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ERROR BOUNDS FOR FINITE ELEMENT METHODS WITH GENERALIZED CUBIC SPLINES FOR A 4-TH ORDER ORDINARY DIFFERENTIAL EQUATION WITH NONSMOOTH DATA

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Abstract — A boundary value problem for a 4-th order self-adjoint ordinary differential equation is considered in the case where the coefficients of the equation and its right-hand side can be nonsmooth (discontinuous, concentrated or rapidly oscillating functions). Generalized cubic splines of deficiency 1 depending on the major coefficient of the equation are applied. An error analysis of finite element methods exploiting such splines is presented in detail including superconvergence error bounds. This is based on general L^q–L^p interpolation error bounds for the splines.

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1. Introduction

The 4-th order ordinary differential equations are important, in particular, in continuum mechanics. Mesh methods to solve them were studied in a lot of publications (see [5,6,9,10]). Especially complicated is the case where the data, that is, the coefficients of the equation and its right-hand side are nonsmooth, in particular, discontinuous, concentrated or rapidly oscillating functions. It is well-known that specific methods have to be constructed to treat this case efficiently. For this case, in [11] projective-grid methods (in other words, finite element methods) with generalized cubic Hermitian splines (of deficiency 2) depending on the major coefficient of the equation were considered, and their error analysis was performed in detail including superconvergence error bounds.

In this paper, we present a similar study of two finite element methods using the generalized cubic splines of deficiency 1. Their advantage is the twice less number of unknowns in the corresponding system of linear algebraic equations. The first method covers the case of nonsmooth coefficients, and the second one is its simplification in the case of the piecewise smooth major coefficient. As a part of the whole study, we present general L^q–L^p interpolation error bounds for the generalized cubic splines. The paper is an abridged English version of our article [12] published in Russian.

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2. Boundary value problem

We consider a boundary value problem for the 4–th order self–adjoint ordinary differential equation

$$D^2(a_2D^2u) - D(a_1Du) + a_0u = f \quad \text{on } \Omega = (0, X), \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \quad Du|_{\partial\Omega} = 0, \tag{2.2}$$

where $D = d/dx$, $\partial\Omega = \{0, X\}$ and in general the free term f has the divergence form $f = D^{2-\ell}f^{(2-\ell)}$, $\ell = 0, 1, 2$. We assume that the major coefficient a_2 satisfies $a_2 \in L^\infty(\Omega)$ and $N^{-1} \leqslant a_2(x) \leqslant N$ on Ω, where $N > 1$ is a parameter. To avoid too many brackets, we adopt the abbreviation $Dw \cdot \varphi = (Dw)\varphi$ below.

A function $u \in W_0^{2,p}(\Omega)$ is called *a weak solution* to the problem (2.1), (2.2) if it satisfies the identity

$$\begin{aligned} \mathscr{L}(u, \varphi) &:= (a_2D^2u, D^2\varphi)_\Omega + \langle a_1, Du \cdot D\varphi\rangle_\Omega + \langle a_0, u\varphi\rangle_\Omega = \\ \langle f, \varphi\rangle_\Omega &:= (-1)^{2-\ell}(f^{(2-\ell)}, D^{2-\ell}\varphi)_\Omega \quad \text{for all } \varphi \in W_0^{2,p'}(\Omega). \end{aligned} \tag{2.3}$$

As usual, here

$$W_0^{m,p}(\Omega) := \{w \in W^{m,p}(\Omega) |\ D^kw\big|_{\partial\Omega} = 0 \ \text{ for } 0 \leqslant k < m\}$$

is a subspace in the Sobolev space $W^{m,p}(\Omega)$ with exponents $m \geqslant 1$, $1 \leqslant p \leqslant \infty$; also $1/p + 1/p' = 1$. Moreover, we use the notation

$$(w, \psi)_\Omega := \int\limits_\Omega w(x)\psi(x)\, dx$$

and, more generally, $\langle w, \psi\rangle_\Omega$ denotes the value of the functional w at the function ψ defined on Ω.

We assume that the bilinear form $\mathscr{L}(\cdot, \cdot)$ is $W_0^{2,2}(\Omega)$–positive definite, that is,

$$N^{-1}\|\varphi\|^2_{W^{2,2}(\Omega)} \leqslant \mathscr{L}(\varphi, \varphi) \quad \text{for all } \varphi \in W_0^{2,2}(\Omega). \tag{2.4}$$

We introduce the dual spaces $W^{-1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, consisting of functionals having the form $w = Dw^{(1)}$, that is,

$$\langle w, \psi\rangle_\Omega = -(w^{(1)}, D\psi)_\Omega \quad \text{for all } \psi \in W_0^{1,p'}(\Omega),$$

where $w^{(1)} \in L^p(\Omega)$ and $\int_\Omega w^{(1)}\, dx = 0$, equipped with the norm $\|w\|_{W^{-1,p}(\Omega)} = \|w^{(1)}\|_{L^p(\Omega)}$. (It is not difficult to verify that $W^{-1,p}(\Omega) = [W_0^{1,p'}(\Omega)]^*$ for $1 < p \leqslant \infty$.) We also set $W^{-2,p}(\Omega) = [W_0^{2,p'}(\Omega)]^*$ for $2 \leqslant p \leqslant \infty$.

Let $\overline{\omega}_0$ be a fixed finite set of points $0 = x_{0,0} < x_{0,1} < \cdots < x_{0,n_0} = X$, $n_0 \geqslant 1$. For $w \in L^1(\Omega)$, we define *the piecewise weak derivative* $\bar{D}^mw$, $m \geqslant 1$, by the identity

$$\langle \bar{D}^mw, \varphi\rangle_\Omega = (-1)^m(w, D^m\varphi)_\Omega \quad \text{for all } \varphi \in C^m(\bar{\Omega}) \ \text{ such that } \ D^k\varphi\big|_{\overline{\omega}_0} = 0,\ 0 \leqslant k < m.$$

If $\bar{D}^mw \in L^1(\Omega)$, then this definition is equivalent to that used in [11].

Below, in the inequalities, $K(N)$, $K_i(N)$, $i = 1, 2, \ldots$, denote the nondecreasing functions of the parameter N; they can also depend on X only. $c^{(0)}$, $c^{(1)}$ denote the absolute constants (that is, fixed numbers).

In [11], the following result on properties (existence, uniqueness and regularity) of the solution to problem (2.1), (2.2) with nonsmooth data was proved. This is essential for deriving results of this paper as well.

Theorem 2.1. *Let $1 \leqslant p \leqslant \infty$.*

1. *If $f = D^2 f^{(2)}$ with $f^{(2)} \in L^p(\Omega)$ and $\|a_1\|_{W^{-1,\widetilde{p}}(\Omega)} + \|a_0\|_{W^{-2,\widetilde{p}}(\Omega)} \leqslant N$ with $\widetilde{p} = \max\{p, p'\}$, then there exists a unique weak solution $u \in W^{2,p}(\Omega)$ to the problem (2.1), (2.2), and the following bound holds:*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant K_1(N)\|f^{(2)}\|_{L^p(\Omega)}.$$

2. *If $f = Df^{(1)}$ with $f^{(1)} \in L^p(\Omega)$ and $\|a_1\|_{L^p(\Omega)} + \|a_0\|_{W^{-1,p}(\Omega)} \leqslant N$, then the solution is more regular*

$$\|D(a_2 D^2 u)\|_{L^p(\Omega)} \leqslant K_2(N)\|f^{(1)}\|_{L^p(\Omega)}.$$

3. *If $f = f^{(0)}$ with $f^{(0)} \in L^p(\Omega)$ and $\|a_1\|_{L^\infty(\Omega)} + \|\bar{D}a_1\|_{L^p(\Omega)} + \|a_0\|_{L^p(\Omega)} \leqslant N$, then the additional piecewise regularity bound holds*

$$\|\bar{D}D(a_2 D^2 u)\|_{L^p(\Omega)} \leqslant K_3(N)\|f^{(0)}\|_{L^p(\Omega)}.$$

3. The space of the generalized cubic splines of deficiency 1

We introduce a mesh $\overline{\omega}^h$ on $\bar{\Omega}$ with nodes $0 = x_0 < x_1 < \dots < x_n = X$ and steps $h_i = x_i - x_{i-1}$. Let $\Omega_i = (x_{i-1}, x_i)$, $i = \overline{1,n}$, as well as $|h| = \max_i h_i$ and $h_{\min} = \min_i h_i$. Hereafter $\overline{1,n} := \{1, \dots, n\}$. We assume that $\overline{\omega}_0 \subset \overline{\omega}^h$. Let the mesh $\overline{\omega}^h$ be called *quasi-uniform* provided that $|h| \leqslant N h_{\min}$.

