seen later that each subclass has its own peculiarities. Further assumptions on the data will of course be required; these will be stated as needed later.

We restrict our considerations to linear problems. Using standard linearization techniques the results can be generalized to certain classes of quasilinear and semilinear problems.

All published convergence results that are uniform in the perturbation parameters are at present confined to methods that apply special layer-adapted meshes — e.g., those of Bakhvalov and Shishkin types — to these problems. Thus we shall confine our attention to methods that use such meshes.

Our focus is on robust numerical methods that converge in the discrete maximum norm, uniformly in the parameters $\varepsilon_1, \ldots, \varepsilon_\ell$. Such methods are said to be *uniformly convergent* in this paper.

Notation. Vectors are assumed to be columns. The superscript $^\top$ means transpose.

We write $\|\cdot\|_\infty$ for the maximum norm on $C(\bar\Omega)$. If $\boldsymbol{v} = (v_1, v_2, \ldots, v_\ell)^\top$ is any vector function in $C(\bar\Omega)^\ell$, set $\|\boldsymbol{v}\|_\infty := \max\limits_{i=1,\ldots,\ell} \|v_i\|_\infty$. On any mesh ω, the discrete analogues of these norms are denoted by $\|\cdot\|_{\infty,\omega}$.

Let $\bar\omega_N : 0 = x_0 < x_1 < x_2 \cdots < x_N = 1$ be an arbitrary mesh on the interval $[0,1]$. The set of interior mesh points is $\omega_N := \{x_1, x_2, \ldots, x_{N-1}\}$. Set $h_i = x_i - x_{i-1}$ and $\hbar_i = (h_i + h_{i+1})/2$ for each x_i. Given an arbitrary mesh function $\{v_i\}_{i=0}^N$, define the difference operators

$$v_{x;i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar x;i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{\hat x;i} = \frac{v_{i+1} - v_i}{\hbar_i}.$$

Let $\mathbb{R}_0^{N+1}$ denote the set of all mesh functions that vanish at x_0 and x_N.

For any function $g \in C[0,1]$ and each $x_i \in \omega_N$ we set $g_i = g(x_i)$; if $\boldsymbol{g} \in C[0,1]^\ell$ then set $\boldsymbol{g}_i = \boldsymbol{g}(x_i) = (g_{1,i}, \ldots, g_{\ell,i})^\top$.

Finally, C denotes a generic constant that is independent of all small diffusion parameters and of any mesh; it can take different values at different places in the paper.

2. Reaction — diffusion problems in one dimension

2.1. Scalar problems. Before investigating systems of reaction — diffusion equations that are posed in one dimension, we remind the reader of the main properties of a single equation of this type. Consider the singularly perturbed reaction — diffusion two-point boundary value problem

$$\mathcal{L}u := -\varepsilon^2 u'' + ru = f \quad \text{in} \quad (0,1), \quad u(0) = u(1) = 0, \tag{2.1}$$

where $0 < \varepsilon \ll 1$, and $r, f \in C^2[0,1]$ with $r(x) \geqslant \varrho^2$, $\varrho > 0$ for $x \in [0,1]$. Note how here and subsequently we use ε^2 instead of ε as the diffusion coefficient when discussing reaction — diffusion problems; this is to simplify the notation, since the analysis of such problems is expressed most naturally in terms of the square root of the diffusion coefficient. The solution of (2.1) typically exhibits a boundary layer of width $\mathcal{O}(\varepsilon \ln 1/\varepsilon)$ at each endpoint of the domain. More precisely, the following bounds for u and its lower-order derivatives can be shown using the techniques from [27, Chapter 6]:

$$\left|u^{(\ell)}(x)\right| \leqslant C\left\{\varepsilon^{\min\{0,2-\ell\}} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho x/\varepsilon} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho(1-x)/\varepsilon}\right\} \quad \text{for} \quad \ell = 0,\dots,4. \tag{2.2}$$

Let (2.1) be discretized on the arbitrary mesh $\bar\omega_N$ by the central difference scheme

$$\left[Lu^N\right]_i := -\varepsilon^2 u^N_{\bar x\hat x;i} + r(x_i)u_i^N = f(x_i) \quad \text{for} \quad i = 1,\dots,N-1, \quad u_0^N = u_N^N = 0, \tag{2.3}$$

where the discrete solution is $u_0^N, u_1^N, \dots, u_N^N$.

2.1.1. Stability. It is well known (see, e.g., [31, 32]) that (2.1) satisfies the following comparison principle:

Lemma 2.1. *Let $v, w \in C^2(0,1)\cap C[0,1]$. Then*

$$\left.\begin{array}{l}\mathcal{L}v \geqslant \mathcal{L}w \quad \text{in} \quad (0,1),\\ v(0)\geqslant w(0), \quad v(1)\geqslant w(1)\end{array}\right\} \quad \Longrightarrow \quad v \geqslant w \quad \text{on} \quad [0,1].$$

As the matrix associated with (2.3) is easily seen to be an M-matrix, it follows that the discrete problem enjoys a similar property:

Lemma 2.2. *Let $v, w \in \mathbb{R}^{N+1}$ Then*

$$\left.\begin{array}{l}[Lv]_i \geqslant [Lw]_i \quad \text{for} \quad i = 1,\dots,N-1,\\ v_0 \geqslant w_0, \quad v_N \geqslant w_N\end{array}\right\} \quad \Longrightarrow \quad v_i \geqslant w_i \quad \text{for} \quad i = 0,\dots,N.$$

To investigate the numerical method, the most effective approach is to work with continuous and discrete Green's functions. Let $\mathcal{G}$ be the Green's function associated with the differential operator $\mathcal{L}$ and the point $\xi \in (0,1)$. Then — see [4, 12, 15] for a detailed derivation — one has

$$\mathcal{G} \geqslant 0, \quad \int_0^1 \left|(r\mathcal{G})(x)\right| dx \leqslant 1, \quad \int_0^1 \left|\mathcal{G}'(x)\right| dx \leqslant \frac{1}{\varepsilon\varrho}, \quad \int_0^1 \left|\mathcal{G}''(x)\right| dx \leqslant \frac{2}{\varepsilon^2}. \tag{2.4}$$

For the discrete analogue G of $\mathcal{G}$ that is associated with the difference operator L and a mesh node $\xi \in \omega_N$, one has

$$G_i \geqslant 0, \quad \sum_{i=1}^{N-1} \hbar_i \left|r_i G_i\right| \leqslant 1, \quad \sum_{i=1}^{N} \left|G_{x;i}\right| \leqslant \frac{1}{\varepsilon\varrho}, \quad \sum_{i=1}^{N-1} \left|G_{\hat x\bar x;i}\right| \leqslant \frac{2}{\varepsilon^2}. \tag{2.5}$$

These bounds imply for example that

$$\|v\|_\infty \leqslant \|\mathcal{L}v/r\|_\infty \quad \text{for all} \quad v \in C^2[0,1] \text{ with } v(0) = v(1) = 0 \tag{2.6}$$

and

$$\|v\|_{\infty,\bar\omega_N} \leqslant \|Lv/r\|_{\infty,\omega_N} \quad \text{for all} \quad v \in \mathbb{R}_0^{N+1}.$$

2.1.2. Error bounds. Various error bounds are given in the literature for the solution of (2.3). The most general of these [15] invokes (2.5) to prove that

$$\left\|u - u^N\right\|_{\infty,\bar\omega_N} \leqslant C\vartheta_{rd,\varepsilon}(\omega_N)^2 \tag{2.7}$$

where

$$\vartheta_{rd,\varepsilon}(\omega_N) := \max_{k=1,\dots,N-1} \int\limits_{x_{k-1}}^{x_k} \left[1 + \varepsilon^{-1}\mathrm{e}^{-\varrho x/(2\varepsilon)} + \varepsilon^{-1}\mathrm{e}^{-\varrho(1-x)/(2\varepsilon)}\right] dx.$$

Furthermore, one can derive an *a posteriori* bound [12, 19] by appealing to (2.4):

$$\left\|u - u^N\right\|_\infty \leqslant 2 \max_{i=1,\dots,N} \frac{h_i^2}{4\varepsilon^2} |q_i| + \max_{i=1,\dots,N} \frac{h_i}{2\varrho\varepsilon} |q_i - q_{i-1}| + \frac{1}{\varrho^2} \left\|q - q^I\right\|_{\infty,\bar\omega}$$

where $q := f - ru^N$.

2.1.3. Mesh construction. We now concentrate on special meshes that are designed specifically for (2.1). Such meshes will — partly or completely — resolve the boundary layers appearing in u in order to attain convergence bounds for (2.3) that are uniform in the singular perturbation parameter ε. Our exposition begins with the Bakhvalov mesh then continues with the more recent Shishkin mesh, which is simpler but yields less accurate computed solutions.

Bakhvalov meshes [7] for the discretization of (2.1) are constructed as follows. Choose parameters $q \in (0, 1/2)$ and $\sigma > 0$; here q is (roughly) the ratio of the number mesh points used to resolve a single layer to the total number of mesh points, while σ determines the grading of the mesh inside the layer. Away from the layer an equidistant mesh in x is used. The transition point τ between the graded and equidistant portions of the mesh is determined by the requirement that the resulting mesh generating function φ is $C^1[0, 1]$. That is, define

$$\varphi(\xi) = \begin{cases} \chi(\xi) := -\dfrac{\sigma\varepsilon}{\beta} \ln\left(1 - \dfrac{\xi}{q}\right) & \text{for } \xi \in [0, \tau], \\ \pi(\xi) := \chi(\tau) + \chi'(\tau)(\xi - \tau) & \text{for } \xi \in [\tau, 1/2], \\ 1 - \varphi(1 - \xi) & \text{for } \xi \in (1/2, 1], \end{cases}$$

where the transition point τ must satisfy the nonlinear equation

$$\chi'(\tau) = \frac{1 - 2\chi(\tau)}{1 - 2\tau},$$

which cannot be solved exactly. The mesh points are then defined by $x_i = \varphi(i/N)$ for $i = 0, \dots, N$. The mesh generating function and the Bakhvalov mesh are shown in Figure 2.1.

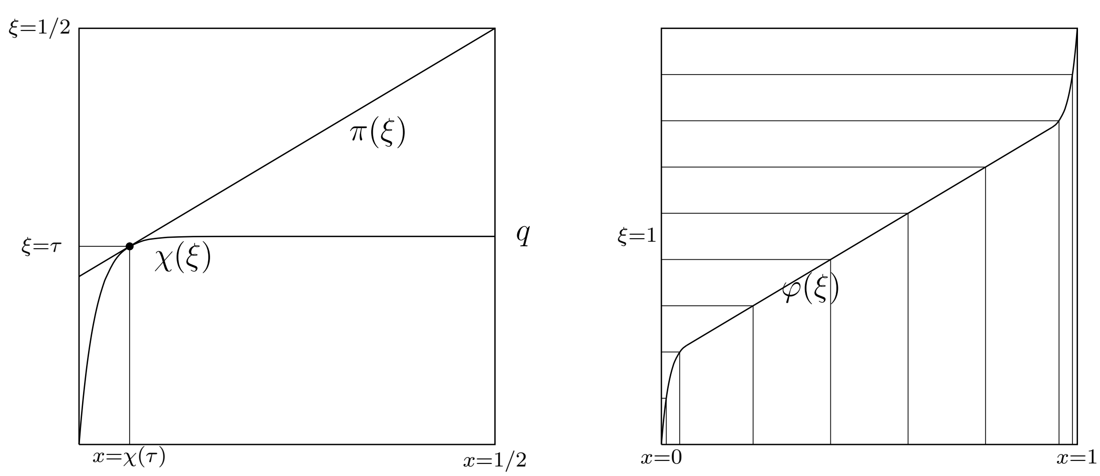

Fig. 2.1. Bakhvalov mesh: the mesh generating function (left) and the x-mesh generated (right)

Alternatively, the Bakhvalov mesh can be generated [14] by equidistributing the function

$$M_{Ba}(x) = \max\Big\{1,\ K\varepsilon^{-1}\mathrm{e}^{-\varrho x/\varepsilon\sigma},\ K\varepsilon^{-1}\mathrm{e}^{-\varrho(1-x)/\varepsilon\sigma}\Big\}$$

with positive constants K and σ, i.e., the mesh points x_i are chosen such that

$$\int\limits_{x_{i-1}}^{x_i} M_{Ba}(x)dx = \frac{1}{N}\int\limits_0^1 M_{Ba}(x)\,dx \quad \text{for all } i.$$

The parameter K determines the number of mesh points used to resolve the layers. For the Bakhvalov mesh ω_N^B we have

$$\vartheta_{rd,\varepsilon}(\omega_N^B) \leqslant CN^{-1} \ \text{ if } \sigma \geqslant 2,$$

because $\int_0^1 M_{Ba}(x)\,dx \leqslant C$.

