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ERROR ESTIMATES ON ARBITRARY GRIDS FOR A 2ND-ORDER MIMETIC DISCRETIZATION OF BOUNDARY-VALUE PROBLEMS FOR LINEAR ODES

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Abstract — We obtain sharp pointwise 2nd-order estimates for both solution and derivative errors on arbitrary grids for a mimetic finite-difference approximation to solutions of one-dimensional linear boundary-value problems with separated boundary conditions. Although the scheme considered is formally inconsistent with the differential equation, it turns out to possess nice convergence properties which make it a good alternative to more standard, consistent discretizations of similar arithmetic complexity, particularly with respect to derivative errors.

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1. Introduction

In this work we derive detailed error estimates for a mimetic finite difference method considered in [13, 14, 18, 19, 22] for the numerical approximation of smooth solutions to one-dimensional boundary-value problems of the form

$$-\frac{d}{dx}\left(\mathsf{K}(x)\,\frac{d\mathsf{u}}{dx}\right) + \mathsf{q}(x)\mathsf{u}(x) \;=\; \mathsf{f}(x), \quad a < x < b, \tag{1.1a}$$

$$\alpha_0\,\mathsf{u}(a) \;-\; \alpha_1\mathsf{K}(a)\,\mathsf{u}'(a) \;=\; \Gamma_a, \quad \beta_0\,\mathsf{u}(b) \;+\; \beta_1\mathsf{K}(b)\,\mathsf{u}'(b) \;=\; \Gamma_b, \tag{1.1b}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \Gamma_a, \Gamma_b$ are given constants satisfying

$$\alpha_0,\, \alpha_1,\, \beta_0,\, \beta_1 \,\geqslant\, 0, \qquad \alpha_0 + \alpha_1 > \,0, \qquad \beta_0 + \beta_1 > \,0, \qquad \alpha_0 + \beta_0 > \,0, \tag{1.1c}$$

and where $\mathsf{K}, \mathsf{q}, \mathsf{f}$ denote certain known (smooth) functions, with $\mathsf{K}(x) > 0$ and $\mathsf{q}(x) \geqslant 0$ everywhere on $[\,a, b\,]$.

Under these conditions, it is well known that problem (1.1) admits a unique solution u, which cannot in general be obtained in closed, exact form. Hence, some sort of approximation must be used to compute u, such as that provided by discrete methods like finite difference or finite element formulae.

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Precise convergence results like those obtained here, especially when they are extended to multiple dimension, are important because mimetic and related methods are now seeing extensive applications and are undergoing rapid theoretical development. An idea of recent developments in mimetic finite difference methods can be found in [2,4,5,16,21]. An overview of related ideas for finite elements can be found in [1,3] and for applications to electromagnetics in [9,10,12]. There has also been extensive development and applications of discrete differential forms [7,8,11], including the construction of an extensive programming library [6].

For the discretization of (1.1), we set up some grid on the interval $[a,b]$, picking $N+1$ points $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$, called *nodes*, which divide $[a,b]$ into N subintervals $[x_{i-1}, x_i]$, or *cells*, with lengths $L_{i-1/2} = x_i - x_{i-1}$, whose central points will be denoted by $x_{i-1/2}$, $1 \leqslant i \leqslant N$. (Here, we follow notation in [14,18].) It will prove convenient to set $x_{-1/2} \equiv x_0$, $L_{-1/2} \equiv 0$, $x_{N+1/2} \equiv x_N$, $L_{N+1/2} \equiv 0$, and define, for each node, the nodal length h_i given by $h_i := x_{i+1/2} - x_{i-1/2}$, i.e.,

$$h_i := \frac{L_{i-1/2} + L_{i+1/2}}{2}, \quad 0 \leqslant i \leqslant N \quad (L_{-1/2} \equiv 0, \quad L_{N+1/2} \equiv 0). \tag{1.2}$$

These quantities are illustrated in the figure 1.1.

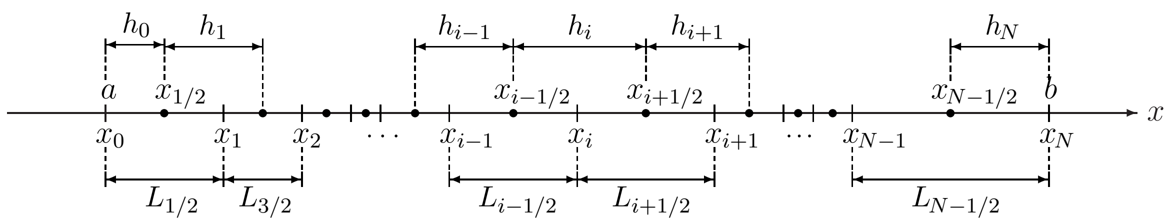


Fig. 1.1. Nodal points x_i, $0 \leqslant i \leqslant N$, cell centers $x_{i-1/2}$, $1 \leqslant i \leqslant N$, and cell and nodal lengths $L_{i-1/2}$, h_i