Let $\varkappa \in L^\infty(\Omega)$ and $0 < \underline{\varkappa} \leqslant \varkappa(x) \leqslant \overline{\varkappa}$ on Ω. We define the space of *generalized cubic splines of deficiency 1*

$$S_1[\varkappa] := \{\varphi \in W^{2,\infty}(\Omega) \big| \varkappa D^2\varphi \in W^{1,\infty}(\Omega),\ D^2(\varkappa D^2\varphi) = 0 \text{ on } \Omega_i,\ i = \overline{1,n}\}. \tag{3.1}$$

In the classical case, $\varkappa(x) \equiv 1$; for the case of the differentiable $\varkappa$ see [7,10]. In [11], the space of generalized cubic splines of deficiency 2 (in other words, of generalized cubic Hermitian splines)

$$S_2[\varkappa] := \{\varphi \in W^{2,\infty}(\Omega) \big| D^2(\varkappa D^2\varphi) = 0 \text{ on } \Omega_i,\ i = \overline{1,n}\}$$

was exploited. It is easy to see that $S_1[\varkappa] = \{\varphi \in S_2[\varkappa] |\, \varkappa D^2\varphi \in C(\bar{\Omega})\}$ is a subspace in $S_2[\varkappa]$.

For $u \in W^{2,1}(\Omega)$, we need *an interpolating generalized cubic spline* $s_\varkappa u \in S_1[\varkappa]$ such that

$$s_\varkappa u(x_i) = u(x_i) \text{ for } i = \overline{0,n}, \quad Ds_\varkappa u(x) = Du(x) \text{ for } x = 0, X. \tag{3.2}$$

Lemma 3.1. *The interpolating spline $s_\varkappa u$ is uniquely defined, and the following projection property holds:*

$$\big(\varkappa D^2(u - s_\varkappa u), D^2\varphi\big)_\Omega = 0 \text{ for all } \varphi \in S_1[\varkappa]. \tag{3.3}$$

Proof. Conditions (3.2) mean that the interpolation error $e := u - s_\varkappa u$ satisfies

$$e(x_i) = 0 \text{ for } i = \overline{0,n}, \quad De\big|_{\partial\Omega} = 0. \tag{3.4}$$

Integrating by parts and applying the property

$$D(\varkappa D^2\varphi)\big|_{\Omega_i} = c_i = \text{const},\ i = \overline{1,n}, \text{ for all } \varphi \in S_1[\varkappa], \tag{3.5}$$

we get

$$(\varkappa D^2 e, D^2\varphi)_\Omega = -(De, D(\varkappa D^2\varphi))_\Omega = -\sum_{i=1}^{n}(De, D(\varkappa D^2\varphi))_{\Omega_i} = -\sum_{i=1}^{n} e|_{x_{i-1}}^{x_i} D(\varkappa D^2\varphi)|_{\Omega_i} = 0.$$

Property (3.3) is proved. We notice that, for $u=0$, by virtue of this property we have $\varkappa D^2 s_\varkappa u = 0$ and since $s_\varkappa u|_{\partial\Omega} = 0$, $s_\varkappa u = 0$ as well.

In accordance with property (3.5), the spline $s_\varkappa u$ belongs to a 4–dimensional space on each interval Ω_i, $i = \overline{1,n}$. Therefore, on Ω it is uniquely defined by $4n$ parameters (which equal 0 in the case of $s_\varkappa u = 0$). The continuity conditions for functions $s_\varkappa u$, $Ds_\varkappa u$, $\varkappa D^2 s_\varkappa u$ in the internal nodes x_i, $i = \overline{1,n-1}$, and conditions (3.2) lead to a system of $3(n-1) + (n+1) + 2 = 4n$ linear algebraic equations for the mentioned $4n$ parameters. If $u = 0$, then this system is homogeneous. But for $u = 0$ we have already known that $s_\varkappa u = 0$, thus, this homogeneous system has only a zero solution. Therefore, in the general case, the nonhomogeneous system for $4n$ parameters is uniquely solvable, that is, the interpolating spline $s_\varkappa u$ is uniquely defined. This argument is similar to that from [7]. □

Notice that

$$\dim S_1[\varkappa] = n+3, \quad \dim S_2[\varkappa] = 2(n+1). \tag{3.6}$$

By the standard argument, the projection property (3.3) implies the following extremal property:

$$\|\sqrt{\varkappa}D^2 s_\varkappa u\|_{L^2(\Omega)} = \min \|\sqrt{\varkappa}D^2 g\|_{L^2(\Omega)},$$

where the minimum is taken over all $g \in W^{2,2}(\Omega)$ such that $g(x_i) = u(x_i)$ for $i = \overline{0,n}$, $Dg(x) = Du(x)$ for $x = 0, X$.

For completeness, we present a global finite support basis in $S_1[\varkappa]$ (see [14]) (a similar basis for a simpler periodic case and for a uniform mesh is also presented in [5]). We supplement the mesh $\overline{\omega}^h$ by auxiliary nodes $x_{-i} = -x_i$ and $x_{n+i} = 2X - x_{n-i}$, $i = 1,2,3$, and set $\tilde{\Omega} := [x_{-3}, x_{n+3}]$. Let now $\Omega_i = (x_{i-1}, x_i)$ and $h_i = x_i - x_{i-1}$ for $i = \overline{-2, n+3}$. We also extend $\varkappa$ evenly with respect to the points $x = 0, X$ beyond Ω. Actually we present a global finite support basis in an auxiliary space of the generalized cubic splines on $\tilde{\Omega}$

$$\tilde{S}_1[\varkappa] = \left\{ \varphi \in W_0^{2,\infty}(\tilde{\Omega}) \,\middle|\, \varkappa D^2\varphi \in W_0^{1,\infty}(\tilde{\Omega}), \quad D^2(\varkappa D^2\varphi) = 0 \ \text{ on } \Omega_i, \ i = \overline{-2, n+3} \right\},$$

(compare with (3.1)). The basis consists of the functions

$$d_i(x) = \begin{cases} \alpha_{i,-1} \displaystyle\int_{x_{i-2}}^{x} (x-\chi)\frac{e_{i-1}(\chi)}{\varkappa(\chi)}\,d\chi + \alpha_{i,0} \int_{x_{i-1}}^{x} (x-\chi)\frac{e_i(\chi)}{\varkappa(\chi)}\,d\chi, & x_{-3} \leqslant x \leqslant x_i, \\ \alpha_{i,0} \displaystyle\int_{x}^{x_{i+1}} (\chi - x)\frac{e_i(\chi)}{\varkappa(\chi)}\,d\chi + \alpha_{i,1} \int_{x}^{x_{i+2}} (\chi - x)\frac{e_{i+1}(\chi)}{\varkappa(\chi)}\,d\chi, & x_i \leqslant x \leqslant x_{n+3}, \end{cases} \tag{3.7}$$

where $i = \overline{-1, n+1}$. Here we use the standard basis of hill functions

$$e_j(x) = \begin{cases} (x - x_{j-1})/h_j, & x_{j-1} \leqslant x \leqslant x_j, \\ (x_{j+1} - x)/h_{j+1}, & x_j \leqslant x \leqslant x_{j+1}, \\ 0, & x \notin [x_{j-1}, x_{j+1}], \end{cases} \qquad j = \overline{-2, n+2},$$

in the space of functions that are continuous on $\tilde{\Omega}$, linear on segments $\overline{\Omega}_i$, $i=\overline{-2,n+3}$, and zero at $x=x_{-3},x_{n+3}$. The coefficients in (3.7) have the form

$$\alpha_{i,-1}=\frac{1}{\gamma_{i-1}\nu_{i-1/2}},\quad \alpha_{i,0}=-\frac{1}{\gamma_i}\left(\frac{1}{\nu_{i-1/2}}+\frac{1}{\nu_{i+1/2}}\right),\quad \alpha_{i,1}=\frac{1}{\gamma_{i+1}\nu_{i+1/2}},$$

where $\nu_{j-1/2}=h_j+\hat{\delta}_j-\hat{\delta}_{j-1}>0$, $j=\overline{-1,n+2}$, and

$$\gamma_j=\int\limits_{x_{j-1}}^{x_{j+1}}\frac{e_j(x)}{\varkappa(x)}\,dx>0,\quad \hat{\delta}_j=\frac{1}{\gamma_j}\int\limits_{x_{j-1}}^{x_{j+1}}(x-x_j)\frac{e_j(x)}{\varkappa(x)}\,dx,\quad j=\overline{-2,n+2}.$$

It is known that $d_i(x)>0$ on (x_{i-2},x_{i+2}), and clearly the values of $d_i(x)$ are zero outside (x_{i-2},x_{i+2}).