Shishkin meshes [27, 33] appear often in the research literature because of their relative simplicity — they are piecewise equidistant. Let $q \in (0, 1/2)$ and $\sigma > 0$ be mesh parameters. Many authors simply take $q = 1/4$. Set

$$\tau = \min\left\{q,\ \frac{\sigma\varepsilon}{\varrho}\ln N\right\}.$$

Assume that qN is an integer. Then the Shishkin mesh ω_N^S for problem (1.1) divides each of the intervals $[0, \tau]$ and $[1-\tau, 1]$ into qN equidistant subintervals, while $[\tau, 1-\tau]$ is divided into $(1-2q)N$ equidistant subintervals. Figure 2.2 depicts a Shishkin mesh for (2.1) with 32 mesh intervals. For this mesh it is straightforward to show that

$$\vartheta_{rd,\varepsilon}(\omega_N^S) \leqslant CN^{-1}\ln N \ \text{ if } \sigma \geqslant 2.$$

In the above estimates for $\vartheta_{rd,\varepsilon}$ on the Bakhvalov and Shishkin meshes, the constants C increase slowly with σ, so in practice σ is not chosen larger than the theory demands — recall that by (2.7) the error in the computed solution is bounded by $C\vartheta_{rd,\varepsilon}(\omega_N)^2$.

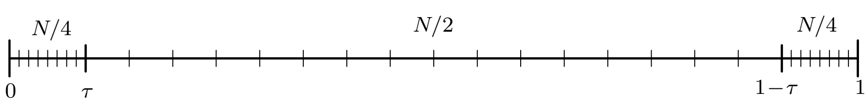

Fig. 2.2. Shishkin mesh for scalar reaction — diffusion problem

2.2. Systems of reaction — diffusion equations. We now leave the scalar equation (2.1) and move on to systems of equations of this type.

2.2.1. The continuous problem. Find $\boldsymbol{u} \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ such that

$$\boldsymbol{\mathfrak{L}u} := -\boldsymbol{E}^2\boldsymbol{u''} + \boldsymbol{Au} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \tag{2.8}$$

where $\boldsymbol{E} = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_\ell)$ and the small parameter ε_k is in $(0, 1]$ for $k = 1, \dots, \ell$. We set $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{f} = (f_i)$. Written out in full, (2.8) is

$$\begin{aligned}
-\varepsilon_1^2 u_1'' + a_{11}u_1 + a_{12}u_2 + \cdots + a_{1\ell}u_\ell &= f_1 \ \text{ in } (0,1), \qquad u_1(0) = u_1(1) = 0,\\
-\varepsilon_2^2 u_2'' + a_{21}u_1 + a_{22}u_2 + \cdots + a_{2\ell}u_\ell &= f_2 \ \text{ in } (0,1), \qquad u_2(0) = u_2(1) = 0,\\
&\ \ \vdots\\
-\varepsilon_\ell^2 u_\ell'' + a_{\ell 1}u_1 + a_{\ell 2}u_2 + \cdots + a_{\ell\ell}u_\ell &= f_\ell \ \text{ in } (0,1), \qquad u_\ell(0) = u_\ell(1) = 0.
\end{aligned}$$

Stability. Assume that all entries a_{ij} of the coupling matrix $\boldsymbol{A}$ lie in $C[0,1]$ and that $\boldsymbol{A}$ has positive diagonal entries. Assume likewise that all f_i lie in $C[0,1]$. Our analysis follows that of [23] and is based on the stability properties of Section 2.1.

For each k, the k^{th} equation of the system (2.8) can be written as

$$-\varepsilon_k^2 u_k'' + a_{kk} u_k = f_k - \sum_{\substack{m=1\\ m\neq k}}^{\ell} a_{km} u_m.$$

The stability inequality (2.6) and a triangle inequality then yield

$$\|u_k\|_\infty - \sum_{\substack{m=1\\ m\neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \|u_m\|_\infty \leqslant \left\| \frac{f_k}{a_{kk}} \right\|_\infty \tag{2.9}$$

Define the $\ell \times \ell$ constant matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{A}) = (\gamma_{ij})$ by

$$\gamma_{ii} = 1 \quad \text{and} \quad \gamma_{ij} = -\left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty \quad \text{for} \quad i \neq j. \tag{2.10}$$

Suppose that $\boldsymbol{\Gamma}$ is inverse-monotone, i.e., that $\boldsymbol{\Gamma}$ is invertible and

$$\boldsymbol{\Gamma}^{-1} \geqslant 0; \tag{2.11}$$

this can often be verified by using the M-criterion [32]. Then (2.9) gives immediately a bound on $\|\boldsymbol{u}\|_\infty$ in terms of the data $\boldsymbol{A}$ and $\boldsymbol{f}$ and one obtains the following stability result for the operator $\boldsymbol{\mathfrak{L}}$:

Theorem 2.1. *Assume that the matrix $\boldsymbol{A}$ has positive diagonal entries. Suppose also that all entries of $\boldsymbol{A}$ lie in $C[0,1]$. Assume that $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for $i = 1,\dots,\ell$ one has*

$$\|v_i\|_\infty \leqslant \sum_{k=1}^{\ell} \left(\boldsymbol{\Gamma}^{-1}\right)_{ik} \left\| \frac{(\boldsymbol{\mathfrak{L}}\boldsymbol{v})_k}{a_{kk}} \right\|_\infty$$

for any function $\boldsymbol{v} = (v_1,\dots,v_\ell)^\top \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ with $\boldsymbol{v}(0) = \boldsymbol{v}(1) = \boldsymbol{0}$.

Corollary 2.1. *Under the hypotheses of Theorem 2.1 the boundary value problem (2.8) has a unique solution $\boldsymbol{u}$, and $\|\boldsymbol{u}\|_\infty \leqslant C \|\boldsymbol{f}\|_\infty$ for some constant C.*

Remark 2.1. Thus the operator $\boldsymbol{\mathfrak{L}}$ is maximum-norm stable although in general it is not inverse-monotone — the hypotheses of Theorem 2.1 do not in general imply that (2.8) obeys a maximum principle.

Remark 2.2. The obvious analogue of the stability inequality (2.6) is also valid for scalar reaction — diffusion problems posed in domains Ω lying in $\mathbb{R}^d$ for $d > 1$. Consequently Theorem 2.1 holds true also for reaction — diffusion systems posed on $\Omega \subset \mathbb{R}^d$ with $d > 1$.

Remark 2.3. Most publications in the literature [9, 22, 24–26, 38] assume that on each row of the coupling matrix $\boldsymbol{A}$ one has $a_{ij} \leqslant 0$ for $i \neq j$ and $\sum_j a_{ij} \geqslant \alpha$ on $[0,1]$ for some positive constant α. These properties imply that $\boldsymbol{A}$ is an M-matrix and the operator $\boldsymbol{\mathfrak{L}}$ obeys a maximum principle. It follows (use the M-criterion with a constant test vector) that the matrix $\mathbf{\Gamma}(\boldsymbol{A})$ is also an M-matrix. Thus the stability result from [23] that is stated in Theorem 2.1 above is more general than the stability results in these publications.

In [7, 9] the coupling matrix $\boldsymbol{A}$ is assumed to be coercive, viz.,

$$\boldsymbol{v}^\top \boldsymbol{A}(x)\boldsymbol{v} \geqslant \mu^2 \boldsymbol{v}^\top \boldsymbol{v} \quad \text{for all } \ \boldsymbol{v} \in \mathbb{R}^\ell \text{ and } x \in [0,1], \tag{2.12}$$

where μ is some positive constant. The following new result, which slightly generalizes [37], establishes a connection between (2.11) and (2.12).

Theorem 2.2. *Assume that $\boldsymbol{A}$ has positive diagonal entries and that $\mathbf{\Gamma}$ is inverse-monotone. Then there exists a constant diagonal matrix $\boldsymbol{D}$ and a constant $\alpha > 0$ such that*

$$\boldsymbol{v}^\top \boldsymbol{D}\boldsymbol{A}(x)\boldsymbol{v} \geqslant \alpha \boldsymbol{v}^\top \boldsymbol{v} \quad \textit{for all } \ \boldsymbol{v} \in \mathbb{R}^\ell, \ x \in [0,1],$$

i.e., the matrix $\boldsymbol{D}\boldsymbol{A}$ is coercive uniformly in x.

Proof. As $\mathbf{\Gamma}^{-1}$ exists, one can define $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^\ell$ by $\mathbf{\Gamma}\boldsymbol{y} = \mathbf{1}$ and $\mathbf{\Gamma}^\top \boldsymbol{z} = \mathbf{1}$. Then $\mathbf{\Gamma}^{-1} \geqslant 0$ implies that $y_i > 0$ and $z_i > 0$ for $i = 1, \dots, \ell$. Define the matrix-valued function $\boldsymbol{G} = (g_{ij})$ by $g_{ij}(x) = z_i a_{ij}(x) y_j$ for all i and j. Observe that both $\boldsymbol{G}$ and $\boldsymbol{G}^\top$ are strictly diagonally dominant:

$$g_{ii}(x) - \sum_{j\neq i} |g_{ij}(x)| \geqslant a_{ii}(x) z_i \left\{ y_i - \sum_{j\neq i} \left\| \frac{a_{ij}}{a_{ii}} \right\|_\infty y_j \right\} = a_{ii}(x) z_i \sum_j \gamma_{ij} y_j = a_{ii}(x) z_i > 0,$$

$$g_{ii}(x) - \sum_{j\neq i} |g_{ji}(x)| \geqslant a_{ii}(x) y_i \left\{ z_i - \sum_{j\neq i} \left\| \frac{a_{ji}}{a_{ii}} \right\|_\infty z_j \right\} = a_{ii}(x) y_i \sum_j \gamma_{ji} z_j = a_{ii}(x) y_i > 0.$$

Thus $(\boldsymbol{G} + \boldsymbol{G}^\top)/2$ is strictly diagonally dominant and symmetric. Hence there exists a constant $\beta > 0$ such that

$$\boldsymbol{v}^\top \boldsymbol{G}\boldsymbol{v} = \boldsymbol{v}^\top \boldsymbol{G}^\top \boldsymbol{v} = \boldsymbol{v}^\top \frac{\boldsymbol{G} + \boldsymbol{G}^\top}{2} \boldsymbol{v} \geqslant \beta \boldsymbol{v}^\top \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \mathbb{R}^\ell.$$

Define the diagonal matrix $\boldsymbol{D} = (d_{ii})$ by $d_{ii} = z_i / y_i$ for all i. Then

$$\boldsymbol{v}^\top \boldsymbol{D}\boldsymbol{A}(x)\boldsymbol{v} = \sum_{i,j} d_{ii} a_{ij} v_i v_j = \sum_{i,j} g_{ij} \frac{v_i}{y_i} \frac{v_j}{y_j} \geqslant \beta \sum_i \left(\frac{v_i}{y_i} \right)^2 \geqslant \alpha \sum_i {v_i}^2.$$

$\square$

Remark 2.4. Our proof of Theorem 2.2 remains valid for reaction — diffusion problems posed in $\Omega \subset \mathbb{R}^d$ with $d > 1$.