The mimetic scheme to be considered here can then be written in the form

$$-\mathcal{D}(K \cdot \mathcal{G} v^h) + q \cdot v^h = f, \tag{1.3a}$$

$$\alpha_0 v_0 - \alpha_1 K_0 \cdot (\mathcal{G} v^h)_0 = \Gamma_a, \quad \beta_0 v_N + \beta_1 K_N \cdot (\mathcal{G} v^h)_N = \Gamma_b, \tag{1.3b}$$

for appropriate difference operators $\mathcal{D}$ ("discrete divergence"), $\mathcal{G}$ ("discrete gradient") and discrete functions K, q, f (or, in fuller notation, K^h, q^h, f^h, where superscript h refers to the grid) that represent (project) $\mathsf{K}, \mathsf{q}, \mathsf{f}$ on appropriate grid points [13]; solving (1.3) for v^h gives the approximation sought for the exact values u^h. In our case, $\mathcal{D}$, $\mathcal{G}$ are defined by

$$(\mathcal{D} w)_{i-1/2} = \frac{w_i - w_{i-1}}{L_{i-1/2}}, \quad 1 \leqslant i \leqslant N, \tag{1.4a}$$

$$(\mathcal{G} z)_i = \frac{z_{i+1/2} - z_{i-1/2}}{h_i}, \quad 0 \leqslant i \leqslant N, \tag{1.4b}$$

for (arbitrary) discrete functions w, z defined at the grid points x_i, $x_{i-1/2}$, respectively. Thus, (1.3a) reads

$$-\frac{K_{i-1}}{h_{i-1}} v_{i-3/2} + \left(\frac{K_{i-1}}{h_{i-1}} + \frac{K_i}{h_i} + q_{i-1/2} L_{i-1/2}\right) v_{i-1/2} - \frac{K_i}{h_i} v_{i+1/2} = L_{i-1/2} f_{i-1/2} \tag{1.5}$$

for $1 \leqslant i \leqslant N$, where $K_i = \mathsf{K}(x_i)$, $K_{i-1} = \mathsf{K}(x_{i-1})$, $q_{i-1/2} = \mathsf{q}(x_{i-1/2})$, and so forth.

Our goal is to investigate the errors $e^h := v^h - u^h$ (solution error), $E^h := \mathcal{G}v^h - (\mathsf{u}')^h$ (derivative or gradient error), which are related to τ^h (truncation error) defined by

$$-\mathcal{D}(K \cdot \mathcal{G}u^h) + q \cdot u^h = f + \tau^h \tag{1.6a}$$

at the cell centers, and

$$\alpha_0 u_0 - \alpha_1 K_0 \cdot (\mathcal{G}u^h)_0 = \Gamma_a + \tau_0, \quad \beta_0 u_N + \beta_1 K_N \cdot (\mathcal{G}u^h)_N = \Gamma_b + \tau_N \tag{1.6b}$$

at the endpoints $x_0 = a$, $x_N = b$. The relevance of τ^h can be seen from the equations

$$-\mathcal{D}(K \cdot \mathcal{G}e^h) + q \cdot e^h = -\tau^h \tag{1.7a}$$

$$\alpha_0 e_0 - \alpha_1 K_0 \cdot (\mathcal{G}e^h)_0 = -\tau_0, \quad \beta_0 e_N + \beta_1 K_N \cdot (\mathcal{G}e^h)_N = -\tau_N \tag{1.7b}$$

relating τ^h to e^h. Likewise, similar steps can be given for discrete methods in general, and it is a fundamental result that, as the grid is infinitely refined ("$h \to 0$"), condition $\tau^h \to 0$ ("consistency") turns out to be sufficient[1] to assure $e^h \to 0$ ("convergence"), although this is by no means necessary [14, 18, 22]. This is the case of our scheme (1.3), (1.4), for which[2]

$$\begin{aligned}\tau_{i-1/2} = {} & K_{i-1/2}\, u''_{i-1/2} \left(1 - \frac{h_i + h_{i-1}}{2\, L_{i-1/2}}\right) - \frac{1}{6}\, K_{i-1/2}\, u'''_{i-1/2}\, \frac{h_i^2 - h_{i-1}^2}{L_{i-1/2}} \\ & - \frac{1}{4}\, K'_{i-1/2}\, u''_{i-1/2}\, (h_i - h_{i-1}) + O(L_{i-3/2}^2) + O(L_{i-1/2}^2) + O(L_{i+1/2}^2)\end{aligned} \tag{1.8}$$

for $1 \leqslant i \leqslant N$: not only τ^h may fail to vanish uniformly as $h \to 0$, it may even grow unboundedly! And yet, as it will be shown in the sequel, the mimetic method happens to have some nice convergence properties, with [2]

$$e_{i-1/2} = -\frac{1}{8}\, \mathsf{u}''(x_{i-1/2})\, L_{i-1/2}^2 + O(\hbar^2), \quad 1 \leqslant i \leqslant N \tag{1.9a}$$

$$e_0 = O(\hbar^2), \quad e_N = O(\hbar^2) \tag{1.9b}$$

$$E_i = O(\hbar^2), \quad 0 \leqslant i \leqslant N \tag{1.9c}$$

uniformly in i, where $\hbar$ is the global grid spacing measure [17, 20] given by

$$\hbar = \sqrt{\sum_{j=1}^{N} L_{j-1/2}^3}\,. \tag{1.10}$$

Insight into the estimator $\hbar$ can be gained by noting that for uniform grids on $[a, b]$ with cells of length h, $\hbar = \sqrt{b-a}\, h$, and if one mimimizes $\hbar$ given in equation (1.10) with respect to the $L_{j-1/2}$'s with the constraint $L_{1/2} + \ldots + L_{N-1/2} = b - a$, one obtains a uniform grid with $h = (b-a)/N$. A common way of generating a grid [15] is to choose a smooth

[1] One must note that, in our present setting, *consistency* is also sufficient for *zero-stability* [22].

[2] Expressions (1.8), (1.9) are valid provided that $\mathsf{K} \in C^3([a, b])$, $\mathsf{u} \in C^4([a, b])$. If K, u are less smooth, then these must be changed accordingly.

monotone function $\Phi = \Phi(\xi)$ that maps [0,1] to $[a, b]$ and then set $h = 1/N$ and $x_i = \Phi(i\,h)$. In this case,

$$\hbar \approx h\Big(\int_0^1 \Phi'(\xi)^3\, d\xi\Big)^{1/2}$$

for small h. For generating functions of the form $\Phi(\xi) = (b-a)\,\xi^p + a,\ p > 0$, convergence is second order in h for $p > 2/3$, that is, convergence is still second order in $1/N$ for such singular grids where $\Phi'(\xi)$ is unbounded.