For calculations using this basis, the following formulas for the derivatives are convenient:

$$(Dd_i)(x)=\begin{cases}\alpha_{i,-1}\displaystyle\int\limits_{x_{i-2}}^{x}\frac{e_{i-1}(\chi)}{\varkappa(\chi)}\,d\chi+\alpha_{i,0}\int\limits_{x_{i-1}}^{x}\frac{e_i(\chi)}{\varkappa(\chi)}\,d\chi, & x_{-3}\leqslant x\leqslant x_i,\\ -\alpha_{i,0}\displaystyle\int\limits_{x}^{x_{i+1}}\frac{e_i(\chi)}{\varkappa(\chi)}\,d\chi-\alpha_{i,1}\int\limits_{x}^{x_{i+2}}\frac{e_{i+1}(\chi)}{\varkappa(\chi)}\,d\chi, & x_i\leqslant x\leqslant x_{n+3},\end{cases}$$

and

$$(\varkappa D^2d_i)(x)=\alpha_{i,-1}e_{i-1}(x)+\alpha_{i,0}e_i(x)+\alpha_{i,1}e_{i+1}(x)\quad \text{on } \tilde{\Omega}.$$

Of course, the values of $Dd_i(x)$ and $\varkappa D^2d_i(x)$ are also zero outside (x_{i-2},x_{i+2}). Recall that exactly the latter formula is the original one for the derivation of (3.7).

The latter formula together with the well–known property of the hill functions imply the useful formula

$$(\varkappa D^2w,D^2d_i)_{\tilde{\Omega}}=h_{i-3/2}\alpha_{i,-1}\hat{\partial}\bar{\partial}w_{i-1}+h_{i-1/2}\alpha_{i,0}\hat{\partial}\bar{\partial}w_i+h_{i+1/2}\alpha_{i,1}\hat{\partial}\bar{\partial}w_{i+1}=$$
$$\frac{1}{\tilde{\gamma}_{i+1}}\frac{\hat{\partial}\bar{\partial}w_{i+1}-\hat{\partial}\bar{\partial}w_i}{\nu_{i+1/2}}-\frac{1}{\tilde{\gamma}_i}\frac{\hat{\partial}\bar{\partial}w_i-\hat{\partial}\bar{\partial}w_{i-1}}{\nu_{i-1/2}}\quad \text{for } w\in W^{2,1}(\tilde{\Omega}),$$

where $\tilde{\gamma}_j:=\gamma_j/h_{j+1/2}$, $h_{j+1/2}:=(h_j+h_{j+1})/2$ and

$$\hat{\partial}\bar{\partial}w_j:=\frac{1}{h_{j+1/2}}\left(\frac{w_{j+1}-w_j}{h_{j+1}}-\frac{w_j-w_{j-1}}{h_j}\right)$$

is the simplest three-point approximation of $(D^2w)(x_j)$, with $w_i=w(x_i)$.

Below we need the subspace

$$S_{1,0}[\varkappa]:=\{\varphi\in S_1[\varkappa]|\ \varphi|_{\partial\Omega}=0,\ D\varphi|_{\partial\Omega}=0\}. \tag{3.8}$$

To form its basis, it is convenient to remove the functions d_{-1}, d_0, d_n, d_{n+1} from the original basis and replace the basis functions d_1, d_{n-1} by the following ones:

$$d_{1,0}(x)=d_0(0)[d_1(x)+d_{-1}(x)]-2d_1(0)d_0(x),$$
$$d_{n-1,0}(x)=d_n(X)[d_{n-1}(x)+d_{n+1}(x)]-2d_{n-1}(X)d_n(x).$$

Since by construction $d_0(x)=d_0(-x)$ and $d_1(x)=d_{-1}(-x)$ as well as $d_n(X-x)=d_n(X+x)$ and $d_{n-1}(X-x)=d_{n+1}(X+x)$, we have $d_{1,0},d_{n-1,0}\in S_{1,0}[\varkappa]$.

4. Interpolation error bounds for generalized cubic splines

We now prove the interpolation error bounds. First we derive not the most general L^q–L^p bounds (but in the case of an arbitrary mesh $\overline{\omega}^h$) and apply the rather standard technique to this end.

Theorem 4.1. 1. *Let* $2 \leqslant q \leqslant \infty$, $k = 0, 1$. *For* $u \in W^{2,2}(\Omega)$, *the interpolation error bound*

$$\|D^k(u - s_\varkappa u)\|_{L^q(\Omega)} \leqslant \underline{\varkappa}^{-1/2}|h|^{2-k-\left(\frac{1}{2}-\frac{1}{q}\right)}\|\sqrt{\varkappa}D^2u\|_{L^2(\Omega)} \tag{4.1}$$

holds.

2. *Let* $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, $k = 0, 1$ *and* $m = 0, 1$. *For* $u \in W^{2,2}(\Omega)$ *and* $\bar{D}^m D(\varkappa D^2 u) \in L^p(\Omega)$, *the interpolation error bound*

$$\|D^k(u - s_\varkappa u)\|_{L^q(\Omega)} \leqslant \underline{\varkappa}^{-1}|h|^{m+3-k-\left(\frac{1}{p}-\frac{1}{q}\right)}\|\bar{D}^m D(\varkappa D^2 u)\|_{L^p(\Omega)} \tag{4.2}$$

holds.

3. *In the case* $q = 2$, *Claims* 1 *and* 2 *hold for* $k = 2$ *as well.*

Proof. The proof comprises three steps.

(a) Let $1 \leqslant r \leqslant q \leqslant \infty$ and $u \in W^{2,r}(\Omega)$. Applying the first property (3.4) of $e = u - s_\varkappa u$, we get $\|e\|_{L^q(\Omega_i)} \leqslant h_i\|De\|_{L^q(\Omega_i)}$, $i = \overline{1,n}$, and thus

$$\|e\|_{L^q(\Omega)} \leqslant |h|\|De\|_{L^q(\Omega)}.$$

Moreover, since $h_i^{-1}\int_{\Omega_i} De\,dx = 0$, we also get

$$\|De\|_{C(\bar{\Omega}_i)} \leqslant \|D^2e\|_{L^1(\Omega_i)}, \ \ i = \overline{1,n}.$$

By virtue of the Hölder inequality and the last one we have

$$\|De\|_{L^q(\Omega_i)} \leqslant h_i^{\frac{1}{q}}\|De\|_{C(\bar{\Omega}_i)} \leqslant h_i^{1-\left(\frac{1}{r}-\frac{1}{q}\right)}\|D^2e\|_{L^r(\Omega_i)}, \quad i = \overline{1,n}.$$

Applying the known number inequality

$$\left(\sum_i |\alpha_i|^q\right)^{1/q} \leqslant \left(\sum_i |\alpha_i|^r\right)^{1/r}$$

(where, for example, for $q = \infty$, the left–hand side should be understood as $\max\limits_i |\alpha_i|$), we derive an auxiliary bound

$$\|D^k e\|_{L^q(\Omega)} \leqslant |h|^{2-k-\left(\frac{1}{r}-\frac{1}{q}\right)}\|D^2e\|_{L^r(\Omega)}, \quad k = 0, 1. \tag{4.3}$$

(b) We first prove the last Claim 3. By virtue of the projection property (3.3), we have

$$\left(\varkappa D^2 e, D^2 e\right)_\Omega = (\varkappa D^2 u, D^2 e)_\Omega, \tag{4.4}$$

thus, for $u \in W^{2,2}(\Omega)$, we get

$$\|D^2e\|_{L^2(\Omega)} \leqslant \underline{\varkappa}^{-1/2}\|\sqrt{\varkappa}D^2e\|_{L^2(\Omega)} \leqslant \underline{\varkappa}^{-1/2}\|\sqrt{\varkappa}D^2u\|_{L^2(\Omega)}, \tag{4.5}$$

that is, bound (4.1) holds for $q = 2$, $k = 2$.