Remark 2.5. As multiplication on the left by a positive diagonal matrix neither changes the structure of (2.8) nor alters $\mathbf{\Gamma}(\boldsymbol{A})$, Theorem 2.2 implies that, without loss of generality, if $\boldsymbol{A}$ has positive diagonal entries then whenever (2.11) is satisfied one can assume that (2.12) holds true also. Thus the hypothesis that (2.12) alone holds true is more general than an assumption that (2.11) is valid, but the only analyses [7, 9] that are based solely on (2.12) are restricted to the special case $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_\ell$; see Section 3.

Derivative bounds and solution decomposition. Let the coupling matrix $\boldsymbol{A}(x)$ be strictly diagonally dominant for all $x \in [0,1]$. Then $\boldsymbol{A}$ has positive diagonal entries and there exists a constant β such that

$$\sum_{\substack{k=1\\k\neq i}}^{\ell} \left\| \frac{a_{ik}}{a_{ii}} \right\|_\infty \leqslant \beta < 1 \quad \text{for} \quad i = 1, \ldots, \ell. \tag{2.13}$$

An application of the M-criterion with a constant test vector shows that $\boldsymbol{\Gamma}^{-1} \geqslant 0$. Define $\kappa = \kappa(\beta) > 0$ by

$$\kappa^2 := (1-\beta) \min_{i=1,\ldots,\ell} \min_{x\in[0,1]} a_{ii}(x).$$

For arbitrary $\varepsilon \in (0,1]$ and $0 \leqslant x \leqslant 1$, set

$$\mathcal{B}_\varepsilon(x) := \mathrm{e}^{-\kappa x/\varepsilon} + \mathrm{e}^{-\kappa(1-x)/\varepsilon}.$$

For simplicity in our presentation we assume that

$$\varepsilon_1 \leqslant \varepsilon_2 \leqslant \cdots \leqslant \varepsilon_\ell \quad \text{and} \quad \varepsilon_\ell \leqslant \frac{\kappa}{4};$$

the first inequality can always be achieved by renumbering the equations while the second provides a threshold value for the validity of our analysis.

The next result generalizes (2.2).

Theorem 2.3 [23]. *Let $\boldsymbol{A}$ and $\boldsymbol{f}$ be twice continuously differentiable. Then the solution $\boldsymbol{u}$ of* (2.8) *can be decomposed as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are defined by*

$$-\boldsymbol{E}^2\boldsymbol{v}'' + \boldsymbol{A}\boldsymbol{v} = \boldsymbol{f} \quad \text{in} \quad (0,1), \quad \boldsymbol{v}(0) = \boldsymbol{A}(0)^{-1}\boldsymbol{f}(0), \quad \boldsymbol{v}(1) = \boldsymbol{A}(1)^{-1}\boldsymbol{f}(1),$$

and

$$-\boldsymbol{E}^2\boldsymbol{w}'' + \boldsymbol{A}\boldsymbol{w} = \boldsymbol{0} \;\; \text{in} \;\; (0,1), \quad \boldsymbol{w}(0) = -\boldsymbol{v}(0), \quad \boldsymbol{w}(1) = -\boldsymbol{v}(1).$$

For all $x \in \Omega$, the derivatives of $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy the bounds

$$\left\|v_i^{(k)}\right\|_{\bar\Omega} \leqslant C\left(1+\varepsilon_i^{2-k}\right) \quad \text{for} \quad k = 0,1,\ldots,4 \quad \text{and} \quad i = 1,\ldots,\ell,$$

$$\left|w_i^{(k)}(x)\right| \leqslant C \sum_{m=i}^{\ell} \varepsilon_m^{-k} \mathcal{B}_{\varepsilon_m}(x) \quad \text{for} \quad k = 0,1,2 \quad \text{and} \quad i = 1,\ldots,\ell,$$

and

$$\left|w_i^{(k)}(x)\right| \leqslant C\varepsilon_i^{2-k} \sum_{m=1}^{\ell} \varepsilon_m^{-2} \mathcal{B}_{\varepsilon_m}(x) \quad \text{for} \quad k = 3,4 \text{and} \quad i = 1,\ldots,\ell.$$

The bounds of Theorem 2.3 say that each component u_i of the solution $\boldsymbol{u}$ can be written as a sum of a smooth part (whose low-order derivatives are bounded independently of the small parameters) and ℓ overlapping layers, though the full effect of these layers is manifested only in derivatives of order at least 3.

Figure 2.3 displays a typical solution in the case $\ell = 2$. The first plot shows the two components on the entire domain $[0,1]$; all that is apparent is that each component has layers at $x = 0$ and $x = 1$. The second plot is a blow-up of the layer at $x = 0$ (the layer at

$x = 1$ is similar) and we observe that while u_2 has a standard layer, the layer in u_1 appears to have a kink close to $x = 0$. Consequently a third plot is provided, which zooms further into the solution at $x = 0$, and now it is clear that the layer in u_1 is a sum of two separate layers — exactly as predicted by Theorem 2.3. This theorem also forecasts that u_2 has no such visible behaviour, since interactions between layers in u_2 appear only in the third-order and higher derivatives, and these are not easily noticed on graphs.

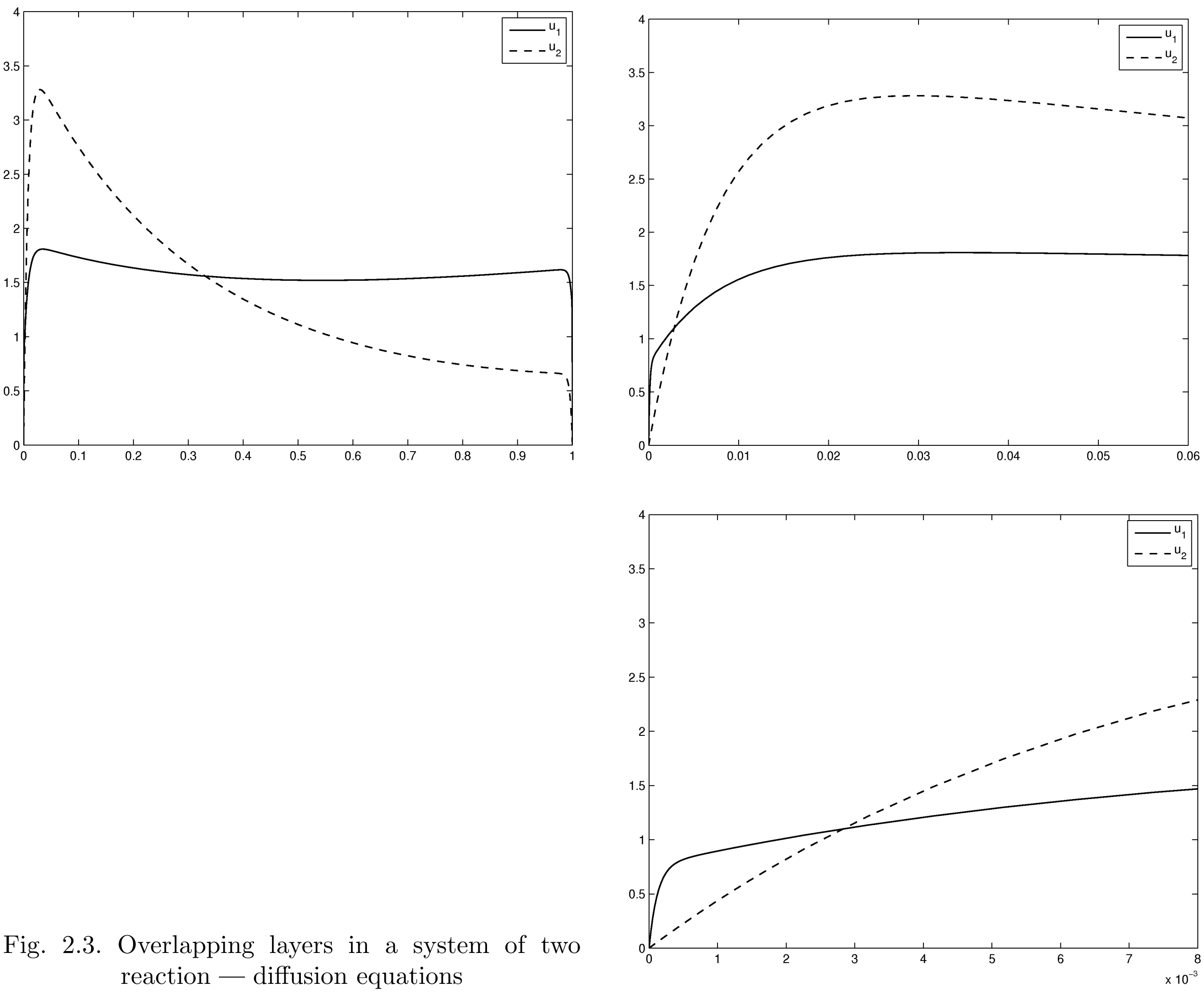

Fig. 2.3. Overlapping layers in a system of two reaction — diffusion equations

2.2.2. Mesh construction. The presence of multiple layers in the solution $\boldsymbol{u}$ at each end of the interval $[0, 1]$, as revealed by Theorem 2.3, forces us to generalize the layer-adapted mesh constructions of Section 2.1.3 by refining the mesh separately for each layer. That is, when approaching an end-point of $[0, 1]$, one requires a fine mesh that undergoes a further refinement as one enters each new layer in $\boldsymbol{u}$. Thus Bakhvalov meshes for (2.8) can be constructed by equidistributing the monitor function M_{Ba} defined by

$$M_{Ba}(t) := \max\left\{1, \frac{K_1}{\varepsilon_1}\mathrm{e}^{-\kappa t/\sigma\varepsilon_1}, \frac{K_1}{\varepsilon_1}\mathrm{e}^{-\kappa(1-t)/\sigma\varepsilon_1}, \dots, \frac{K_\ell}{\varepsilon_\ell}\mathrm{e}^{-\kappa t/\sigma\varepsilon_\ell}, \frac{K_\ell}{\varepsilon_\ell}\mathrm{e}^{-\kappa(1-t)/\sigma\varepsilon_\ell}\right\}$$

with positive user chosen constants σ and K_m.

Shishkin meshes for problem (2.8) are still piecewise equidistant, but now each layer in $\boldsymbol{u}$ requires its own fine mesh. They are constructed as follows. Let N, the number of mesh intervals, be divisible by $2(\ell+1)$. Let $\sigma > 0$ be arbitrary. Fix the mesh transition points τ_k by setting

$$\tau_{\ell+1} = 1/2, \quad \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \; \frac{\sigma\varepsilon_k}{\kappa}\ln N\right\} \text{ for } k = \ell, \dots, 1, \quad \text{and} \quad \tau_0 = 0.$$

Then the mesh is obtained by dividing each of the intervals $[\tau_k, \tau_{k+1}]$ and $[1-\tau_{k+1}, 1-\tau_k]$, for $k = 0, \dots, \ell$, into $N/(2\ell+2)$ subintervals of equal length. Figure 2.4 depicts a Shishkin mesh with 24 mesh intervals for a system of 2 equations.