By (1.9), the error $e_{i-1/2} = v_{i-1/2} - u_{i-1/2}$ at the i-th cell is made of two components: one local, with size $O(L^2_{i-1/2})$, and a global component $\varepsilon^{\mathtt{glob}}_{i-1/2} = e_{i-1/2} + 1/8\, u''_{i-1/2} L^2_{i-1/2}$ whose size depends on the entire grid. At the endpoints $x_0 = a$, $x_N = b$, however, e^h behaves globally (to second order accuracy), as do the derivative errors $E_i = (\mathcal{G}v^h)_i - u'_i$ everywhere. This is precisely the behavior observed in numerical experiments [13, 18, 22], including the case of negative or sign-changing q, as illustrated in Figures 1.2 and 1.3 below.

2. Error analysis for $\mathbf{q} = 0$

We first derive (1.9) in the fundamental case $\mathsf{q} = 0$, and then extend the results to more general q in Section 3. It will be sufficient to consider the boundary conditions to be, say, of Dirichlet type at one end and Robin or Neumann type at the other, since the other cases can be handled in an entirely similar way. Thus, we set

$$\mathsf{u}(a) \;=\; \Gamma_a, \quad \beta\, \mathsf{u}(b) \;+\; \mathsf{K}(b)\, \mathsf{u}'(b) \;=\; \Gamma_b \tag{2.1}$$

for some given β, Γ_a, $\Gamma_b \in \mathbb{R}$, with $\beta \geqslant 0$.[1] In particular, we take $v_0 = \Gamma_a$, and so $e_0 = 0$. The key point to determine the other errors is to obtain the quantity $\mathcal{G}e^h$ first [14, 22], which is achieved in (2.13) below. To do this, we observe that

$$K_i(\mathcal{G}e^h)_i \;=\; K_N(\mathcal{G}e^h)_N \;-\; \sum_{j=i+1}^{N} L_{j-1/2}\,(\mathcal{D}(K\cdot\mathcal{G}e^h))_{j-1/2} \tag{2.2}$$

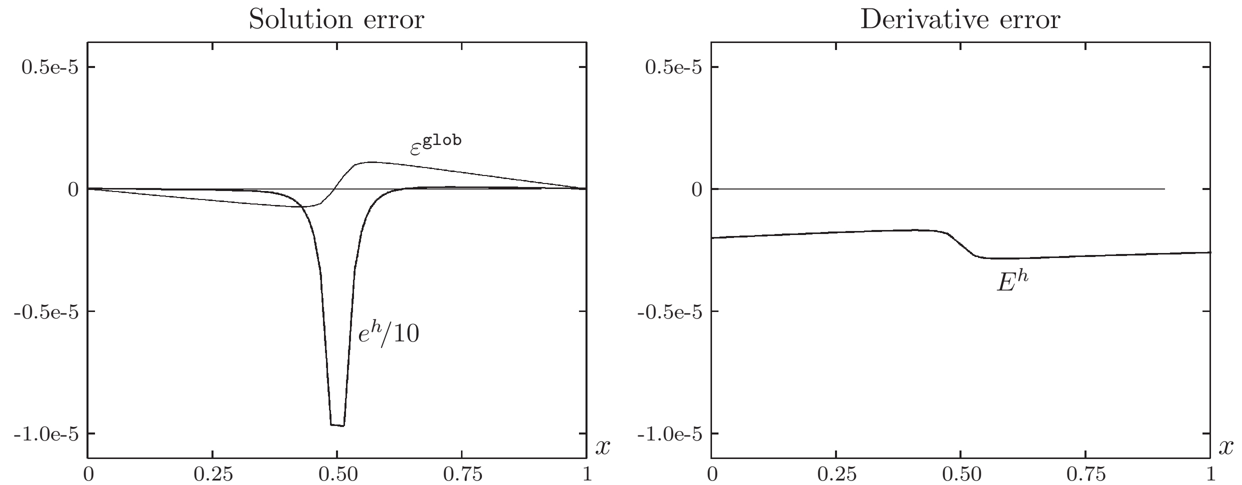
for all $0 \leqslant i \leqslant N$, so that we obtain, by (1.7a) and recalling that $\mathsf{q} = 0$,

$$K_i(\mathcal{G}e^h)_i \;=\; K_N(\mathcal{G}e^h)_N \;-\; \sum_{j=i+1}^{N} L_{j-1/2}\,\tau_{j-1/2}, \quad 0 \leqslant i \leqslant N. \tag{2.3}$$