Let in addition $\bar{D}^m D(\varkappa D^2 u) \in L^p(\Omega)$ ($m = 0$ or 1). Then, integrating by parts and taking into account properties (3.4) and applying the Hölder inequality, from equality (4.4) we derive:

$$\underline{\varkappa}\|D^2 e\|^2_{L^2(\Omega)} \leqslant \|\sqrt{\varkappa} D^2 e\|^2_{L^2(\Omega)} = (-1)^{m+1}\left(\bar{D}^m D(\varkappa D^2 u), D^{1-m} e\right)_\Omega$$
$$\leqslant \|D^{1-m} e\|_{L^{p'}(\Omega)} \|\bar{D}^m D(\varkappa D^2 u)\|_{L^p(\Omega)}.$$

Applying bound (4.3) (with $k = 1-m$, $q = p'$, $r = 2$), we get

$$\|D^2 e\|_{L^2(\Omega)} \leqslant \underline{\varkappa}^{-1} |h|^{m+1-\left(\frac{1}{p}-\frac{1}{2}\right)} \|\bar{D}^m D(\varkappa D^2 u)\|_{L^p(\Omega)}, \quad 1 \leqslant p \leqslant 2, \tag{4.6}$$

that is, bound (4.2) holds for $q = 2$, $k = 2$.

(c) Claims 1 and 2 follow straightforwardly from bound (4.3) (with $r = 2$) and also from bounds (4.5) and (4.6) respectively. □

Remark 4.1. Bound (4.2) for $p = 1$ (and the values of parameters q, k and m from Claims 2 and 3) can be strengthened. Namely, let $u \in W^{2,2}(\Omega)$ and if $m = 1$, then $D(\varkappa D^2 u) \in L^1(\Omega)$ as well. Then in bound (4.2) for $p = 1$, the term $\|\bar{D}^m D(\varkappa D^2 u)\|_{L^p(\Omega)}$ can be replaced by $\mathrm{var}_{\bar{\Omega}} D^m(\varkappa D^2 u)$; here $\mathrm{var}_{\bar{\Omega}} w$ is a variation of the function w over $\bar{\Omega}$. To verify this, it is sufficient to justify the possibility of the same replacement in inequality (4.6). This can be accomplished similarly to Remark 4.2 in [11] (by using the Stieltjes integral over $\bar{\Omega}$).

In contrast to Theorem 4.1, to derive the L^q–L^p interpolation error bounds in the general case $1 \leqslant p \leqslant q \leqslant \infty$ (but for the quasi-uniform mesh $\overline{\omega}^h$ only), we need a more delicate technique.

Theorem 4.2. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $u \in W^{2,p}(\Omega)$. Let the mesh $\overline{\omega}^h$ be quasi-uniform.*

1. *Let $k = 0, 1, 2$ and $q = p$ in the case $k = 2$. Then the interpolation error bound*

$$\|D^k(u - s_\varkappa u)\|_{L^q(\Omega)} \leqslant \underline{\varkappa}^{-1/p'} \left[c^{(0)}\sqrt{N}\left(\frac{\overline{\varkappa}}{\underline{\varkappa}}\right)^2\right]^{|1-\frac{2}{p}|} |h|^{2-k-\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varkappa^{1/p'} D^2 u\|_{L^p(\Omega)} \tag{4.7}$$

holds. In the particular case $p = 2$, the bound coincides with (4.1) and holds for an arbitrary mesh $\overline{\omega}^h$.

2. *Let $k = 0, 1, 2$, $m = 0, 1$ and $\bar{D}^m D(\varkappa D^2 u) \in L^p(\Omega)$. Then the interpolation error bound*

$$\|D^k(u - s_\varkappa u)\|_{L^q(\Omega)} \leqslant \underline{\varkappa}^{-1} \left[c^{(0)}\sqrt{N}\left(\frac{\overline{\varkappa}}{\underline{\varkappa}}\right)^2\right]^{\sigma_k(p,\,q)} |h|^{m+3-k-\left(\frac{1}{p}-\frac{1}{q}\right)} \|\bar{D}^m D(\varkappa D^2 u)\|_{L^p(\Omega)} \tag{4.8}$$

holds, where

$$\sigma_k(p,\,q) := \max\left\{\frac{2}{q}, 1\right\} - \min\left\{\frac{2}{p}, 1\right\} = \begin{cases} 0, & 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty, \\ \dfrac{2}{q} - 1, & 1 \leqslant p \leqslant q \leqslant 2, \\ 1 - \dfrac{2}{p}, & 2 \leqslant p \leqslant q \leqslant \infty, \end{cases} \qquad \text{for } k = 0, 1,$$

and $\sigma_2(p,\,q) := |1 - \frac{2}{q}|$.

In the particular case $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, the bound coincides with (4.2) and holds for an arbitrary mesh $\overline{\omega}^h$.

Proof. The proof comprises four steps.

(a) We introduce the space $\widehat{S}$ of functions that are continuous on $\bar{\Omega}$ and linear on segments $\overline{\Omega}_i$, $i=\overline{1,n}$. Let $1 \leqslant p \leqslant q \leqslant \infty$ and $m=0,1$. In the case $w,\, \bar{D}^m Dw \in L^p(\Omega)$, the following bound is known:

$$\|w-\widehat{w}\|_{L^q(\Omega)} \leqslant |h|^{m+1-\left(\frac{1}{p}-\frac{1}{q}\right)} \|\bar{D}^m Dw\|_{L^p(\Omega)}, \tag{4.9}$$

where $\widehat{w} \in \widehat{S}$ and $\widehat{w}(x_i)=w(x_i)$ for $i=\overline{0,n}$. (Recall that this is derived from the basic bound

$$\|w-\widehat{w}\|_{C(\bar{\Omega}_i)} \leqslant |h|^m \|\bar{D}^m Dw\|_{L^1(\Omega_i)},\ i=\overline{1,n},$$

in the same manner as for bound (4.3).)

Clearly $S_1[\varkappa] = \{\varphi \in W^{2,\infty}(\Omega) \,|\, \varkappa D^2\varphi \in \widehat{S}\}$, and the equation $\varkappa D^2\varphi = \psi \in \widehat{S}$ has solutions $\varphi \in S_1[\varkappa]$: one can solve, successively for $i=\overline{1,n}$, the auxiliary Cauchy problems $D^2\varphi = \psi/\varkappa$ on Ω_i for arbitrary $\varphi(0)$, $D\varphi(0)$ (for $i=1$) or for $\varphi(x_{i-1}^+)=\varphi(x_{i-1}^-)$, $D\varphi(x_{i-1}^+) = D\varphi(x_{i-1}^-)$ (for $i=\overline{2,n}$). Therefore the projection property (3.3) can be transformed to the form

$$\left(\frac{1}{\varkappa}\,(\varkappa D^2 s_\varkappa u), \psi\right)_\Omega = \left(\frac{1}{\varkappa}\,w, \psi\right)_\Omega \quad \text{for all } \psi \in \widehat{S} \tag{4.10}$$

with $w=\varkappa D^2 u$, and $\varkappa D^2 s_\varkappa u \in \widehat{S}$.

(b) Let $\rho \in L^1(\Omega)$ and $0<\rho(x)$ on Ω. We introduce the weighted Lebesgue space $L^{q,\rho}(\Omega)$, $1 \leqslant q \leqslant \infty$, consisting of functions w that are measurable on Ω and have the finite norm

$$\|w\|_{L^{q,\rho}(\Omega)} = \|\rho^{1/q} w\|_{L^q(\Omega)};$$

we consider this space as complex in this item of the proof and as real in the next one. In this item of the proof, we also assume that

$$(y,\psi)_\Omega := \int_\Omega y\bar{\psi}\,dx.$$

Clearly $L^{\infty,\rho}(\Omega) = L^\infty(\Omega)$.