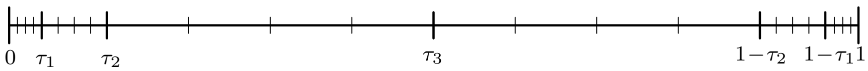

Fig. 2.4. Shishkin mesh for system of 2 reaction — diffusion equations

2.2.3. Finite difference approximation. We consider the discretization of (2.8) by standard central differencing on meshes $\bar{\omega}_N$ that for the moment are arbitrary. That is, we seek $\boldsymbol{u}^N \in \left(\mathbb{R}_0^{N+1}\right)^\ell$ such that

$$\left[\boldsymbol{L}\boldsymbol{u}^N\right]_i := -\mathrm{diag}\,(\boldsymbol{E})^2 \boldsymbol{u}^N_{\bar{x}\hat{x};i} + \boldsymbol{A}(x_i)\boldsymbol{u}^N_i = \boldsymbol{f}(x_i) \quad \text{for} \;\; i = 1, \dots, N-1. \tag{2.14}$$

Stability. By imitating the argument of Theorem 2.1 and appealing to Lemma 2.2, one gets the following stability result:

Theorem 2.4. *Assume that the matrix $\boldsymbol{A}$ has positive diagonal entries. Suppose also that all entries of $\boldsymbol{A}$ and components of $\boldsymbol{f}$ lie in $C[0,1]$. Assume that $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for $i = 1, \dots, \ell$ one has*

$$\|v_i\|_{\infty,\bar{\omega}_N} \leqslant \sum_{k=1}^{\ell} \left(\boldsymbol{\Gamma}^{-1}\right)_{ik} \left\| \frac{(\boldsymbol{L}\boldsymbol{v})_k}{a_{kk}} \right\|_{\infty,\bar{\omega}_N} \qquad \text{for} \;\; i = 1, \dots, \ell,$$

for any mesh function $\boldsymbol{v} \in \mathbb{R}_0^{N+1}$.

Error analysis. Let $\boldsymbol{\eta} := \boldsymbol{u} - \boldsymbol{u}^N$ denote the error of the discrete solution and $\boldsymbol{\tau} := \boldsymbol{L}\boldsymbol{\eta}$ the truncation error. We decompose the solution error as $\boldsymbol{\eta} = \boldsymbol{\varphi} + \boldsymbol{\psi}$, where the components φ_i and ψ_i of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ respectively are the solutions of

$$-\varepsilon_i^2 \varphi_{i,\bar{x}\hat{x}} + a_{ii}\varphi_i = \tau_i = -\varepsilon_i^2\left(u_{i,x\hat{x}} - u_i''\right) \quad \text{on} \;\; \omega_N, \quad \varphi_{i,0} = \varphi_{i,N} = 0$$

and

$$-\varepsilon_i^2 \psi_{i,\bar{x}\hat{x}} + a_{ii}\psi_i = -\sum_{\substack{k=1\\k\neq i}}^{\ell} a_{ik}\eta_k \;\; \text{on} \;\; \omega_N, \quad \psi_{i,0} = \psi_{i,N} = 0.$$

Assume that the matrix $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for each i one has

$$\|\eta_i\|_{\infty,\bar{\omega}_N} \leqslant \|\varphi_i\|_{\infty,\bar{\omega}_N} + \|\psi_i\|_{\infty,\bar{\omega}_N} \leqslant \|\varphi_i\|_{\infty,\bar{\omega}_N} + \sum_{\substack{k=1\\k\neq i}}^{\ell} \left\|\frac{a_{ik}}{a_{ii}}\right\|_{\infty} \|\eta_k\|_{\infty,\bar{\omega}_N}$$

by (2.6). Gathering together the η terms and invoking the inverse-monotonicity of $\boldsymbol{\Gamma}(\boldsymbol{A})$, we get

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}_N} \leqslant C\left\|\boldsymbol{\varphi}\right\|_{\infty,\bar{\omega}_N}.$$

Each component φ_i of $\boldsymbol{\varphi}$ is the solution of a scalar problem and can be analysed using standard techniques. In [23, § 3.2] the derivative bounds of Theorem 2.3 and the estimates (2.5) for the discrete Green's function are used to deduce the following result:

Theorem 2.5. *Let the matrix $\boldsymbol{A}$ be diagonally dominant. Then the error in the solution of* (2.14) *satisfies*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\omega_N} \leqslant C\vartheta_{rd}(\omega_N)^2,$$

where

$$\vartheta_{rd}(\omega_N) := \max_{k=1,\dots,N}\int\limits_{x_{k-1}}^{x_k}\left(1+\sum_{m=1}^{\ell}\varepsilon_m^{-1}\mathcal{B}_{2\varepsilon_m}(t)\right)dt.$$

Corollary 2.2. *(Convergence on layer-adapted meshes) If $\sigma \geqslant 2$, then*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}_N} \leqslant \begin{cases} CN^{-2} & \text{for Bakhvalov meshes,}\\ CN^{-2}\ln^2 N & \text{for Shishkin meshes.}\end{cases}$$

Remark 2.6. An alternative analysis based on comparison principles with special barrier functions was used in [22, 25, 26]. This technique has the more restrictive condition that $\boldsymbol{A}$ be an M-matrix and up to now has been applied successfully only to Shishkin meshes. □

By interpreting the components of $\boldsymbol{u}^N$ as piecewise linear functions, one can conduct an *a posteriori* error analysis that combines the analysis from [12, 19] with Theorem 2.1 to give

$$\left\|u_j-u_j^N\right\|_\infty \leqslant \sum_{k=1}^{\ell}\left(\boldsymbol{\Gamma}^{-1}\right)_{jk}\left(\max_{i=1,\dots,N}\frac{h_i^2\left|q_{k;i}\right|}{2\varepsilon_k^2}+\max_{i=1,\dots,N}\frac{h_i\left|q_{k;i}-q_{k;i-1}\right|}{2\varrho_k\varepsilon_k}+\frac{\left\|q_k-q_k^I\right\|_\infty}{\varrho_k^2}\right),$$

where

$$q_k = f_k - \sum_{\nu=1}^{\ell} a_{k\nu}u_\nu^N \quad\text{and}\quad \varrho_k = \min_{x\in[0,1]} a_{kk}(x)^{1/2}.$$

2.2.3. Finite element methods. Linear finite elements for the discretization of a system of two equations were first considered in [21]. Here we summarize the more general theoretical results for (2.8) from [20]. See also [40] for numerical results.

Assume that the coupling matrix $\boldsymbol{A}$ has positive diagonal entries and satisfies (2.13). By virtue of Theorem 2.2 we can then assume without loss of generality that $\boldsymbol{A}$ is coercive: $\boldsymbol{v}^\top\boldsymbol{A}\boldsymbol{v} \geqslant \mu^2\boldsymbol{v}^\top v$ for all $v\in\mathbb{R}^\ell$, where μ is a positive constant.

Consider the weak formulation of (2.8): Find $\boldsymbol{u}\in H_0^1(0,1)^\ell$ such that

$$B(\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{v}) \quad\text{for all}\quad \boldsymbol{v}\in H_0^1(0,1)^\ell, \tag{2.15}$$

with

$$B(\boldsymbol{u},\boldsymbol{v}) := \sum_{m=1}^{\ell}\varepsilon_m^2(u_m',v_m') + \sum_{m=1}^{\ell}\sum_{i=1}^{\ell}(a_{mi}u_i,v_m) \quad\text{and}\quad F(\boldsymbol{v}) := \sum_{m=1}^{\ell}(f_m,v_m),$$

where $(v,w) := \int_0^1 vw$ is the standard scalar product in $L_2(0,1)$.

The natural norm on $H_0^1(0,1)^\ell$ that is associated with the bilinear form $B(\cdot,\cdot)$ is the energy norm $|||\cdot|||$ defined by

$$|||\boldsymbol{v}|||^2 = \sum_{m=1}^{\ell} \varepsilon_m^2 \, |v_m|_1^2 + \mu^2 \, \|\boldsymbol{v}\|_0^2 , \quad \|\boldsymbol{v}\|_0^2 := \sum_{m=1}^{\ell} \|v_m\|_0^2 ,$$

where μ^2 is the coercivity constant, $\|v\|_0 := (v,v)^{1/2}$ is the standard norm on $L_2(0,1)$ and $|v|_1 := \|v'\|_0$ is the usual H^1 seminorm.

The bilinear form $B(\cdot,\cdot)$ is coercive with respect to the energy norm, viz.,

$$|||\boldsymbol{v}|||^2 \leqslant B(\boldsymbol{v},\boldsymbol{v}) \ \text{ for all } \quad \boldsymbol{v} \in H_0^1(0,1)^\ell .$$

If $\boldsymbol{f} \in L_2(0,1)^\ell$ then F is a bounded linear functional on $H_0^1(0,1)^\ell$. Consequently (2.15) has a unique solution $\boldsymbol{u} \in H_0^1(0,1)^\ell$.

Given an arbitrary mesh $\bar\omega_N$ on $[0,1]$, let $V \subset H_0^1(0,1)$ be the space of piecewise linear functions on $\bar\omega_N$ that vanish at $x=0$ and $x=1$. Then our discretization is: Find $\boldsymbol{u}^N \in V^\ell$ such that

$$B(\boldsymbol{u}^N,\boldsymbol{v}) = f(\boldsymbol{v}) \ \text{ for all } \quad \boldsymbol{v} \in V^\ell .$$

Let $\boldsymbol{u}^{\boldsymbol{I}}$ be the piecewise linear nodal interpolant to $\boldsymbol{u}$ on the mesh $\bar\omega_N$. Then [20] one can show that

$$\|\boldsymbol{u}-\boldsymbol{u}^{\boldsymbol{I}}\|_0 \leqslant C\vartheta_{rd}(\omega_N)^2 \quad \text{and} \quad |||\boldsymbol{u}-\boldsymbol{u}^{\boldsymbol{I}}||| \leqslant C\vartheta_{rd}(\omega_N),$$

and furthermore, the discrete solution satisfies the uniform error bounds

$$\|\boldsymbol{u}-\boldsymbol{u}^N\|_0 + |||\boldsymbol{u}^{\boldsymbol{I}}-\boldsymbol{u}^N||| \leqslant C\vartheta_{rd}(\omega_N)^2 \quad \text{and} \quad |||\boldsymbol{u}-\boldsymbol{u}^N||| \leqslant C\vartheta_{rd}(\omega_N).$$

When the mesh parameters satisfy $\sigma \geqslant 2$, these inequalities imply that

$$\|\boldsymbol{u}-\boldsymbol{u}^N\|_0 + |||\boldsymbol{u}^{\boldsymbol{I}}-\boldsymbol{u}^N||| \leqslant C(N^{-1}\ln N)^2 \quad \text{and} \quad |||\boldsymbol{u}-\boldsymbol{u}^N||| \leqslant CN^{-1}\ln N$$

on the Shishkin mesh, and the Bakhvalov mesh gives a similar result without the $\ln N$ factors.

3. Reaction — diffusion systems in two dimensions

In this section we examine reaction — diffusion systems that are posed on the unit square. Recall that Theorems 2.1 and 2.2 hold true for systems of this type. Compatibility conditions at the corners of the domain will play a role in the general analysis.

3.1.1. One parameter. Consider first, as in [8,9], a system of ℓ equations where the same small diffusion parameter ε appears in each equation. That is, the problem is

$$\boldsymbol{\mathfrak{L}u} := -\varepsilon^2\Delta\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{on } \Omega := (0,1)^2, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{3.1}$$

where $\varepsilon \in (0,1]$, $\boldsymbol{A} : \Omega \to \mathbb{R}^{\ell,\ell}$, $\boldsymbol{f} : \Omega \to \mathbb{R}^\ell$ and $\boldsymbol{g} : \partial\Omega \to \mathbb{R}^\ell$. As usual it is assumed that all data of the problem are continuous functions. We shall present here a unified analysis that includes the results of [8] and [9] as special cases.

3.1.2. Stability. Assume that a diagonal constant matrix $\boldsymbol{D} = \operatorname{diag}(d_{11}, \ldots, d_{\ell\ell})$ exists with $d_{ii} > 0$ for all i and a constant $\mu > 0$ such that

$$\boldsymbol{v}^\top \boldsymbol{D}\boldsymbol{A}\boldsymbol{v} \geqslant \mu^2 \boldsymbol{v}^\top \boldsymbol{v} \quad \text{in} \quad \Omega \quad \text{for all } \boldsymbol{v} \in \mathbb{R}^\ell. \tag{3.2}$$

Under this hypothesis it is not true in general that the system (3.1) satisfies a maximum principle.