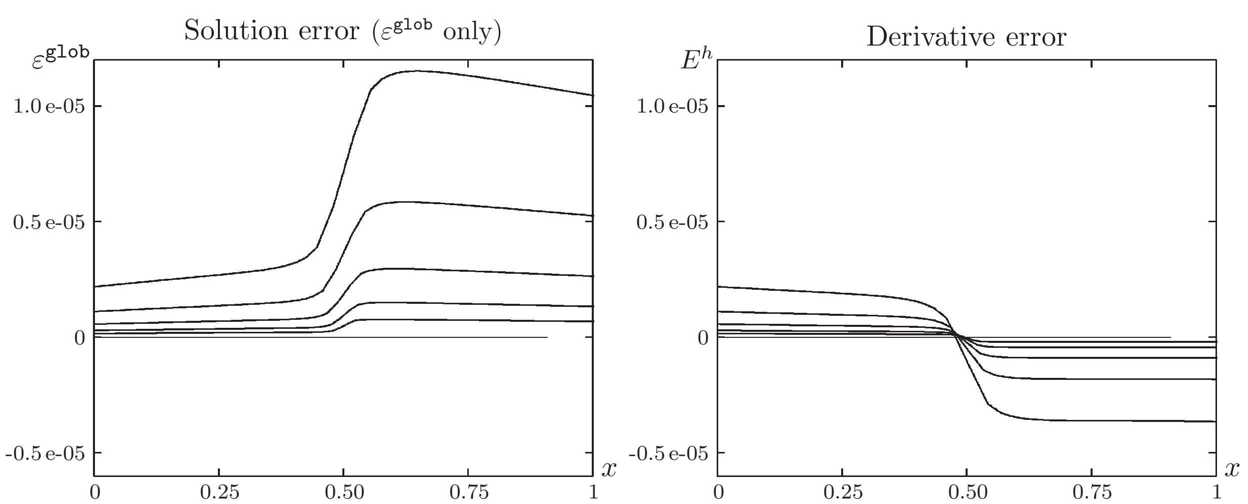
Now, from (1.8), we get

$$\sum_{j=i+1}^{N} L_{j-1/2}\,\tau_{j-1/2} \;=\; O(\hbar^2) \;+\; \sum_{j=i+1}^{N} K_{j-1/2}\,u''_{j-1/2}\Big(L_{j-1/2} \;-\; \frac{h_{j-1}+h_j}{2}\Big) -$$
$$-\,\frac{1}{4}\sum_{j=i+1}^{N} K'_{j-1/2}\,u''_{j-1/2}\,L_{j-1/2}\,(\,h_j - h_{j-1}) \;-\; \frac{1}{6}\sum_{j=i+1}^{N} K_{j-1/2}\,u'''_{j-1/2}\,(\,h_j^2 - h_{j-1}^2) \tag{2.4}$$

[1] Actually, as will be clear in the analysis below, we need only assume $\beta > -\Big(\int_a^b \mathsf{K}(x)^{-1} dx\Big)^{-1}$.



F i g. 1.2. Errors $e^h_{i-1/2}$, $\varepsilon^{\texttt{glob}}_{i-1/2} = e^h_{i-1/2} + 1/8\, u''_{i-1/2} L^2_{i-1/2}$, E^h_i for the Dirichlet problem (1.1) on $[0,1]$ with $\mathsf{K}(x) = 2 + \sin x$, $\mathsf{q}(x) = -\mathrm{e}^x/10$, $\mathsf{u}(x) = 1 + \cosh x$ and grid points $x_0 = 0$, $x_i = x_{i-1} + H/(1001 - i)^{3/4}$, $1 \leqslant i \leqslant 1000$, $x_i = x_{i-1} + H/(i - 1000)^{3/4}$, $1001 \leqslant i \leqslant 2000$, $H = 0.262\text{e-}1$; for this grid, $\hbar = 0.726\text{e-}2$. The values of e^h are shown divided by a factor of 10 to fit the picture



F i g. 1.3. Errors $\varepsilon^{\texttt{glob}}_{i-1/2}$, E^h_i for five successively refined grids in the case of problem (1.1) on $[0,1]$ with $\mathsf{K}(x) = 2+\sin x$, $\mathsf{q}(x) = -\mathrm{e}^x/10$, $\mathsf{u}(x) = 1+\cosh x$ and boundary conditions of Robin type $-2\mathsf{u}(0) + \mathsf{K}(0)\mathsf{u}'(0) = \Gamma_0$, $\mathsf{u}(1) + \mathsf{K}(1)\mathsf{u}'(1) = \Gamma_1$, showing the $O(\hbar^2)$ behavior as these errors are halved each time $\hbar^2$ is halved. Grid points are $x_0 = 0$, $x_i = x_{i-1} + H_N/(1001-i)^{3/4}$, $1 \leqslant i \leqslant N/2$, $x_i = x_{i-1} + H_N/(i-1000)^{3/4}$, $N/2+1 \leqslant i \leqslant N$, with $N = 444$, 926, 2000, 4440 and 10064, corresponding to $\hbar = 1.452\text{e-}2$. $1.028\,\text{e-}2$, $0.726\,\text{e-}2$, $0.513\,\text{e-}2$ and $0.363\,\text{e-}2$, respectively and we proceed by estimating the sums on the right hand side of (2.4). For the first sum, setting $\mathsf{w}(x) := \mathsf{K}(x)\mathsf{u}''(x)$, we have

$$\sum_{j=i+1}^{N} K_{j-1/2}\, u''_{j-1/2}\Big(L_{j-1/2} \;-\; \frac{h_{j-1}+h_j}{2}\Big) \;=$$

$$-\frac{1}{4}\sum_{j=i+1}^{N} w_{j-1/2}\big\{\,(\,L_{j+1/2}-L_{j-1/2}\,)\;-\;(\,L_{j-1/2}-L_{j-3/2}\,)\,\big\} =$$

$$-\frac{1}{4}\sum_{j=i+1}^{N} \big\{\, w_j\,(\,L_{j+1/2}-L_{j-1/2}\,)\;-\;w_{j-1}\,(\,L_{j-1/2}-L_{j-3/2}\,)\,\big\}+$$

$$\frac{1}{4}\sum_{j=i+1}^{N} \big\{\,(\,w_j-w_{j-1/2}\,)\,(\,L_{j+1/2}-L_{j-1/2}\,)\;-\;(\,w_{j-1}-w_{j-3/2}\,)\,(\,L_{j-1/2}-L_{j-3/2}\,)\,\big\}+$$