Let S be a finite-dimensional subspace in $L^\infty(\Omega)$. We consider the operator $P=P_\rho[S]$: $L^{1,\rho}(\Omega) \to S$ such that

$$(\rho P w, \psi)_\Omega = (\rho w, \psi)_\Omega \quad \text{for all } w \in L^{1,\rho}(\Omega),\ \psi \in S. \tag{4.11}$$

This identity defines the operator P uniquely; moreover, it is a projector, that is, $P^2=P$. This operator extends the orthogonal projector in $L^{2,\rho}(\Omega)$ on S; thus

$$\|P\|_{\mathscr{L}[L^{2,\rho}(\Omega)]} = \|I-P\|_{\mathscr{L}[L^{2,\rho}(\Omega)]} = 1. \tag{4.12}$$

Hereafter $\mathscr{L}[B]$ is the space of linear bounded operators acting in a normed space B, and I is the identity operator.

In the spirit of the technique in [3], for $1 \leqslant q \leqslant \infty$, we prove the inequality

$$\|w-Pw\|_{L^{q,\rho}(\Omega)} \leqslant \|I-P\|_{\mathscr{L}[L^\infty(\Omega)]}^{|1-\frac{2}{q}|} \inf_{\psi \in S} \|w-\psi\|_{L^{q,\rho}(\Omega)} \quad \text{for } w \in L^{q,\rho}(\Omega). \tag{4.13}$$

Clearly

$$\|w-Pw\|_{L^{q,\rho}(\Omega)} \leqslant \|I-P\|_{\mathscr{L}[L^{q,\rho}(\Omega)]} \inf_{\psi \in S} \|w-\psi\|_{L^{q,\rho}(\Omega)} \tag{4.14}$$

(since $w - Pw = w - \psi - P(w - \psi)$ for any $\psi \in S$). Moreover, the following equality holds:

$$\|I - P\|_{\mathscr{L}[L^{q,\rho}(\Omega)]} = \|I - P\|_{\mathscr{L}\left[L^{q',\rho}(\Omega)\right]}, \quad 1 \leqslant q \leqslant \infty. \tag{4.15}$$

Actually, by the inverse Hölder inequality (see [2], Section 1.2.6) (its original real version in [2] rather easily implies the corresponding complex version), we can write down the following relations:

$$\|I - P\|_{\mathscr{L}[L^{q,\rho}(\Omega)]} = \sup_{\|w\|_{L^{q,\rho}(\Omega)}=1} \|(I - P)w\|_{L^{q,\rho}(\Omega)} = \sup_{\|w\|_{L^{q,\rho}(\Omega)}=1} \sup_{\|g\|_{L^{q'}(\Omega)}=1} |(\rho^{1/q}(w - Pw), g)_\Omega| =$$

$$\sup_{\|w\|_{L^{q,\rho}(\Omega)}=1} \sup_{\|v\|_{L^{q',\rho}(\Omega)}=1} |(\rho(w - Pw), v)_\Omega| =$$

$$\sup_{\|v\|_{L^{q',\rho}(\Omega)}=1} \sup_{\|w\|_{L^{q,\rho}(\Omega)}=1} |(\rho w, v - Pv)_\Omega| = \|I - P\|_{\mathscr{L}[L^{q',\rho}(\Omega)]}.$$

Here we have applied the equalities

$$(\rho(w - Pw), v)_\Omega = (\rho(w - Pw), v - Pv)_\Omega = (\rho w, v - Pv)_\Omega$$

that are valid by virtue of the original identity (4.11).

By the classical Riesz — Thorin interpolation theorem [1] we have

$$\|I - P\|_{\mathscr{L}[L^{q,\rho}(\Omega)]} \leqslant \|I - P\|^{2/q}_{\mathscr{L}[L^{2,\rho}(\Omega)]} \|I - P\|^{1-2/q}_{\mathscr{L}[L^{\infty,\rho}(\Omega)]}, \quad 2 \leqslant q \leqslant \infty.$$

Applying equalities (4.12) and (4.15), we get

$$\|I - P\|_{\mathscr{L}[L^{q,\rho}(\Omega)]} \leqslant \|I - P\|^{|1-2/q|}_{\mathscr{L}[L^{\infty,\rho}(\Omega)]}, \quad 1 \leqslant q \leqslant \infty. \tag{4.16}$$

Inequalities (4.14) and (4.16) imply inequality (4.13).

Remark 4.2. The content of Item (b) is valid in the case where Ω is any set of finite measure in $\mathbb{R}^k$, $k \geqslant 1$.

(c) Let $\rho \in L^\infty(\Omega)$ and $0 < \underline{\rho} \leqslant \rho(x) \leqslant \overline{\rho}$ on Ω. Under the assumption that the mesh $\overline{\omega}^h$ is quasi-uniform, the following estimate holds:

$$\|I - P_\rho[\widehat{S}\,]\|_{\mathscr{L}[L^\infty(\Omega)]} \leqslant 1 + \|P_\rho[\widehat{S}\,]\|_{\mathscr{L}[L^\infty(\Omega)]} \leqslant c^{(0)}\sqrt{N}\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2. \tag{4.17}$$

This is proved in the same manner as the similar bound in [3, pp. 194–196], (see also [4]). To this end, the following inequality is taken into account:

$$\|\widehat{\varphi\chi}\|^2_{L^{2,\rho}(\Omega)} \leqslant 2\,\frac{\overline{\rho}}{\underline{\rho}}\,\|\varphi\|^2_{L^{2,\rho}(\Omega)} \quad \text{for all } \varphi \in \widehat{S},$$

where $\chi = \chi_{[x_i,x_j]}$ is the characteristic function of the segment $[x_i, x_j]$, $0 \leqslant i < j \leqslant n$. The inequality is a consequence of the elementary relations

$$\int\limits_{x_{j-1}}^{x_j} \varphi^2\,dx = \frac{1}{3}\left(\varphi_{j-1}^2 + \varphi_{j-1}\varphi_j + \varphi_j^2\right) \geqslant \frac{1}{4}\,\varphi_j^2.$$

Note that in [3] the case of nonweighted ($\rho(x) \equiv 1$) projectors P onto the general multidimensional finite element subspaces S was considered; it is not difficult to understand that the result in [3] can be generalized to the case of weighted projectors as well.

By virtue of bounds (4.13) and (4.17) for $1 \leqslant q \leqslant \infty$ we have

$$\|\rho(w - P_\rho[\widehat{S}\,]w)\|_{L^q(\Omega)} \leqslant \overline{\rho}^{\,1/q'} \left[c^{(0)} \sqrt{N} \left(\frac{\overline{\rho}}{\underline{\rho}} \right)^2 \right]^{|1-2/q|} \inf_{\psi \in \widehat{S}} \|\rho^{1/q}(w - \psi)\|_{L^q(\Omega)}. \tag{4.18}$$

(d) Comparing identities (4.10) and (4.11), we get the formula $\varkappa D^2 s_\varkappa u = P_{1/\varkappa}[\widehat{S}]w$ and then, according to bound (4.18), the inequality

$$\|D^2(u - s_\varkappa u)\|_{L^q(\Omega)} \leqslant \underline{\varkappa}^{-1/q'} \left[c^{(0)} \sqrt{N} \left(\frac{\overline{\varkappa}}{\underline{\varkappa}} \right)^2 \right]^{|1-2/q|} \inf_{\psi \in \widehat{S}} \|\varkappa^{-1/q}(w - \psi)\|_{L^q(\Omega)} \ \text{ with } w = \varkappa D^2 u. \tag{4.19}$$

In the particular case $q = 2$, bounds (4.18) and (4.19) do not exploit estimate (4.17), and thus they hold for an arbitrary mesh $\overline{\omega}^h$.

Bound (4.7) for $k = 2$, $q = p$ follows from the last one for $q = p$, $\psi = 0$. The case $k = 0, 1$ is reduced to the case $k = 2$ with the help of bound (4.3) for $r = p$.

Bound (4.8) for $k = 2$ follows from bound (4.19) with $\psi = \widehat{w}$ and also from bound (4.9). The case $k = 0, 1$ is reduced to the case $k = 2$ once again with the help of bound (4.3) with $r = 2, q$ and p respectively for $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, $1 \leqslant p \leqslant q \leqslant 2$ and $2 \leqslant p \leqslant q \leqslant \infty$. Theorem 4.2 is completely proved. □

Clearly, Theorem 4.2 generalizes Theorem 4.1 (notice that its proof contains another justification of bound (4.6)).