Remark 3.1. In [8], which generalizes the one-dimensional analysis of [7], the authors analyse the case $\boldsymbol{D} = \boldsymbol{I}$. In [9], the coupling matrix $\boldsymbol{A}$ is assumed to be a strictly diagonally dominant M-matrix, so Theorem 2.2 ensures that (3.2) is satisfied. □

We now extend the analysis of [7, 8] by showing that it remains valid under the more general hypothesis (3.2). Set $\delta^2 = \max_{i=1,\ldots,\ell} d_{ii}$. For any closed and bounded $Q \subset \mathbb{R}^2$ and any $\boldsymbol{z} \in C(Q)^\ell$, set

$$\|\boldsymbol{z}\|_Q := \sup_Q \left|\boldsymbol{z}^\top \boldsymbol{D}\boldsymbol{z}\right|^{1/2}.$$

This defines a norm on $C(Q)^\ell$.

Lemma 3.1. *Let* $\boldsymbol{w} \in C^2(\Omega)^\ell \cap C(\bar{\Omega})^\ell$. *Then*

$$\|\boldsymbol{w}\|_{\bar{\Omega}} \leqslant \max\Big\{\delta^2\mu^{-2}\,\|\boldsymbol{\mathfrak{L}}\boldsymbol{w}\|_\Omega\,,\ \|\boldsymbol{w}\|_{\partial\Omega}\Big\}.$$

Proof. Set $\varphi = \boldsymbol{w}^\top \boldsymbol{D}\boldsymbol{w}/2$. Observe that $2\varphi \leqslant \delta^2 \boldsymbol{w}^\top \boldsymbol{w}$ on $\bar{\Omega}$ and

$$-\boldsymbol{w}^\top \boldsymbol{D}\Delta\boldsymbol{w} = -\Delta\varphi + \partial_x \boldsymbol{w}^\top \boldsymbol{D}\partial_x \boldsymbol{w} + \partial_y \boldsymbol{w}^\top \boldsymbol{D}\partial_y \boldsymbol{w} \geqslant -\Delta\varphi.$$

Taking the scalar product of $\boldsymbol{D}\boldsymbol{w}$ with $-\varepsilon^2\Delta\boldsymbol{w} + \boldsymbol{A}\boldsymbol{w} = \boldsymbol{\mathfrak{L}}\boldsymbol{w}$ and using the coercivity of $\boldsymbol{D}\boldsymbol{A}$, we get

$$-\varepsilon^2\Delta\varphi + \frac{2\mu^2}{\delta^2}\varphi \leqslant \boldsymbol{w}^\top \boldsymbol{D}\boldsymbol{\mathfrak{L}}\boldsymbol{w} \quad \text{in } \Omega.$$

Hence, by the standard maximum principle for *scalar* problems,

$$\|\varphi\|_\infty \leqslant \max\left\{\frac{\delta^2}{2\mu^2}\left\|\boldsymbol{w}^\top \boldsymbol{D}\boldsymbol{\mathfrak{L}}\boldsymbol{w}\right\|_\infty,\ \frac{1}{2}\|\boldsymbol{w}\|_{\partial\Omega}^2\right\}.$$

It follows from the Cauchy-Schwarz inequality that

$$\|\boldsymbol{w}\|_{\bar{\Omega}}^2 = 2\,\|\varphi\|_\infty \leqslant \|\boldsymbol{w}\|_{\bar{\Omega}} \max\Big\{\delta^2\mu^{-2}\,\|\boldsymbol{\mathfrak{L}}\boldsymbol{w}\|_\Omega\,, \|\boldsymbol{w}\|_{\partial\Omega}\Big\},$$

and we are done. □

Corollary 3.1. *Lemma* 3.1 *implies that* $\boldsymbol{\mathfrak{L}}$ *is maximum-norm stable.*

Corollary 3.2. *The solution* $\boldsymbol{u}$ *of* (3.1) *satisfies* $\|\boldsymbol{u}\|_\infty \leqslant C$, *where the constant* C *depends only on* $\boldsymbol{A}$, $\boldsymbol{f}$ *and* $\boldsymbol{g}$.

3.1.3. Derivative bounds. In problems posed on domains with corners, regularity of the solution must be carefully monitored. Assume that $\boldsymbol{f} \in C^{2,\alpha}(\bar{\Omega})^\ell$, $\boldsymbol{g} \in C^{4,\alpha}(\partial\Omega)^\ell$, and that $\boldsymbol{\mathfrak{L}g} = \boldsymbol{f}$ at each corner. Then in [9] it is shown that $\boldsymbol{u} \in C^3(\bar{\Omega})^\ell$ with

$$\|\partial^m \boldsymbol{u}\|_\infty \leqslant C\varepsilon^{-m} \text{ for } m = 0, 1, 2, 3,$$

and additionally, using a result of Volkov [39] (cf. [5]), one obtains the further bound

$$\left\|\partial_x^{2j}\partial_y^{4-2j}\boldsymbol{u}\right\|_\infty \leqslant C\varepsilon^{-4} \text{ for } \; j = 0, 1, 2.$$

Using ideas from the proof of Lemma 3.1, an inductive argument [7,8] sharpens the above bounds to

$$\left|\partial_x^k \boldsymbol{u}(x,y)\right| \leqslant C\left[1 + \varepsilon^{-k}\left(e^{-\varrho x/\varepsilon} + e^{-\varrho(1-x)/\varepsilon}\right)\right],$$
$$\left|\partial_y^k \boldsymbol{u}(x,y)\right| \leqslant C\left[1 + \varepsilon^{-k}\left(e^{-\varrho y/\varepsilon} + e^{-\varrho(1-y)/\varepsilon}\right)\right],$$

for all $(x, y) \in \bar{\Omega}$ and $k = 1, \dots, 4$, where $\varrho \in (0, \mu/\delta)$ is arbitrary. These estimates show clearly that the solution $\boldsymbol{u}$ has smooth and layer components.

3.1.4. Discretization. The problem (3.1) is solved numerically using central differencing (the 2-dimensional analogue of (2.14)) on a Shishkin mesh ω_N^S with N intervals in each coordinate direction; this is the tensor product of the 1-dimensional mesh described in Section 2.1.3. To analyse this method one decomposes $\boldsymbol{u}$ as a sum of regular and layer components, and the truncation error is decomposed similarly. Then [8] one obtains the almost-second-order convergence result

$$\left\|\boldsymbol{u} - \boldsymbol{u}^N\right\|_{\infty,\omega_N^S} \leqslant CN^{-2}\ln^2 N, \tag{3.3}$$

where $\boldsymbol{u}^N$ is the computed solution and C is some positive constant.

In [24] a finite element method with piecewise bilinears is applied. When errors are measured in the usual energy norm, this standard method achieves $\mathcal{O}\left(N^{-2}\ln^2 N\right)$ convergence; a sparse-grid variant of the method is shown to attain the same order of convergence using only $\mathcal{O}\left(N^{3/2}\right)$ degrees of freedom instead of the $\mathcal{O}\left(N^2\right)$ of the original method. Numerical experiments confirm the efficiency of this approach.

A somewhat different numerical approach is adopted in [9], where a Jacobi-type iteration is combined with central differencing. Once again one obtains the convergence result (3.3).

If the Shishkin mesh is replaced by a 2-dimensional Bakhvalov mesh ω_N^B that is a tensor product of the 1-dimensional N-interval meshes of Section 2.1.3 (see [8,32] for further details), then in [8] it is shown — without decomposing $\boldsymbol{u}$ — that (3.3) can be strengthened to

$$\|\boldsymbol{u} - \boldsymbol{U}\|_{\infty,\omega_N^B} \leqslant CN^{-2}.$$

3.2. Multiple parameters. In [36], Shishkin considers a 2-dimensional analogue of (2.8):

$$\boldsymbol{\mathfrak{L}u} := -\begin{pmatrix} \varepsilon^2\Delta & 0 \\ 0 & \mu^2\Delta \end{pmatrix}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{on} \;\; \Omega := (0,1)^2, \tag{3.4}$$

where the parameters ε and μ lie in $(0, 1]$, $\boldsymbol{A} : \bar{\Omega} \to \mathbb{R}^{2,2}$, $\boldsymbol{f} : \bar{\Omega} \to \mathbb{R}^2$ and $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$. Assume that on $\bar{\Omega}$ we have

$$a_{11} \geqslant c_0 \;\text{ and }\; a_{22} \geqslant c_0 \;\text{ for some constant }\; c_0 > 0,$$

$$a_{11} > |a_{12}|, \quad a_{22} > |a_{21}| \text{ at all points.}$$

This problem does not satisfy a maximum principle. Once again the key difficulty is the construction of a decomposition of the solution of (3.4), which is achieved in [36] via a complicated analysis that examines separately the cases $\mu^2 \leqslant \varepsilon$ and $\mu^2 \geqslant \varepsilon$.

To solve (3.4) numerically one constructs a 2-dimensional Shishkin mesh ω_N^S, with N intervals in each coordinate direction, that is modified in each coordinate direction as in Section 2.2.2 to handle two overlapping layers. Using the 2-dimensional central differencing analogue of (2.14) on this mesh, it is shown in [36] that one obtains again (3.3). It is stated in [36] that this result generalizes to systems of $\ell > 2$ equations.

4. Convection — diffusion systems

In this section convection comes into the picture. That is, we return to the general problem

$$\boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\Delta \boldsymbol{u} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \boldsymbol{Au} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}. \tag{1.1}$$

To ensure that this is not a reaction — diffusion problem, we assume that all diagonal entries of $\boldsymbol{B}$ are non-zero; stronger hypotheses will be placed later on $\boldsymbol{B}$.

4.1. The scalar problem in one dimension. Consider first the scalar reaction — convection — diffusion two-point boundary value problem

$$\mathcal{L}u := -\varepsilon u'' - qu' + ru = f \quad \text{on } (0,1), \quad u(0) = u(1) = 0, \tag{4.1}$$

where $0 < \varepsilon \ll 1$, $q, r \in C[0,1]$ while f is allowed to be a generalized function like the Dirac δ distribution.

Assume that $\beta := \min\limits_{x\in[0,1]} q(x) > 0$. Then (4.1) has a unique solution, which typically exhibits a boundary layer of width $\mathcal{O}(\varepsilon \ln 1/\varepsilon)$ at $x = 0$. More precisely, we have the following bounds for u and its lower-order derivatives from [10, Lemma 2.3]:

$$\left|u^{(k)}(x)\right| \leqslant C\left\{1 + \varepsilon^{-k}\mathrm{e}^{-\beta x/\varepsilon}\right\} \quad \text{for } x \in [0,1], \text{ and } k = 0,1,\dots,K, \tag{4.2}$$

where the maximal order K depends on the regularity of the data. If instead q is negative on $[0,1]$ then we have a similar layer at $x = 1$.

Consider the following family of difference schemes on an arbitrary mesh for (4.1). Fix $\nu \in [0,1]$. Setting

$$\chi_i := \nu h_{i+1} + (1-\nu)h_i \quad \text{and} \quad v_{\check{x};i} := \frac{v_{i+1} - v_i}{\chi_i},$$

our discretization of (4.1) is: Find $u^N \in \mathbb{R}_0^{N+1}$ such that

$$\left[Lu^N\right]_k := -\varepsilon u^N_{\bar{x}\check{x};k} - q_k u^N_{\check{x};k} + r_k u^N_k = f_k \quad \text{for } k = 1,\dots,N-1. \tag{4.3}$$

For $\nu = 1$ we obtain the simple upwind scheme of [6] with an unusual discretization of the diffusion term. The scheme with $\nu = 1/2$ was considered in [2]. This time the discretization of the convection term is unusual.

4.1.1. Stability. The following lemma provides a means of establishing uniform maximum-norm stability estimates for the operator $\mathcal{L}$.