$$\frac{1}{4}\sum_{j=i+1}^{N} (\,w_{j-1/2}-w_{j-3/2}\,)\,(\,L_{j-1/2}-L_{j-3/2}\,) =$$

$$\frac{1}{4}w_{N-1/2}L_{N-1/2}+\frac{1}{4}w_{i-1/2}(L_{i+1/2}-L_{i-1/2})+\frac{1}{8}\sum_{j=i+1}^{N} w'_{j-1}(L^2_{j-1/2}-L^2_{j-3/2})+O(\hbar^2) =$$

$$\frac{1}{4}w_{N-1/2}L_{N-1/2}+\frac{1}{8}w'_{N-1/2}L^2_{N-1/2}+\frac{1}{4}w_{i-1/2}(L_{i+1/2}-L_{i-1/2})-\frac{1}{8}w'_iL^2_{i-1/2}+O(\hbar^2) =$$

$$\frac{1}{4}\,w_N\,L_{N-1/2}\;+\;\frac{1}{4}\,w_i(\,L_{i+1/2}-L_{i-1/2}\,)\;-\;\frac{1}{8}\,w'_i\,L_{i-1/2}L_{i+1/2}\;+\;O(\hbar^2),$$

so that we obtain

$$\sum_{j=i+1}^{N} K_{j-1/2}u''_{j-1/2}(L_{j-1/2}-\frac{h_{j-1}+h_j}{2}) = \frac{1}{4}K_N u''_N L_{N-1/2}+\frac{1}{4}\,K_i u''_i(L_{i+1/2}-L_{i-1/2})-$$

$$\frac{1}{8}\,K'_i\,u''_i\,L_{i-1/2}\,L_{i+1/2}\;-\;\frac{1}{8}\,K_i\,u'''_i\,L_{i-1/2}\,L_{i+1/2}\;+\;O(\hbar^2). \tag{2.5a}$$

In a similar way, for the second sum in (2.4), setting $\tilde{\mathsf{w}}(x) := \mathsf{K}'(x)\mathsf{u}''(x)$, we get

$$\sum_{j=i+1}^{N} K'_{j-1/2}u''_{j-1/2}L_{j-1/2}(h_j-h_{j-1}) = -\frac{1}{4}\sum_{j=i+1}^{N}\big\{\tilde{w}_jL_{j-1/2}h_j-\tilde{w}_{j-1}L_{j-1/2}h_{j-1}\big\}+O(\hbar^2) =$$

$$\tilde{w}_N\,L_{N-1/2}\,h_N\;-\;\tilde{w}_i\,L_{i-1/2}\,h_i\;-\sum_{j=i+1}^{N}\tilde{w}_{j-1}\,(\,L_{j-1/2}-L_{j-3/2}\,)\,h_{j-1}\;+\;O(\hbar^2) =$$

$$\tilde{w}_N\,L_{N-1/2}\,h_N\;-\;\tilde{w}_i\,L_{i-1/2}\,h_i\;-\;\frac{1}{2}\sum_{j=i+1}^{N}\Big(\tilde{w}_j\,L^2_{j-1/2}-\tilde{w}_{j-1}\,L^2_{j-3/2}\Big)\;+\;O(\hbar^2) =$$

$$= -\frac{1}{2}\,\tilde{w}_i\,L_{i-1/2}\,L_{i+1/2}\;\;+\;O(\hbar^2),$$

$$\sum_{j=i+1}^{N} K'_{j-1/2}\, u''_{j-1/2}\, L_{j-1/2}\,(\,h_j - h_{j-1}\,) \;=\; -\,\frac{1}{2}\, K'_i\, u''_i\, L_{i-1/2}\, L_{i+1/2} \;+\; O(\hbar^2). \tag{2.5b}$$

Finally, for the third sum in (2.4), we obtain

$$\sum_{j=i+1}^{N} K_{j-1/2}\, u'''_{j-1/2}\,(\,h_j^2 - h_{j-1}^2\,) \;=\; \frac{1}{4}\, K_{N-1/2}\, u'''_{N-1/2}\, L^2_{N-1/2} \;-\; K_i\, u'''_i\, h_i^2 \;+\; O(\hbar^2). \tag{2.5c}$$

Hence, from (2.4) and (2.5a)–(2.5c), we get

$$\sum_{j=i+1}^{N} L_{j-1/2}\, \tau_{j-1/2} \;=\; \frac{1}{4}\, K_N\, u''_N\, L_{N-1/2} \;-\; \frac{1}{24}\, K_N\, u'''_N\, L^2_{N-1/2} \;+\frac{1}{4}\, K_i\, u''_i\,(\,L_{i+1/2} - L_{i-1/2}\,) \;-$$

$$\frac{1}{8}\, K_i\, u'''_i\, L_{i-1/2}\, L_{i+1/2} \;+\; \frac{1}{6}\, K_i\, u'''_i\, h_i^2 \;+\; O(\hbar^2).$$

Now, for (2.3) to be useful, there still remains to estimate $(\mathcal{G}e^h)_N = \big(e_N - e_{N-1/2}\big)/h_N$. By (1.7b)[1], we have $K_N\,(\mathcal{G}e^h)_N = -\,\tau_N - \beta e_N$, where τ_N is the truncation error at $x_N = b$,

$$\tau_N \;=\; -\,\frac{1}{4}\, K_N u''_N\, L_{N-1/2} \;+\; \frac{1}{24}\, K_N u'''_N\, L^2_{N-1/2} \;+\; O(L^3_{N-1/2}); \tag{2.7}$$

thus, $(\mathcal{G}e^h)_N$ is easily obtained when $\beta = 0$. For general $\beta \geqslant 0$, the following procedure can be used: solving (1.7a), (1.7b) for e_N, we obtain

$$e_N \;=\; -\left(\tau_N \;+\; \frac{c_N\, \theta^{[0]}_{N-1}\, c_{N-1}}{c_N + \theta^{[0]}_{N-1}\, c_{N-1}} \sum_{j=1}^{N} \frac{1}{\theta^{[0]}_{j-1}\, c_{j-1}}\; L_{j-1/2}\, \tau_{j-1/2}\right)\bigg/\left(\theta^{[0]}_N\, c_N \;+\; \beta\right) \tag{2.8a}$$

where

$$c_i \;=\; \frac{K_i}{h_i}, \qquad \theta^{[0]}_i \;=\; \frac{h_i}{K_i}\left(\sum_{\ell=0}^{i} \frac{h_\ell}{K_\ell}\right)^{-1}, \qquad 0 \leqslant i \leqslant N. \tag{2.8b}$$