Note that, for general polynomial (but not generalized) interpolating splines, error bounds of the type like in Theorem 4.2 were proved in [8] (by another technique).

5. Finite element methods with generalized cubic splines

We consider a family of finite element methods for solving problem (2.1), (2.2). Following the Galerkin method, we define *an approximate solution* v as a function from the subspace $S_{1,0}[\varkappa]$ (see (3.8)) satisfying the identity

$$\mathscr{L}(v, \varphi) = \langle f, \varphi \rangle_\Omega \ \text{ for all } \varphi \in S_{1,0}[\varkappa]. \tag{5.1}$$

Under the hypotheses of Theorem 2.1, Claim 1 such a function v exists and is unique (for $p = 2$ it is standard while for $p = 1$ it is proved quite similarly to [11]).

The advantage of exploiting generalized splines of deficiency 1 over similar splines of deficiency 2 is that the corresponding system of linear algebraic equations has a twice less number of unknowns as follows from formulas (3.6) (we mean exploiting bases from [14], in particular, see the above Section 3). Note that matrices of the systems are symmetric, positive definite and 7-diagonal in both cases (more precisely, 2×2-block three-diagonal in the latter case).

The choice of the function $\varkappa$ defines the particular method. In the case $\varkappa = a_2$, the error bounds are the strongest and general. We first get the superconvergence bounds for $s_{a_2}u - v$, that is, for the difference between the interpolating generalized cubic spline and the approximate solution.

Theorem 5.1. *Let $\varkappa = a_2$, $1 \leqslant p \leqslant 2$ and $f = D^{2-\ell} f^{(2-\ell)}$, $\ell = 0, 1, 2$. Let the mesh $\overline{\omega}^h$ be quasi-uniform.*

1. *Let $\ell = 0, 1$ and $\|a_1\|_{L^r(\Omega)} + \|a_0\|_{W^{-1,r}(\Omega)} \leqslant N$ for some $r \in [1, p']$ (for $\ell = 0$) or $r \in [p, p']$ (for $\ell = 1$). Then the following superconvergence bound holds:*

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_1(N) |h|^{\ell+1-\left(\frac{1}{p}-\frac{1}{r'}\right)} \|f^{(2-\ell)}\|_{L^p(\Omega)}. \tag{5.2}$$

2. *Let $\|a_1\|_{L^\infty(\Omega)} + \|\bar{D} a_1\|_{L^r(\Omega)} + \|a_0\|_{L^r(\Omega)} \leqslant N$ for some $r \in [1, 2]$ (for $\ell = 0, 1$) or $r \in [p, 2]$ (for $\ell = 2$). Then the following stronger superconvergence bound holds:*

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_2(N) |h|^{\ell+2-\left(\frac{1}{p}-\frac{1}{r'}\right)} \|f^{(2-\ell)}\|_{L^p(\Omega)}. \tag{5.3}$$

In the particular case $a_1 = 0$, the admissible values of r are $r \in [1, p']$ (for $\ell = 0, 1$) or $r \in [p, p']$ (for $\ell = 2$).

In addition, in the case where $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$, Claims 1 and 2 hold for an arbitrary mesh $\overline{\omega}^h$.

Proof. In a standard manner, for any $\varphi \in S_{1,0}[\varkappa]$, we have

$$\mathscr{L}(s_\varkappa u - v, \varphi) = \mathscr{L}(s_\varkappa u - u, \varphi) =$$
$$-(a_2 D^2(u - s_\varkappa u), D^2 \varphi)_\Omega - \langle a_1, D(u - s_\varkappa u) \cdot D\varphi \rangle_\Omega - \langle a_0, (u - s_\varkappa u)\varphi \rangle_\Omega \tag{5.4}$$

by the definitions of v and u (see identities (5.1) and (2.3)).

Let $\varkappa = a_2$ and $s = s_{a_2}$. Due to the projection property (3.3), the first summand on the right-hand side of (5.4) equals zero. Thus, quite similarly to the proof of Theorem 5.1 in [11], under the hypotheses of Claim 1, we derive

$$|\mathscr{L}(su - v, \varphi)| \leqslant \|a_1\|_{L^r(\Omega)} \|D(u - su)\|_{L^{r'}(\Omega)} \|D\varphi\|_{L^\infty(\Omega)} +$$
$$\|a_0\|_{W^{-1,r}(\Omega)} \|D[(u - su)\varphi]\|_{L^{r'}(\Omega)} \leqslant K_1(N) \|D(u - su)\|_{L^{r'}(\Omega)} \|\varphi\|_{W^{2,2}(\Omega)}. \tag{5.5}$$

Furthermore, under the hypotheses of Claim 2, integrating by parts on the right-hand side of (5.4), and taking into account the first property (3.4), we also derive, for $1 \leqslant r \leqslant 2$,

$$|\mathscr{L}(su - v, \varphi)| = |(u - su, a_1 D^2 \varphi + \bar{D} a_1 \cdot D\varphi - a_0 \varphi)_\Omega| \leqslant$$
$$\|u - su\|_{L^{r'}(\Omega)} \left(\|a_1\|_{L^\infty(\Omega)} \|D^2 \varphi\|_{L^r(\Omega)} + \|\bar{D} a_1\|_{L^r(\Omega)} \|D\varphi\|_{L^\infty(\Omega)} + \|a_0\|_{L^r(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \right) \leqslant$$
$$K_2(N) \|u - su\|_{L^{r'}(\Omega)} \|\varphi\|_{W^{2,2}(\Omega)}. \tag{5.6}$$

In the particular case $a_1 = 0$, the values $r \in [1, \infty]$ are admissible in (5.6).

Setting $\varphi = su - v$ and using the $W_0^{2,2}(\Omega)$-positive definiteness property (2.4), from bounds (5.5) and (5.6) we get bounds for $su - v$ respectively

$$\|su - v\|_{W^{2,2}(\Omega)} \leqslant K_3(N) \|D(u - su)\|_{L^{r'}(\Omega)}, \quad r \in [1, \infty], \tag{5.7}$$

and

$$\|su - v\|_{W^{2,2}(\Omega)} \leqslant K_4(N) \|u - su\|_{L^{r'}(\Omega)}, \quad r \in [1, 2]; \tag{5.8}$$

if $a_1 = 0$, the values $r \in [1, \infty]$ are admissible in the latter bound as well. Therefore, applying first Theorem 4.2 (if $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$, then the mesh $\overline{\omega}^h$ can be arbitrary according to this theorem) and next Theorem 2.1, we derive the final bounds (5.2) and (5.3) for $su - v$. $\square$

Notice that we have covered a more general case $1 \leqslant p \leqslant 2$ than the standard one $p = 2$. Clearly, the orders of the superconvergence bounds (5.2) and (5.3) are the highest, respectively, up to $O(|h|^{\ell+1})$ and $O(|h|^{\ell+2-(\frac{1}{p}-\frac{1}{2})})$ (or even $O(|h|^{\ell+2})$ in the particular case $a_1 = 0$), for the maximal admissible values of the parameter r. On the other hand, smaller values of r allow to cover broader classes of the junior coefficients a_1 and a_0.

Corollary 5.1. *Let $\varkappa = a_2$, $1 \leqslant p \leqslant 2$ and $f = D^{2-\ell} f^{(2-\ell)}$, $\ell = 0, 1, 2$. Let the mesh $\overline{\omega}^h$ be quasi-uniform. The following bounds for the error $u - v$ hold:*

$$\|D^k(u-v)\|_{L^q(\Omega)} \leqslant K(N)\|D^k(u-su)\|_{L^q(\Omega)} \leqslant K_1(N)|h|^{\ell+2-k-\left(\frac{1}{p}-\frac{1}{q}\right)}\|f^{(2-\ell)}\|_{L^p(\Omega)}, \quad (5.9)$$

where $q \in [r', \infty]$, and $k = 1$ under the hypotheses of Claim 1 or $k = 0$ under the hypotheses of Claim 2;

$$\|D(u-v)\|_{L^q(\Omega)} \leqslant K_2(N)|h|^{3-\left(\frac{1}{p}-\frac{1}{q}\right)}\|f^{(0)}\|_{L^p(\Omega)}, \quad (5.10)$$

where $q \in [p, \infty]$, under the hypotheses of Claim 2 for $r = p$, $\ell = 2$;

$$\|u-v\|_{W^{2,2}(\Omega)} \leqslant K_3(N)|h|^{\ell-\left(\frac{1}{p}-\frac{1}{2}\right)}\|f^{(2-\ell)}\|_{L^p(\Omega)}$$

under the hypotheses of Claim 1 for $r = p$, $\ell = 1$ and under the hypotheses of Claim 2 for $r = p$, $\ell = 2$.