Lemma 4.1. *Assume that the operator $\mathfrak{L}$ satisfies a comparison principle. Suppose that there exists a function $\varphi \in C^2[0,1]$ with $\varphi \geqslant 0$, $\varphi'' \leqslant 0$ and $-q\varphi' + r\varphi > 0$ in $[0,1]$.*

Then for any $v \in C^2[0,1]$ with $v(0) = v(1) = 0$ we have

$$\|v\|_\infty \leqslant \|\varphi\|_\infty \, \|\mathfrak{L}v/\mathfrak{L}\varphi\|_\infty .$$

Proof. Divide (4.1) by $\mathfrak{L}\varphi$ then use the comparison principle with the barrier function $\varphi \, \|\mathfrak{L}v/\mathfrak{L}\varphi\|_\infty$. □

Remark 4.1. (i) If $r > 0$ on $[0,1]$, then Lemma 4.1 with $\varphi \equiv 1$ yields

$$\|v\|_\infty \leqslant \|\mathfrak{L}v/r\|_\infty . \tag{4.4a}$$

(ii) If $q > 0$ and $r \geqslant 0$ on $[0,1]$, use $\varphi(x) = 1 - x$ to get

$$\|v\|_\infty \leqslant \|\mathfrak{L}v/q\|_\infty . \tag{4.4b}$$

(iii) If $q > 0$ on $[0,1]$, then $r \geqslant 0$ can be ensured by the transformation $u(x) = e^{\delta x}\tilde{u}(x)$ with an appropriate constant δ, but this transformation affects the right-hand side of the differential equation, while in systems different equations may need different transformations that cannot all be executed simultaneously.

(iv) If both $q > 0$ and $r > 0$, one can then derive stability results for $\mathfrak{L}$ that generalize both (4.4a) and (4.4b). This can be done, for example, by using as barrier function a general linear or quadratic function φ. But the resulting stability equality will be more difficult to use when applied to systems.

The presence of the convective term means that $\mathfrak{L}$ also satisfies some less elementary stability estimates. Assume that

$$0 < \beta \leqslant q(x) \leqslant Q \quad \text{and} \quad 0 \leqslant r(x) \leqslant R \quad \text{for} \quad x \in [0,1]. \tag{4.5}$$

Set

$$Q^* = \int_0^1 \left| \left(\frac{1}{q(x)} \right)' \right| dx \quad \text{and} \quad \tilde{Q} = \left(1 + \frac{R}{\beta}\right) \left(Q^* + \frac{2}{\beta}\right).$$

Andreev [3] makes a careful study of the Green's function associated with $\mathfrak{L}$ in showing that

$$\|v\|_\infty \leqslant \frac{1}{\beta} \|\mathfrak{L}v\|_1 \tag{4.6a}$$

and

$$|||v|||_{\varepsilon,\infty} \leqslant \tilde{Q} \|\mathfrak{L}v\|_{-1,\infty}. \tag{4.6b}$$

where $\|\cdot\|_1$ is the usual norm on $L_1[0,1]$ and we set $\|w\|_{-1,\infty} := \inf\{\|W\|_\infty : W' = w\}$ and $|||w|||_{\varepsilon,\infty} := \varepsilon\kappa\|w'\|_\infty + \|w\|_\infty$ with a certain positive constant $\kappa = \kappa(q,r)$ that is given explicitly in [3]. Furthermore, in [17] it is shown by related arguments that

$$\|v'\|_1 \leqslant \frac{2}{\beta} \|\mathfrak{L}v\|_1.$$

Remark 4.2. (i) Note that

$$2\|v\|_{-1,\infty} \leqslant \|v\|_1 \leqslant \|v\|_\infty. \tag{4.7}$$

Thus the weakness of the norm $\|\cdot\|_{-1,\infty}$ means that (4.6b) is much stronger than (4.4) and (4.6a). It is the key to our later analysis for systems with strong coupling in the convection term.

(ii) From (4.6b) one also has a bound on the weighted first-order derivative. While this information is not exploited here, it can be used to study certain forms of coupling in the diffusion term.

(iii) An alternative proof of (4.6) using maximum principles is given in [16] under the additional hypothesis that $q' + r \geqslant 0$; it is shown that (4.6) holds true with $\tilde{Q} = 2/\beta$ and $\kappa = 1/\beta$. In general (see [3]) it is easier to analyse (4.1) if it is written in the conservative form $-\varepsilon u'' - (qu)' + (q' + r)u = f$ provided that $q' + r \geqslant 0$.

In the case of constant q the results from [16] and (4.6b) differ by a factor of $(1 + R/\beta)$. We conclude that the constant factor $\tilde{Q}$ in (4.6b) is not sharp.

The discrete operator L enjoys similar stability properties:

$$\|v\|_{\infty,\omega_N} \leqslant \|Lv/r\|_{\infty,\omega_N} \quad \text{if } r > 0 \text{ on } [0,1]$$

and

$$\|v\|_{\infty,\omega_N} \leqslant \|Lv/q\|_{\infty,\omega_N} \quad \text{if } q > 0,\ r \geqslant 0 \text{ on } [0,1].$$

Assume (4.5) holds true. Then, for $0 \leqslant \nu \leqslant 1$ one has from [2] discrete analogues of (4.6):

$$\|v\|_{\infty,\omega_N} \leqslant \frac{1}{\beta}\|Lv\|_{1,\omega_N}, \tag{4.8a}$$

$$|||v|||_{\varepsilon,\infty,\omega_N} \leqslant \tilde{Q}\|Lv\|_{-1,\infty,\omega_N}, \tag{4.8b}$$

where

$$\|v\|_{1,\omega_N} := \sum_{k=1}^{N-1} \chi_k |v_k|, \quad \|w\|_{-1,\infty,\omega_N} := \inf\left\{\|W\|_{\infty,\omega_N} : W_x = w\right\}$$

and $|||w|||_{\varepsilon,\infty,\omega_N} := \varepsilon\kappa\|w_x\|_{\infty,\omega_N} + \|w\|_{\infty,\omega_N}$. If in addition to (4.5) one has

$$q' \geqslant 0 \text{ on } [0,1] \quad \text{and} \quad \nu = 1, \tag{4.9}$$

then (4.8) holds true with $\tilde{Q} = 2/\beta$ and $\kappa = 1/\beta$; see [16].

4.1.2. Error bounds. By [6, 11, 16] we have the following *a priori* and *a posteriori* error bounds for (4.3) applied to (4.1): Assume that (4.5) and (4.9) hold. Then

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega_N} \leqslant \max_{k=0,\dots,N-1} \int\limits_{x_k}^{x_{k+1}} \left(C_1 \left|u'(x)\right| + C_2|u(x)| + C_3\right) dx$$

and, interpreting u^N as a piecewise linear function,

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty} \leqslant \max_{k=0,\dots,N-1} h_{k+1}\left(C_1 \left|u^N_{x;k}\right| + C_2 \left|u^N_k\right| + C_3\right)$$

with constants

$$C_1 := \|r\|_\infty + \|q'\|_\infty + \|q\|_\infty, \quad C_2 := \|r\|_\infty + \|r'\|_\infty, \quad C_3 := \|f\|_\infty + \|f'\|_\infty.$$

Note that if (4.9) is not satisfied then these error estimates remain true but with slightly different constants C_i.

The derivative bounds (4.2) imply that

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega_N} \leqslant \vartheta_{cd,\varepsilon}(\omega_N) := \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} \left[1 + \varepsilon^{-1} \mathrm{e}^{-\beta x/\varepsilon}\right] dx.$$

4.1.3. Mesh construction. *Bakhvalov meshes* for the discretization of (4.1) are generated by equidistributing the function

$$M_{Ba}(x) = \max\left\{1,\ \frac{K}{\varepsilon}\exp\left(-\frac{\beta x}{\varepsilon\sigma}\right)\right\}$$

with positive constants K and σ. The parameter K determines the number of mesh points used to resolve the layer. For the Bakhvalov mesh ω_N^B it can be shown [14] that

$$\vartheta_{cd,\varepsilon}(\omega_N^B) \leqslant CN^{-1} \quad \text{if } \sigma \geqslant 1.$$

Shishkin meshes. Let $q \in (0,1)$ and $\sigma > 0$ be mesh parameters. Set

$$\tau = \min\left\{q,\ \frac{\sigma\varepsilon}{\beta}\ln N\right\}.$$

Assume that qN is an integer. Then the Shishkin mesh ω_N^S for problem (1.1) divides the interval $[0,\tau]$ into qN equidistant subintervals, while $[\tau,1]$ is divided into $(1-q)N$ equidistant subintervals. For this mesh we have

$$\vartheta_{cd,\varepsilon}(\omega_N^S) \leqslant CN^{-1}\ln N \quad \text{if } \sigma \geqslant 1.$$

4.2. Weakly coupled systems in one dimension. We now leave the scalar convection — diffusion equation and move on to systems of equations of this type.

4.2.1. The continuous problem. The system (1.1) is said to be weakly coupled if the convective coupling matrix $\boldsymbol{B}$ is diagonal, so the system is coupled only through the lower-order reaction terms. In one dimension such systems can be written as

$$\boldsymbol{\mathcal{L}u} := -\mathrm{diag}\,(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \mathrm{diag}\,(\boldsymbol{b})\boldsymbol{u}' + \boldsymbol{Au} = \boldsymbol{f} \quad \text{on} \quad (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}. \tag{4.10}$$

For $i = 1,\ldots,\ell$, the i^{th} equation of (4.10) is

$$-\varepsilon_i u_i'' - b_i u_i' + \sum_{j=1}^{\ell} a_{ij}u_j = f_i \quad \text{on} \quad (0,1), \quad u_i(0) = u_i(1) = 0. \tag{4.11}$$

Assume that for each i one has $b_i \geqslant \beta_i > 0$ and $a_{ii} \geqslant 0$ on $[0,1]$. (In [18] the weaker hypothesis $|b_i| \geqslant \beta_i > 0$ is used, which permits layers at both ends of [0,1], but for simplicity we won't consider this here.)

Stability and bounds on derivatives We follow the argument of [18]. Rewrite (4.11) as

$$-\varepsilon_i u_i'' - b_i u_i' + a_{ii} u_i = -\sum_{j \neq i} a_{ij} u_j + f_i. \tag{4.12}$$

Then (4.4) yields

$$\|u_i\|_\infty + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \tilde{\gamma}_{ij} \|u_j\|_\infty \leqslant \min\left\{\left\|\frac{f_i}{a_{ii}}\right\|_\infty, \left\|\frac{f_i}{b_i}\right\|_\infty\right\} \quad \text{for } i = 1, \ldots, \ell,$$

where the $\ell \times \ell$ constant matrix $\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{\Gamma}}(\boldsymbol{A}, \boldsymbol{b}) = (\tilde{\gamma}_{ij})$ is defined by

$$\tilde{\gamma}_{ii} = 1 \quad \text{and} \quad \tilde{\gamma}_{ij} = -\min\left\{\left\|\frac{a_{ij}}{a_{ii}}\right\|_\infty, \left\|\frac{a_{ij}}{b_i}\right\|_\infty\right\} \quad \text{for} \quad i \neq j.$$

Repeating the analysis of Section 2.2.1 we reach the following stability result.

Theorem 4.1. *Assume that the matrix $\boldsymbol{A}$ has non-negative diagonal entries. Suppose also that all entries of $\boldsymbol{A}$ lie in $C[0,1]$. Assume that $\tilde{\mathbf{\Gamma}}(\boldsymbol{A}, \boldsymbol{b})$ is inverse-monotone. Then for $i = 1, \ldots, \ell$ one has*

$$\|v_i\|_\infty \leqslant \sum_{k=1}^{\ell} \left(\tilde{\mathbf{\Gamma}}^{-1}\right)_{ik} \min\left\{\left\|\frac{(\boldsymbol{\mathcal{L}} \boldsymbol{v})_k}{a_{kk}}\right\|_\infty, \left\|\frac{(\boldsymbol{\mathcal{L}} \boldsymbol{v})_k}{b_k}\right\|_\infty\right\}$$

for any function $\boldsymbol{v} = (v_1, \ldots, v_\ell)^\top \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ with $\boldsymbol{v}(0) = \boldsymbol{v}(1) = \mathbf{0}$. In this inequality the first term in $\min\{\ldots\}$ should be omitted if $a_{kk}(x) = 0$ for any $x \in [0,1]$.