Setting

$$I_i \;:=\; \sum_{\ell=0}^{i} \frac{h_\ell}{K_\ell}\,, \qquad 0 \leqslant i \leqslant N, \tag{2.9}$$

so that, in particular, $I_N = \int_a^b \frac{1}{\mathsf{K}(x)}\, dx + O(\hbar^2)$, (2.8a) reduces to

$$e_N \;=\; -\left(\tau_N \;+\; \frac{1}{I_N} \sum_{j=1}^{N} I_{j-1}\, L_{j-1/2}\, \tau_{j-1/2}\right)\bigg/\left(\beta \;+\; \frac{1}{I_N}\right). \tag{2.10}$$

Similarly to (2.5a) – (2.5c) above, we can show

$$\sum_{j=1}^{N} I_{j-1}\, K_{j-1/2}\, u''_{j-1/2}\left(L_{j-1/2} \;-\; \frac{h_{j-1} + h_j}{2}\right) \;=\; \frac{1}{4}\, I_N\, K_N\, u''_N\, L_{N-1/2} \;+\; O(\hbar^2),$$

[1]Here, by (2.1), one has $\alpha_0 = 1$, $\alpha_1 = 0$, $\tau_0 = 0$, $\beta_0 = \beta$, $\beta_1 = 1$ in equation (1.7b).

$$\sum_{j=1}^{N} I_{j-1}\, K'_{j-1/2}\, u''_{j-1/2}\, L_{j-1/2}\,(\, h_j - h_{j-1}\,) \;=\; O(\hbar^2),$$

$$\sum_{j=1}^{N} I_{j-1}\, K_{j-1/2}\, u'''_{j-1/2}\,(\, h_j^2 - h_{j-1}^2\,) \;=\; \frac{1}{4}\, I_N\, K_N\, u'''_N\, L_{N-1/2}^2 \;+\; O(\hbar^2),$$

so that we have

$$\sum_{j=1}^{N} I_{j-1}\, L_{j-1/2}\, \tau_{j-1/2} \;=\; \frac{1}{4}\, I_N\, K_N\, u''_N\, L_{N-1/2} \;-\; \frac{1}{24}\, I_N\, K_N\, u'''_N\, L_{N-1/2}^2 \;+\; O(\hbar^2). \tag{2.11}$$

Hence, by (2.7), (2.10) and (2.11), we obtain

$$e_N \;=\; O(\hbar^2), \tag{2.12}$$

and so $K_N\,(\,\mathcal{G}e^h\,)_N \;=\; -\,\tau_N \;-\; \beta e_N$ gives

$$(\,\mathcal{G}e^h\,)_N \;=\; \frac{1}{4}\, u''_N\, L_{N-1/2} \;-\; \frac{1}{24}\, u'''_N\, L_{N-1/2}^2 \;+\; O(\hbar^2). \tag{2.13a}$$

Recalling (2.3) and (2.6), this yields the fundamental estimate

$$(\,\mathcal{G}e^h\,)_i \;=\; -\frac{1}{4}\, u''_i\,(\, L_{i+1/2} -\; L_{i-1/2}\,) \;-\; \frac{1}{24}\, u'''_i\,(\, L_{i-1/2}^2 \;-\; L_{i-1/2}L_{i+1/2} \;+\; L_{i+1/2}^2\,) \;+\; O(\hbar^2) \tag{2.13b}$$

for all $0 \leqslant i \leqslant N$.

Once $\mathcal{G}e^h$ and one of the errors e_0 or e_N have been estimated, it becomes simple to obtain the errors e^h, E^h by the following procedure [14, 22]. Starting with E^h, we note that

$$E_i = (\mathcal{G}e^h)_i + (\mathcal{G}u^h)_i - u'_i = (\mathcal{G}e^h)_i + \frac{1}{4}u''_i(L_{i+1/2} - L_{i-1/2}) + \frac{1}{48}u'''_i\frac{L_{i-1/2}^3 + L_{i+1/2}^3}{h_i} + O(\hbar^2),$$

so that we obtain, by (2.13),

$$E_i \;=\; O(\hbar^2), \quad 0 \leqslant i \leqslant N. \tag{2.14}$$

Now, given $0 \leqslant i \leqslant N$, we have, by definition of E^h,

$$v_{i-1/2} \;=\; v_N \;-\sum_{j=i}^{N} h_j\, E_j \;-\sum_{j=i}^{N} h_j\, u'_j,$$

while, by the trapezoidal quadrature rule,

$$u_{i-1/2} \;=\; u_N \;+\; \frac{1}{8}\, u''_{i-1/2}L_{i-1/2}^2 \;-\sum_{j=i}^{N} h_j\, u'_j \;+\; O(\hbar^2).$$

Therefore, for $0 \leqslant i \leqslant N$,[1]

$$e_{i-1/2} \;=\; e_N \;-\; \frac{1}{8}\, u''_{i-1/2}L_{i-1/2}^2 \;-\sum_{j=i}^{N} h_j\, E_j \;+\; O(\hbar^2), \tag{2.15}$$

and we obtain (1.9a), (1.9b) from (2.12), (2.14) and (2.15), concluding the argument. $\square$

[1]Similarly, we can show that $e_{i-1/2} \;=\; e_0 \;-\; \frac{1}{8}\, u''_{i-1/2}L_{i-1/2}^2 \;+\sum_{j=0}^{i-1} h_j\, E_j \;+\; O(\hbar^2)$ for $1 \leqslant i \leqslant N+1$.