In addition, if $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$ and $q \in [2, \infty]$, then the bounds hold for an arbitrary mesh $\overline{\omega}^h$.

Proof. The inequality

$$\|D^k(u-v)\|_{L^q(\Omega)} \leqslant \|D^k(u-su)\|_{L^q(\Omega)} + c\|su-v\|_{W^{2,2}(\Omega)}$$

together with (5.7) and (5.8) imply the left-hand inequality (5.9). Then the error bounds follow from bounds (5.2) and (5.3) taking into account Theorems 4.2 and 2.1. □

Clearly, the orders of the error bounds (5.9) and (5.10) are the highest, respectively, for $q = r'$ and $q = p$; on the other hand, they are bounds in the especially interesting uniform norm for $q = \infty$.

We introduce interpolation spaces $(L^p(\Omega), W^{1,p}(\Omega))_{\alpha,\infty}$, $1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$, by the $K_{\alpha,\theta}$-method of real interpolation with $\theta = \infty$ [1]. It is known that they coincide with the Nikolskii spaces $H^{\alpha,p}(\Omega)$ (up to the equivalence of norms) for $0 < \alpha < 1$, whereas $(L^p(\Omega), W^{1,p}(\Omega))_{\alpha,\infty}$ contains the space $BV(\bar{\Omega})$ of functions of bounded variation on $\bar{\Omega}$ for $\alpha = 1$, $p = 1$. Therefore these spaces contain discontinuous piecewise smooth functions for $\alpha \leqslant 1/p$.

Corollary 5.2. *Let $\varkappa = a_2$, $1 \leqslant p \leqslant 2$, $0 < \alpha \leqslant 1$ and $f = D^{2-\ell} f^{(2-\ell)}$, $\ell = 0, 1$. Let the mesh $\overline{\omega}^h$ be quasi-uniform.*

1. *Let $\ell = 0$ and $\|a_1\|_{L^r(\Omega)} + \|a_0\|_{W^{-1,r}(\Omega)} \leqslant N$ for some $r \in [p, p']$. Then the following superconvergence bound holds:*

$$\|s_{a_2}u - v\|_{W^{2,2}(\Omega)} \leqslant K_1(N)|h|^{1+\alpha-\left(\frac{1}{p}-\frac{1}{r'}\right)}\|f^{(2)}\|_{(L^p(\Omega),W^{1,p}(\Omega))_{\alpha,\infty}}.$$

2. *Let $\ell = 0, 1$ and $\|a_1\|_{L^\infty(\Omega)} + \|\bar{D}a_1\|_{L^r(\Omega)} + \|a_0\|_{L^r(\Omega)} \leqslant N$ for some $r \in [1, 2]$ (for $\ell = 0$) or $r \in [p, 2]$ (for $\ell = 1$). Then the following stronger superconvergence bound holds:*

$$\|s_{a_2}u - v\|_{W^{2,2}(\Omega)} \leqslant K_2(N)|h|^{\ell+2+\alpha-\left(\frac{1}{p}-\frac{1}{r'}\right)}\|f^{(2-\ell)}\|_{(L^p(\Omega),W^{1,p}(\Omega))_{\alpha,\infty}}.$$

In the particular case $a_1 = 0$, the admissible values of r are $r \in [1, p']$ (for $\ell = 0$) or $r \in [p, p']$ (for $\ell = 1$).

In addition, in the case where $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$, Claims 1 and 2 hold for an arbitrary mesh $\overline{\omega}^h$.

Proof. The result straightforwardly follows from Theorem 5.1 by virtue of the interpolation theorem for linear operators [1]. □

Now we complement Theorem 5.1 and the latter corollary in the case $r = 1$ and consider the case of the broadest assumptions on both the junior coefficients and f. We define the subspace $C_0(\bar{\Omega})$ of functions in $C(\bar{\Omega})$ that equal zero on $\partial\Omega$ and the corresponding conjugate space $[C_0(\bar{\Omega})]^*$. Clearly this conjugate space contains δ-functions; they can be also represented as Dg with $g \in BV(\bar{\Omega})$.

Corollary 5.3. *Let $\varkappa = a_2$, $1 \leqslant p \leqslant 2$ and $f = D^{2-\ell} f^{(2-\ell)}$, $\ell = 0, 1$. Let the mesh $\overline{\omega}^h$ be quasi-uniform for $1 \leqslant p < 2$.*

1. *Let $\ell = 0$ and $\|a_1\|_{[C_0(\bar{\Omega})]^*} + \|a_0^{(1)}\|_{[C_0(\bar{\Omega})]^*} \leqslant N$, where $a_0 = Da_0^{(1)}$ in the sense that $\langle a_0, \varphi\rangle_\Omega = -\langle a_0^{(1)}, D\varphi\rangle_\Omega$ for all $\varphi \in W_0^{2,1}(\Omega)$. Then the following superconvergence bounds hold:*

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_1(N)|h|^{1-1/p}\|f^{(2)}\|_{L^p(\Omega)},$$

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_2(N)|h|\|f^{(2)}\|_{BV(\bar{\Omega})}.$$

2. *Let $\ell = 0, 1$ and $\|a_1\|_{BV(\bar{\Omega})} + \|a_0\|_{[C_0(\bar{\Omega})]^*} \leqslant N$. Then the following stronger superconvergence bounds hold:*

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_1(N)|h|^{2-1/p}\|f^{(2)}\|_{L^p(\Omega)},$$

$$\|s_{a_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_2(N)|h|^{\ell+2}\|f^{(2-\ell)}\|_{BV(\bar{\Omega})}.$$

Proof. The argument is similar to that for the corresponding Statement 3.2 in [11]. □

Now we consider the case where the major coefficient a_2 is continuous at any point of $\overline{\Omega} \setminus (\overline{\omega}_0 \setminus \{0, X\})$ whereas it can have discontinuities of the first kind at the points of $\overline{\omega}_0 \setminus \{0, X\}$. In this case, the form of the coefficients of the mesh system of equations can be essentially simplified by another choice of $\varkappa$. Let $\widehat{a}_2$ be the linear function over intervals Ω_i such that

$$\widehat{a}_2(x_{i-1}^+) = a_2(x_{i-1}^+), \quad \widehat{a}_2(x_i^-) = a_2(x_i^-) \quad \text{for } i = \overline{1, n}.$$

This is the piecewise linear interpolant for a_2 which is *discontinuous* for *discontinuous* a_2.

Remark 5.1. The interpolation error bound (4.9) can be easily generalized for piecewise continuous functions (for the same p, q and m) as follows: for w, $\bar{D}^{m+1} w \in L^p(\Omega)$ we have

$$\|w - \widehat{w}\|_{L^q(\Omega)} \leqslant |h|^{m+1-\left(\frac{1}{p}-\frac{1}{q}\right)}\|\bar{D}^{m+1} w\|_{L^p(\Omega)}.$$

We turn to the superconvergence bounds for $s_{\widehat{a}_2} u - v$ and establish the same bounds as in Theorem 5.1 under a suitable piecewise regularity of a_2.