Corollary 4.1. *Under the hypotheses of Theorem 4.1 the boundary value problem (4.10) has a unique solution $\boldsymbol{u}$, and $\|\boldsymbol{u}\|_\infty \leqslant C \|\boldsymbol{f}\|_\infty$ for some constant C.*

Remark 4.3. (*i*) The argument in [18] appeals to (4.4a) but not to (4.4b). Thus our modified analysis above can handle a larger class of problems.

(*ii*) Alternatively, one can use (4.6a) and (4.6b) when bounding the source term to establish that

$$\|\boldsymbol{u}\|_\infty \leqslant C \max_{k=1,\ldots,\ell} \|f_k\|_1 \quad \text{and} \quad \|\boldsymbol{u}\|_\infty \leqslant C \max_{k=1,\ldots,\ell} \|f_k\|_{-1,\infty}.$$

for the solution of (4.10). The latter inequality allows the right-hand side to have singularities like a Dirac δ distribution.

(*iii*) Andreev's results (4.6) can also be applied to the coupling terms $a_{ij} u_j$ in (4.12), in particular when a_{ij} has a singularity or when $\beta_i^{-1} \|a_{ij}\|_1 \leqslant \|a_{ij}/a_{ii}\|_\infty$.

(*iv*) From the above it is apparent that any improvement in the stability inequalities for scalar operators will immediately broaden the class of systems that can be analysed.

Next, by applying the scalar-equation analysis of [10, Lemma 2.3] to (4.12), it is shown in [18] that for $i = 1, \ldots, \ell$ one has

$$|u_i^{(k)}(x)| \leqslant C\left[1 + \varepsilon_i^{-k} e^{-\beta_i x/\varepsilon_i}\right] \quad \text{for } x \in [0,1] \text{ and } k = 0, 1. \tag{4.13}$$

Thus there are boundary layers in the solution at $x = 0$, but no strong interaction between the layers in different components u_i is apparent when first-order derivatives are considered, which is quite unlike the reaction — diffusion case of Theorem 2.3. This bound on the u_i' also reveals a lack of sharpness in the results of some previously published papers that gave derivative bounds for this problem with stronger interactions between different components.

4.2.2. Mesh construction. Meshes for the weakly coupled problem (4.10) with $b_i > 0$ for all i need accommodate layers only at $x = 0$, according to (4.13).

Bakhvalov meshes for (4.10) are constructed by equidistributing the monitor function

$$\max\left\{1, \frac{K_1}{\varepsilon_1}\exp\left(\frac{-\beta_1 t}{\sigma\varepsilon_1}\right), \dots, \frac{K_\ell}{\varepsilon_\ell}\exp\left(\frac{-\beta_\ell t}{\sigma\varepsilon_\ell}\right)\right\}$$

with positive user chosen constants σ and K_m.

Shishkin meshes are equidistant and coarse away from $x = 0$, and piecewise equidistant, with successively finer meshes, as one approaches $x = 0$. The mesh resembles that of Section 2.2.2, modified by removing the mesh refinement at $x = 1$, but one should note that the diffusion coefficient was ε^2 in Section 2.2.2 while here it is ε.

Let N, the number of mesh intervals, be divisible by $\ell + 1$. Let $\sigma > 0$ be arbitrary. Fix the mesh transition points τ_k by setting

$$\tau_{\ell+1} = 1, \quad \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{\sigma\varepsilon_k}{\kappa}\ln N\right\} \text{ for } k = \ell, \dots, 1, \quad \text{and} \quad \tau_0 = 0.$$

Then for $k = 0, \dots, \ell$, the Shishkin mesh is obtained by dividing each of the intervals $[\tau_k, \tau_{k+1}]$ into $N/(\ell + 1)$ subintervals of equal length.

4.2.3. Finite difference scheme. Let $\bar{\omega}_N$ be an arbitrary mesh on $[0, 1]$. Recall that the solution $\boldsymbol{u}$ of (4.10) has layers only at $x = 0$ according to (4.13). In the vast literature on numerical methods for scalar convection — diffusion equations it is well-known [32] that to approximate the convection term one should use some form of differencing that is upwinded away from the boundary layer. Thus it is natural that one approximates (4.10) by simple upwinding (viz., (4.3) with $\nu = 1$) in each equation: we seek $\boldsymbol{u}^N \in \left(\mathbb{R}_0^{N+1}\right)^\ell$ such that

$$\left[\boldsymbol{L}\boldsymbol{u}^N\right]_k := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}^N_{\bar{x}\breve{x};k} - \operatorname{diag}(\boldsymbol{b})(x_k)\boldsymbol{u}^N_{\breve{x};k} + \boldsymbol{A}(x_k)\boldsymbol{u}^N_k = \boldsymbol{f}(x_k) \quad \text{for} \quad k = 1, \dots, N-1. \tag{4.14}$$

The stability analysis for the discrete operator can be conducted along the lines of the continuous analysis.

Theorem 4.2. *Assume that the matrix $\boldsymbol{A}$ has non-negative diagonal entries. Assume that $\tilde{\boldsymbol{\Gamma}}(\boldsymbol{A})$ is inverse-monotone. Then for $i = 1, \dots, \ell$ one has*

$$\|v_i\|_{\infty,\omega_N} \leqslant \sum_{k=1}^{\ell}\left(\tilde{\boldsymbol{\Gamma}}^{-1}\right)_{ik}\min\left\{\left\|\frac{(\boldsymbol{L}\boldsymbol{v})_k}{a_{kk}}\right\|_{\infty,\omega_N}, \left\|\frac{(\boldsymbol{L}\boldsymbol{v})_k}{b_k}\right\|_{\infty,\omega_N}\right\}$$

for any mesh function $\boldsymbol{v} \in \left(\mathbb{R}_0^{N+1}\right)^\ell$. In this inequality the first term in $\min\{\dots\}$ *should be omitted if $a_{kk}(x) = 0$ for any $x \in [0, 1]$.*

We also have for the solution $\boldsymbol{u}^N$ of (4.14)

$$\left\|\boldsymbol{u}^N\right\|_{\infty,\omega_N} \leqslant C\left\|\boldsymbol{f}\right\|_{\infty,\omega_N}, \quad \left\|\boldsymbol{u}^N\right\|_{\infty,\omega_N} \leqslant C\max_{k=1,\dots,\ell}\|f_k\|_{1,\omega_N}$$

and
$$\left\|\boldsymbol{u}^N\right\|_{\infty,\omega_N} \leqslant C \max_{k=1,\ldots,\ell} \|f_k\|_{-1,\infty,\omega_N}. \tag{4.15}$$

A priori error analysis. In [18] the author draws on the strong stability result (4.15) in proving the general error bound
$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}_N} \leqslant C \max_{k=0,\ldots,N-1} \int\limits_{x_k}^{x_{k+1}} \left[1+\sum_{m=1}^{\ell}|u_m'(s)|\right] ds$$
on an arbitrary mesh. After constructing a suitable mesh, one can combine this estimate with (4.13) to get more explicit error bounds, for example
$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}_N} \leqslant \begin{cases} CN^{-1} & \text{for Bakhvalov meshes with } \sigma \geqslant 1, \\ CN^{-1}\ln N & \text{for Shishkin meshes with } \sigma \geqslant 1. \end{cases}$$

A posteriori error analysis. Alternatively, one can appeal to the strong stability (4.6b) of the continuous operator to get the *a posteriori* bound
$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty} \leqslant C \max_{k=0,\ldots,N-1} h_{k+1}\left[1+\sum_{m=1}^{\ell}\left|u^N_{m,x;k}\right|\right].$$
The constant(s) involved in this error bound can be specified more explicitly; cf. Section 4.1.2.

4.3. Weakly coupled systems in two dimensions. Two papers by Shishkin [34, 35] consider weakly coupled systems on a strip of the form $[0,1]\times\mathbb{R}$. The systems take the form
$$\boldsymbol{\mathcal{L}u} := -\begin{pmatrix}\varepsilon\Delta & 0 \\ 0 & \mu\Delta\end{pmatrix}\boldsymbol{u} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{on} \quad \Omega := [0,1]\times\mathbb{R},$$
where $\boldsymbol{B} = (b_1, b_2)$ is diagonal. The hypotheses in [34] are that $b_1 > 0 > b_2$ on $\bar{\Omega}$, so the solution has an exponential boundary layer along both edges of the strip, while in [35] one has $b_1(0,\cdot) = b_1(1,\cdot) = 0$ with $b_1 > 0$ on Ω and $b_2 > 0$ on $\bar{\Omega}$, which leads to a parabolic boundary layer in the solution along the edges of the strip.

One can construct from the entries of $\boldsymbol{A}$ a constant matrix $\boldsymbol{\Gamma}$ similar to (2.10), and it is assumed as in (2.11) that $\boldsymbol{\Gamma}$ is inverse-monotone.

In [34], bounds on derivatives of the solution are derived and appropriate meshes are constructed; these are piecewise uniform in the x_1 variable with N mesh intervals in total, and uniform in the x_2 variable with mesh spacing $1/N$. A difference scheme based on simple upwinding is considered. In the two special cases $\varepsilon = \mu$ and $\mu = 1$, it is shown that one obtains convergence of $\mathcal{O}\left(N^{-1}\ln N\right)$ at all mesh points, while in the case of general positive ε and μ the nodal convergence bound is $\mathcal{O}\left(N^{-1/10}\right)$. The two special cases are easier to analyse as their layers depend only on the parameter ε, while the general case has two overlapping layers. The practical implications of the unboundedness of the domain and mesh are not discussed and numerical results are not provided.

The analysis in [35] is broadly similar in that the same three (ε,μ) regimes are considered, but the convergence results obtained are more complicated and we shall not state them here.

4.4. Strongly coupled systems. We now return to the general problem (1.1). Assume that this system is *strongly coupled*: this means that for each i one has $b_{ij} \not\equiv 0$ for some $j \neq i$.

For strongly coupled convection — diffusion systems, the only known theoretical convergence results that are uniform in the singular perturbation parameters are for problems posed in one dimension. Strong coupling causes interactions between boundary layers that are not fully understood at present. The main papers on this problem are [1, 17, 28–30].

The general strongly coupled two-point boundary value problem is

$$\mathcal{L}\boldsymbol{u} := -\mathrm{diag}\,(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{on } \Omega := (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \tag{4.16}$$

where as before $\boldsymbol{u} = (u_1, u_2, \dots, u_\ell)^\top$, $\boldsymbol{f} = (f_1, \dots, f_\ell)^\top$, while $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are $\ell \times \ell$ matrices, and the $\ell \times \ell$ matrix $\mathrm{diag}\,(\boldsymbol{\varepsilon})$ is diagonal with i^{th} entry ε_i for all i.