3. Error analysis for $\mathsf{q} \geqslant 0$

We will now obtain (1.9) for arbitrary $\mathsf{q} \geqslant 0$, and same assumptions as in Section 1. Given q, it will be convenient here to change notation slightly and denote by $\mathsf{u}^{[\mathsf{q}]}$, $v^{[\mathsf{q}]}$ the solutions of problems (1.1), (1.3), respectively, with corresponding errors $e^{[\mathsf{q}]}$, $E^{[\mathsf{q}]}$ given by

$$e^{[\mathsf{q}]}_{i-1/2} \;=\; v^{[\mathsf{q}]}_{i-1/2} \;-\; \mathsf{u}^{[\mathsf{q}]}(x_{i-1/2}), \quad 0 \leqslant i \leqslant N+1, \tag{3.1}$$

$$E^{[\mathsf{q}]}_{i} \;=\; (\,\mathcal{G} v^{[\mathsf{q}]})_i \;-\; \frac{d\mathsf{u}^{[\mathsf{q}]}}{dx}(x_i), \quad 0 \leqslant i \leqslant N, \tag{3.2}$$

i.e., we make explicit their dependence on q. In Section 2, estimates (1.9), (2.13) were derived for $e^{[0]}$, $E^{[0]}$; extension to $e^{[\mathsf{q}]}$, $E^{[\mathsf{q}]}$ follows directly from Theorems 3.1 and 3.2 below.

Theorem 3.1. $e^{[\mathsf{q}]} \;=\; e^{[0]} \;+\; O(\hbar^2)$, i.e., $\|\, e^{[\mathsf{q}]} \;-\; e^{[0]} \,\|_{\sup} \;=\; O(\hbar^2)$.

Proof. We assume that the boundary conditions are given by (2.1), since the other cases can be treated in a similar way. (Thus, as $e^{[\mathsf{q}]}_0 = 0$ here, we set $e^{[\mathsf{q}]} = \{\, e^{[\mathsf{q}]}_{i-1/2}, 1 \leqslant i \leqslant N+1 \,\}$ in the following argument.) Writing (1.7a), (1.7b) as $A_h(\mathsf{q}) e^{[\mathsf{q}]} = -\, b^h$, where

$$A_h(\mathsf{q}) = \begin{bmatrix} a^{[\mathsf{q}]}_{1/2} & -\,c_1 & 0 & \cdots & 0 & 0 \\ -\,c_1 & a^{[\mathsf{q}]}_{3/2} & -\,c_2 & \cdots & 0 & 0 \\ 0 & -\,c_2 & a^{[\mathsf{q}]}_{5/2} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a^{[\mathsf{q}]}_{N-1/2} & -\,c_N \\ 0 & 0 & 0 & \cdots & -\,c_N & a_N \end{bmatrix}, \quad b^h = \begin{bmatrix} L_{1/2}\,\tau_{1/2} \\ L_{3/2}\,\tau_{3/2} \\ L_{5/2}\,\tau_{5/2} \\ \cdot \\ \cdot \\ \cdot \\ L_{N-1/2}\,\tau_{N-1/2} \\ \tau_N \end{bmatrix}$$

with $\tau_{1/2}, \ldots, \tau_{N-1/2}$ and τ_N given in (1.8), (2.7), $c_i = K_i/h_i$, $0 \leqslant i \leqslant N$, $a_N = \beta + c_N$ and $a^{[\mathsf{q}]}_{i-1/2} = c_{i-1} + c_i + q_{i-1/2}\, L_{i-1/2}$, $1 \leqslant i \leqslant N$, we get, recalling that $A_h(0) e^{[0]} = -\, b^h$,

$$A_h(\mathsf{q})\,(e^{[\mathsf{q}]} - e^{[0]}) \;=\; -\,Q_h(\mathsf{q})\, e^{[0]}, \tag{3.3}$$

where $Q_h(\mathsf{q})$ is a diagonal matrix, given by $Q_h(\mathsf{q}) = \operatorname{diag}(\, q_{1/2}\, L_{1/2}, \ldots, q_{N-1/2}\, L_{N-1/2}\,, 0\,)$. Reducing (3.3) to triangular form, we obtain $\hat{A}_h(\mathsf{q})\,(e^{[\mathsf{q}]} - e^{[0]}) \;=\; \hat{\zeta}^{[\mathsf{q}]}$, where $\hat{\zeta}^{[\mathsf{q}]} \in \mathbb{R}^{N+1}$ is given by

$$\hat{\zeta}^{[\mathsf{q}]}_{i-1/2} \;=\; -\sum_{j=1}^{i} \frac{\theta^{[\mathsf{q}]}_{i-1} c_{i-1}}{\theta^{[\mathsf{q}]}_{j-1} c_{j-1}}\, \nu^{[\mathsf{q}]}_{ij}\, L_{j-1/2}\, q_{j-1/2}\, e^{[0]}_{j-1/2}, \quad 1 \leqslant i \leqslant N+1,$$

with $|\,\nu^{[\mathsf{q}]}_{ij}\,| \leqslant 1$ for all i, j, and positive $\theta^{[\mathsf{q}]}_0$, $\theta^{[\mathsf{q}]}_1, \ldots, \theta^{[\mathsf{q}]}_N$ given recursively by $\theta^{[\mathsf{q}]}_0 = 1$ and

$$\theta^{[\mathsf{q}]}_i \;=\; \frac{c_{i-1}\,\theta^{[\mathsf{q}]}_{i-1} \;+\; q_{i-1/2}\, L_{i-1/2}}{c_{i-1}\,\theta^{[\mathsf{q}]}_{i-1} \;+\; c_i \;+\; q_{i-1/2}\, L_{i-1/2}}, \quad 1 \leqslant i \leqslant N.$$