4.4.1. The continuous problem. Our analysis follows [17] as it is in several ways the most general of the above papers. For each i assume that

$$b_{ii} \geqslant \beta_i > 0, \quad a_{ii} \geqslant 0 \quad \text{and} \quad b_{ii}' \geqslant 0 \quad \text{on} \quad [0,1]. \tag{4.17}$$

(In [17] the weaker hypothesis $|b_{ii}| \geqslant \beta_i > 0$ is used, but for simplicity we won't consider this here.) Rewrite the i^{th} equation of the system (4.16) as

$$\mathcal{L}_i u_i := -\varepsilon_i u_i'' - b_{ii} u_i' + a_{ii} u_i = f_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \left[\left(b_{ij} u_j\right)' - (b_{ij}' + a_{ij}) u_j \right], \tag{4.18a}$$

$$u_i(0) = u_i(1) = 0. \tag{4.18b}$$

In (4.18) write $u_i = v_i + w_i + z_i$, where

$$\mathcal{L}_i v_i = \sum_{\substack{j=1 \\ j \neq i}}^{\ell} (b_{ij} u_j)', \quad \mathcal{L}_i w_i = -\sum_{\substack{j=1 \\ j \neq i}}^{\ell} \left[(b_{ij}' + a_{ij}) u_j \right], \quad \mathcal{L}_i z_i = f_i,$$

with homogeneous Dirichlet boundary conditions for v_i, w_i and z_i. Apply the stability bound of (4.6b) and (4.7) to obtain

$$\frac{\beta_i}{2} |||u_i|||_{\varepsilon_i,\infty} \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \left\{ \|b_{ij}\|_\infty + \frac{1}{2} \left\|b_{ij}' + a_{ij}\right\|_1 \right\} \|u_j\|_\infty + \|f_i\|_{-1,\infty} ;$$

see also Remark 4.2(iii). Then, after some minor manipulation, gather the u_j terms to the left-hand side in the manner of Section 2.2. Define the $\ell \times \ell$ matrix $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(\boldsymbol{A}, \boldsymbol{B}) = (\gamma_{ij})$ by

$$\gamma_{ii} = 1 \quad \text{and} \quad \gamma_{ij} = -\frac{2\|b_{ij}\|_\infty + \left\|b_{ij}' + a_{ij}\right\|_1}{\beta_1} \quad \text{for} \quad i \neq j.$$

Theorem 4.3. *Assume that* $\boldsymbol{B}$ *and* $\boldsymbol{A}$ *satisfy* (4.17). *Suppose* $\boldsymbol{\Upsilon}(\boldsymbol{A}, \boldsymbol{B})$ *is inverse-monotone. Then*

$$|||u_i|||_{\varepsilon_i,\infty} \leqslant 2 \sum_{k=1}^{\ell} \left(\boldsymbol{\Upsilon}^{-1}\right)_{ik} \|(\mathcal{L}\boldsymbol{u})_k\|_{-1,\infty} \quad \textit{for} \quad i = 1, \dots, \ell.$$

Corollary 4.2. *Under the hypotheses of Theorem* 4.3 *the boundary value problem* (4.16) *possesses a unique solution* $\boldsymbol{u}$*, and*

$$|||\boldsymbol{u}|||_{\boldsymbol{\varepsilon},\infty} := \max_{i=1,\dots,\ell} |||u_i|||_{\varepsilon_i,\infty} \leqslant C \max_{i=1,\dots,\ell} \|f_i\|_{-1,\infty}$$

for some constant C. This implies the maximum-norm stability bound $\|\boldsymbol{u}\|_\infty \leqslant C\|\boldsymbol{f}\|_\infty$.

Remark 4.4. In the above argument the only stability inequality that can be used to bound v_i is (4.6b), while for w_i and z_i one can also apply (4.4) or (4.6a). (The latter was for example employed in [28].) One then loses the additional control over the derivative, but sharper results in the maximum norm may follow, and the definition of the matrix $\boldsymbol{\Upsilon}$ has to be modified.

For the analysis of numerical methods, bounds on derivatives of the solution $\boldsymbol{u}$ of (4.16) are required. Under the assumption of Theorem 4.3 we have

$$\max_{i=1,\dots,\ell} \|u_i'\|_1 \leqslant C \max_{i=1,\dots,\ell} \|f_i\|_1 \,, \tag{4.19}$$

see [17].

The derivative bounds (4.19) are used in [17] — see also Section 4.4.2 below — to derive an abstract error estimate which essentially states that for a simple upwind difference scheme there exists a mesh giving nodal convergence of $\mathcal{O}\left(N^{-1}\right)$, the optimal order for this method. But this L_1-type information on the u_i' is insufficiently precise to allow the construction of reliable layer-adapted meshes *a priori*.

The special case where all diffusion coefficients ε_i are the same is studied in [28]. A lengthy analysis with some further assumptions on the data of the problem leads to the following result:

Theorem 4.4. *Suppose that* $\varepsilon_i = \varepsilon$ *for* $i = 1, \dots, \ell$*. Then for any* $\beta \in (0, \min_i \beta_i)$*, there exists a constant* C *such that the solution* $\boldsymbol{u}$ *of* (4.16) *can be decomposed as*

$$\boldsymbol{u} = \boldsymbol{v} + \sum_{i=1}^{\ell} [(u_i - v_i)(0)]\boldsymbol{w}_i$$

where

$$\left\|\boldsymbol{v}^{(j)}\right\|_\infty \leqslant C\left(1 + \varepsilon^{2-j}\right) \quad \textit{for} \;\; j = 0, 1, 2, 3,$$

and for each $\boldsymbol{w}_i = (w_{i1}, \dots, w_{i\ell})^\top$ *and* $x \in [0,1]$ *one has*

$$\left|w_{ik}^{(j)}(x)\right| \leqslant C\varepsilon^{-j} \mathrm{e}^{-\beta x/\varepsilon} \quad \textit{for} \;\; j = 0, 1, 2, 3 \quad \textit{and} \; k = 1, \dots, \ell.$$

4.4.2. Finite difference scheme. We continue to follow [17]. Discretize (4.16) using the scheme (4.14) for each equation of the system: Find $\boldsymbol{u}^N \in \left(\mathbb{R}_0^{N+1}\right)^\ell$ such that

$$\left[\boldsymbol{L}\boldsymbol{u}^N\right]_k := -\mathrm{diag}\;(\boldsymbol{\varepsilon})\, \boldsymbol{u}^N_{\bar{x}\breve{x};k} - \boldsymbol{B}_k \boldsymbol{u}^N_{\breve{x};k} + \boldsymbol{A}_k \boldsymbol{u}^N_{j;k} = \boldsymbol{f}_k \quad \text{for } k = 1, \dots, N-1, \tag{4.20}$$

where the notation is implicitly defined by stating that for $i = 1, \dots, \ell$ the i^{th} equation, evaluated at x_k, is

$$\left[L_i \boldsymbol{u}^N\right]_k = -\varepsilon_i u^N_{i,\bar{x}\breve{x};k} - \sum_{j=1}^{\ell} b_{ij;k} u^N_{j,\breve{x};k} + \sum_{j=1}^{\ell} a_{ij;k} u^N_{j;k} = f_{i;k}.$$

The stability analysis for the difference operator $\boldsymbol{L}$ is analogous to that for the continuous operator $\boldsymbol{\mathcal{L}}$ in Section 4.4.1.

Theorem 4.5. *Assume that $\boldsymbol{B}$ and $\boldsymbol{A}$ satisfy* (4.17). *Suppose $\boldsymbol{\Upsilon}(\boldsymbol{A},\boldsymbol{B})$ is inverse-monotone. Then*

$$|||u_i^N|||_{\varepsilon_i,\infty,\omega_N} \leqslant 2\sum_{j=1}^{\ell}\left(\boldsymbol{\Upsilon}^{-1}\right)_{ij}\left\|(\boldsymbol{L}^N\boldsymbol{u})_j\right\|_{-1,\infty,\omega_N} \quad \textit{for} \ \ i=1,\dots,\ell.$$

Corollary 4.3. *Under the hypotheses of Theorem* 4.5*, the difference equation* (4.20) *has a unique solution $\boldsymbol{u}^N$, with*

$$|||\boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\bar\omega_N} := \max_{i=1,\dots,\ell}|||u_i|||_{\varepsilon_i,\infty,\bar\omega_N} \leqslant C\max_{i=1,\dots,\ell}\|f_i\|_{-1,\infty,\bar\omega_N}$$

for some constant C.

This implies the maximum-norm stability bound $\|\boldsymbol{u}\|_{\infty,\bar\omega_N} \leqslant C\|\boldsymbol{f}\|_{\infty,\bar\omega_N}$.

Using the strong stability results of Theorems 4.3 and 4.5, we can follow [17] to obtain the *a priori* and *a posteriori* error bounds

$$\left|\left|\left|\boldsymbol{u}-\boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon},\infty,\bar\omega_N} \leqslant C\max_{k=0,\dots,N-1}\int_{x_k}^{x_{k+1}}\left(1+\sum_{i=1}^{\ell}|u_i'(x)|\right)dx$$

and

$$\left|\left|\left|\boldsymbol{u}-\boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon},\infty} \leqslant C\max_{k=0,\dots,N-1}h_{k+1}\left(1+\sum_{i=1}^{\ell}|u_{i,x;k}^N|\right).$$

By (4.19) we have $\|u_i'\|_1 \leqslant C$ for $i=1,\dots,\ell$. Therefore there exists a mesh ω^* such that

$$\int_{x_k}^{x_{k+1}}\left(1+\sum_{i=1}^{\ell}|u_i'(x)|\right)dx = \frac{1}{N}\int_0^1\left(1+\sum_{i=1}^{\ell}|u_i'(x)|\right)dx \leqslant CN^{-1}$$

and on this mesh one has consequently

$$\left|\left|\left|\boldsymbol{u}-\boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon},\infty,\bar\omega_N^*} \leqslant CN^{-1}.$$

Remark 4.5. (i) As satisfactory pointwise bounds on $|u_i'|$ are unavailable, this result does not give an immediate explicit convergence result on, e.g., a Bakhvalov or Shishkin mesh.

(ii) When $\varepsilon_i=\varepsilon$ for $i=1,\dots,\ell$, Theorem 4.4 yields

$$\left|\left|\left|\boldsymbol{u}-\boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon},\infty,\bar\omega_N} \leqslant C\max_{k=0,\dots,N-1}\int_{x_k}^{x_{k+1}}\left(1+\mathrm{e}^{-\beta x/\varepsilon}\right)dx.$$

The system behaves like the scalar equation of Section 4.1 and appropriately-adapted meshes can be constructed as in Section 4.1.3.

In [28] one also finds an error analysis for a system with a single parameter, but the analysis is limited to Shishkin meshes and uses a more traditional truncation error and barrier function argument. Furthermore, higher regularity of the solution is required. On the other hand, in certain situations the analysis of [28] is valid under less restrictive hypotheses on the entries of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ than the requirement that $\boldsymbol{\Upsilon}$ be inverse-monotone.

(iii) In [17] the above *a posteriori* bound motivates a generalization of the adaptive arc-length equidistribution procedure of [13] to systems. Numerical examples are presented in [17], but a complete analysis of the adaptive algorithm is not given.

As regards the other papers that we mentioned at the beginning of Section 4.4, [29] modifies the iterative approach of [9] from reaction — diffusion to convection — diffusion systems, [30] deals only with the special case $\ell = 2$ and $\varepsilon_1 = \varepsilon_2$, while [1] gives a very different analysis that draws on deeper results from classical analysis but considers only upwinding on an equidistant coarse mesh $\bar{\omega}_N^c$, which — even for a scalar convection — diffusion equation — cannot yield a convergence result for $\left\|\boldsymbol{u} - \boldsymbol{u}^N\right\|_{\infty,\bar{\omega}_N^c}$ that is uniform in ε.

5. Conclusions

In this survey we have seen that for finite differences the numerical analysis of systems of reaction — diffusion equations in one dimension is well developed. For such systems in two dimensions, when all equations share the same diffusion parameter we have reasonably satisfactory results, but when different diffusion parameters are present and the number of equations exceeds two then the derivation of sharp *a priori* bounds on derivatives is an unsolved problem, which forbids any effective numerical analysis of such problems.

Progress has been made in the finite difference solution of convection — diffusion systems that are weakly coupled and posed in one dimension, but further work on stability bounds is needed to improve our understanding of these problems. For such systems in two dimensions, much remains to be done. For strongly coupled convection — diffusion systems, we have only a limited grasp of the situation. Even for one-dimensional problems there are still basic difficulties: when different diffusion parameters are present, can sharp pointwise bounds on derivatives be proved? Meanwhile in two dimensions, no convergence result is known.

The survey does not pretend to be exhaustive: we have ignored time-dependent singularly perturbed systems and we have focussed almost entirely on finite difference methods (though in fact there have been few convergence results for other numerical methods that are uniform in the singular perturbation parameters). Nevertheless we hope that this summary of what is currently known will whet readers' appetites and encourage them to fill some of the gaps in our knowledge.

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