Moreover, because $\mathsf{q} \geqslant 0$, we have

$$\frac{1}{\theta_i^{[\mathsf{q}]} c_i} \leqslant I_i, \quad 0 \leqslant i \leqslant N,$$

with I_i defined in (2.9), so that

$$|\,\hat{\zeta}^{[\mathsf{q}]}_{i-1/2}\,| \leqslant \theta^{[\mathsf{q}]}_{i-1} c_{i-1}\, I_N \,\|\, \mathsf{q} \,\|_{\sup} \sum_{j=1}^{N} L_{j-1/2}\, |\, e^{[0]}_{j-1/2}\,| \tag{3.4}$$

for all $1 \leqslant i \leqslant N+1$, where $\|\, \mathsf{q} \,\|_{\sup}$ denotes the supnorm of q on $[\,a, b\,]$. Now, as can be readily checked, if $\hat{z} = (\,\hat{z}_{1/2}, \ldots, \hat{z}_{N-1/2},\, \hat{z}_{N+1/2}) \in \mathbb{R}^{N+1}$ is such that $|\,\hat{z}_{i-1/2}\,| \leqslant \theta^{[\mathsf{q}]}_{i-1} c_{i-1}\, \Gamma$ for all $1 \leqslant i \leqslant N+1$, and some $\Gamma \geqslant 0$ independent of i, the solution $\hat{w}$ of $\hat{A}_h(\mathsf{q})\, \hat{w} = \hat{z}$ verifies $\|\, \hat{w} \,\|_{\sup} \leqslant \Gamma$. Therefore, (3.4) gives

$$\|\, e^{[\mathsf{q}]} - e^{[0]} \,\|_{\sup} \leqslant I_N \,\|\, \mathsf{q} \,\|_{\sup} \sum_{j=1}^{N} L_{j-1/2}\, |\, e^{[0]}_{j-1/2}\,|, \tag{3.5}$$

from which we obtain, by our previous results, $\|\, e^{[\mathsf{q}]} - e^{[0]} \,\|_{\sup} = O(\hbar^2)$, as claimed. □

Theorem 3.2. $\mathcal{G}e^{[\mathsf{q}]} = \mathcal{G}e^{[0]} + O(\hbar^2)$, $E^{[\mathsf{q}]} = E^{[0]} + O(\hbar^2)$.

Proof. Starting with $\mathcal{G}e^{[\mathsf{q}]}$, one has, by (1.7a), (2.2) and Theorem 3.1 above,

$$K_i(\mathcal{G}e^{[\mathsf{q}]})_i = K_N(\mathcal{G}e^{[\mathsf{q}]})_N - \sum_{j=i+1}^{N} L_{j-1/2}\tau_{j-1/2} + O(\hbar^2), \quad 0 \leqslant i \leqslant N-1,$$

uniformly in i, so that, by (2.3), we need only to show that $(\mathcal{G}e^{[\mathsf{q}]})_N = (\mathcal{G}e^{[0]})_N + O(\hbar^2)$.

For Robin or Neumann condition at $x_N = b$, cf. (2.1), we get, by Theorem 3.1, $K_N(\mathcal{G}e^{[\mathsf{q}]})_N = -\beta e^{[\mathsf{q}]}_N - \tau_N = -\beta e^{[0]}_N - \tau_N + O(\hbar^2) = K_N(\mathcal{G}e^{[0]})_N + O(\hbar^2)$, and we are done. (In case of Dirichlet condition, we proceed as in the proof of Theorem 3.1: writing $e^{[\mathsf{q}]} = e^{[0]} + w^{[\mathsf{q}]}$, we have $w^{[\mathsf{q}]}_N = 0$ and, in the notation of (3.3), $A_h(0)w^{[\mathsf{q}]} = -Q_h(\mathsf{q})e^{[\mathsf{q}]}$ for the other components, from which we get, similarly to (3.5) above, solving for $w^{[\mathsf{q}]}_{N-1/2}$,

$$|\, w^{[\mathsf{q}]}_{N-1/2}\,| \leqslant \frac{\theta^{[0]}_{N-1} c_{N-1}}{c_N + \theta^{[0]}_{N-1} c_{N-1}}\, I_N \|\, \mathsf{q} \,\|_{\sup} \sum_{j=1}^{N} L_{j-1/2}\, |\, e^{[\mathsf{q}]}_{j-1/2}\,| =$$

$$\frac{h_N}{K_N} \|\, \mathsf{q} \,\|_{\sup} \sum_{j=1}^{N} L_{j-1/2}\, |\, e^{[\mathsf{q}]}_{j-1/2}\,| = h_N O(\hbar^2)$$

by (2.8b), (2.9) and Theorem 3.1. This gives $(\,\mathcal{G}e^{[\mathsf{q}]})_N = (\,\mathcal{G}e^{[0]})_N + O(\hbar^2)$, as before). Therefore, (2.13) remains valid for $\mathsf{q} \geqslant 0$, and (2.14) follows, i.e., $E^{[\mathsf{q}]} = O(\hbar^2)$. □

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