Theorem 5.2. *Let $\varkappa = \widehat{a}_2$, $1 \leqslant p \leqslant 2$ and $f = D^{2-\ell} f^{(2-\ell)}$, $\ell = 0, 1, 2$. Let the mesh $\overline{\omega}^h$ be quasi-uniform.*

1. *Let $\ell = 0, 1$ and $\|a_1\|_{L^r(\Omega)} + \|a_0\|_{W^{-1,r}(\Omega)} \leqslant N$ for some $r \in [1, 2]$ (for $\ell = 0$) or $r \in [p, 2]$ (for $\ell = 1$), and also $\|\bar{D} a_2\|_{L^\infty(\Omega)} \leqslant N$. Then the following superconvergence bound holds:*

$$\|s_{\widehat{a}_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_1(N)|h|^{\ell+1-\left(\frac{1}{p}-\frac{1}{r'}\right)}\|f^{(2-\ell)}\|_{L^p(\Omega)}. \tag{5.11}$$

If instead of $\|\bar{D}a_2\|_{L^\infty(\Omega)} \leqslant N$ we assume that $\|\bar{D}^2 a_2\|_{L^\infty(\Omega)} \leqslant N$ and $\|\bar{D}a_2\|_{L^p(\Omega)} \leqslant N$ (the latter condition for $\ell = 1$ only), then the admissible values of r are $r \in [1, p']$ (for $\ell = 0$) or $r \in [p, p']$ (for $\ell = 1$).

2. Let $\|a_1\|_{L^\infty(\Omega)} + \|\bar{D}a_1\|_{L^r(\Omega)} + \|a_0\|_{L^r(\Omega)} \leqslant N$ for some $r \in [1, 2]$ (for $\ell = 0, 1$) or $r \in [p, 2]$ (for $\ell = 2$) as well as $\|\bar{D}^2 a_2\|_{L^\infty(\Omega)} \leqslant N$ and $\|\bar{D}a_2\|_{L^{\ell p}(\Omega)} \leqslant N$ (the latter condition for $\ell = 1, 2$ only). Then the following stronger superconvergence bound holds:

$$\|s_{\widehat{a}_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K_2(N)|h|^{\ell+2-(\frac{1}{p}-\frac{1}{r'})}\|f^{(2-\ell)}\|_{L^p(\Omega)}. \tag{5.12}$$

In the particular case $a_1 = 0$, the admissible values of r are $r \in [1, p']$ (for $\ell = 0, 1$) or $r \in [p, p']$ (for $\ell = 2$).

In addition, in the case where $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$, Claims 1 and 2 hold for an arbitrary mesh $\overline{\omega}^h$.

Proof. We set $e = u - s_\varkappa u$ once more and bound the major summand on the right-hand side of identity (5.4) (other summands are bounded quite similarly to the proof of the previous theorem). Let $1 \leqslant t \leqslant 2$ and $h(x) = h_i$ on Ω_i, $i = \overline{1,n}$. Applying the projection property (3.3) and the Hölder inequality, we get

$$(a_2 D^2 e, D^2\varphi)_\Omega = ((a_2 - \varkappa)D^2 e, D^2\varphi)_\Omega \leqslant c^{(1)} \left\| h^{-(\frac{1}{2}-\frac{1}{t'})}\left(\frac{a_2}{\varkappa} - 1\right)\right\|_{L^\infty(\Omega)} \|D^2 e\|_{L^t(\Omega)}\|\varkappa D^2\varphi\|_{L^2(\Omega)}; \tag{5.13}$$

here the inequality

$$\|h^{\frac{1}{2}-\frac{1}{t'}}\varkappa D^2\varphi\|_{L^{t'}(\Omega)} \leqslant c^{(1)}\|\varkappa D^2\varphi\|_{L^2(\Omega)} \quad \text{for } \varphi \in S_2[\varkappa],$$

(see [11], has also been exploited).

Let $\varkappa = \widehat{a}_2$ and $\|\bar{D}^j a_2\|_{L^\infty(\Omega)} \leqslant N$, where $j = 1$ or 2. It is easy to see that

$$\|h^{-\lambda}(a_2 - \widehat{a}_2)\|_{L^\infty(\Omega)} \leqslant \|\bar{D}^j a_2\|_{L^\infty(\Omega)}|h|^{j-\lambda} \leqslant N|h|^{j-\lambda} \quad \text{for } 0 \leqslant \lambda \leqslant 1$$

(compare with Remark 5.1 for $q = p = \infty$). Therefore, relations (5.4) and (5.13) imply the bound

$$\|s_{\widehat{a}_2} u - v\|_{W^{2,2}(\Omega)} \leqslant K(N)\left(|h|^{j-\left(\frac{1}{t}-\frac{1}{2}\right)}\|D^2 e\|_{L^t(\Omega)} + \|D^{i_0} e\|_{L^{r'}(\Omega)}\right), \tag{5.14}$$

where $i_0 = 1$ or 0 for a_1 and a_0 satisfying the condition from respectively Claim 1 or 2 of the theorem.

Since $(\widehat{a}_2/a_2)(x^\pm) = 1$ for $x \in \overline{\omega}_0 \setminus \{0, X\}$, we have $\widehat{a}_2/a_2 \in C(\bar{\Omega})$ and $D(\widehat{a}_2/a_2) = \bar{D}(\widehat{a}_2/a_2)$. Therefore, taking into account the relations $\widehat{a}_2 D^2 u = (\widehat{a}_2/a_2)\, a_2 D^2 u$ and $N^{-1} \leqslant a_2 \leqslant N$, we get the bounds

$$\|D(\widehat{a}_2 D^2 u)\|_{L^p(\Omega)} \leqslant K_1(N)\left(1 + \|\bar{D}a_2\|_{L^p(\Omega)}\right)\|a_2 D^2 u\|_{W^{1,p}(\Omega)}, \tag{5.15}$$

$$\|\bar{D}D(\widehat{a}_2 D^2 u)\|_{L^p(\Omega)} \leqslant K_2(N)(1 + \|\bar{D}a_2\|^2_{L^{2p}(\Omega)} + \|\bar{D}^2 a_2\|_{L^p(\Omega)})\times$$
$$(\|a_2 D^2 u\|_{W^{1,\infty}(\Omega)} + \|\bar{D}D(a_2 D^2 u)\|_{L^p(\Omega)}). \tag{5.16}$$

Here we have also exploited the simple formulas

$$\bar{D}\frac{\widehat{a}_2}{a_2} = \frac{1}{a_2}\,\bar{D}\widehat{a}_2 - \frac{\widehat{a}_2}{a_2^2}\,\bar{D}a_2, \quad \bar{D}D\frac{\widehat{a}_2}{a_2} = -\frac{2}{a_2^2}\,\bar{D}\widehat{a}_2 \cdot \bar{D}a_2 + \widehat{a}_2\left(-\frac{1}{a_2^2}\,\bar{D}^2 a_2 + \frac{2}{a_2^3}\,(\bar{D}a_2)^2\right)$$

and the inequality $\|\bar{D}\widehat{a}_2\|_{L^q(\Omega)} \leqslant \|\bar{D}a_2\|_{L^q(\Omega)}$, $1 \leqslant q \leqslant \infty$.

To derive bounds (5.11) and (5.12), we apply successively bound (5.14) with $t = p$ (for $\ell = 0$) or $t = 2$ (for $\ell = 1, 2$), Theorem 4.2 (for $k = i_0, 2$), then bound (5.15) (for $\ell = 1$) or (5.16) (for $\ell = 2$), and finally Theorem 2.1. In the particular case where $p = 2$ for $\ell = 0$ as well as $r \leqslant 2$, the mesh $\overline{\omega}^h$ can be arbitrary. □

Remark 5.2. According to formula (3.7), the form of the coefficients of the mesh system of equations is simplified even more if we choose $\varkappa = \widehat{(a_2^{-1})}^{-1}$ instead of $\varkappa = \widehat{a}_2$. In this case, it is rather easy to see that Theorem 5.2 remains valid.

Remark 5.3. Theorem 5.2 is also valid for the method exploiting the space $S_2[\varkappa]$ (instead of $S_1[\varkappa]$) with $\varkappa = \widehat{a}_2$, $\widehat{(a_2^{-1})}^{-1}$; the mesh $\overline{\omega}^h$ can be arbitrary in this case. This supplements the results of [11].

Remark 5.4. We have confined ourselves only to the homogeneous boundary conditions (2.2) for brevity. In the case of the nonhomogeneous boundary conditions

$$u(0) = u_0, \quad u(X) = u_X, \quad Du(0) = u_0^{(1)}, \quad Du(X) = u_X^{(1)}, \tag{5.17}$$

Theorem 2.1 (for the obviously modified definition of the weak solution) remains valid after the addition of the summand $|u_0| + |u_X| + |u_0^{(1)}| + |u_X^{(1)}|$ to the norms $\|f^{(2-\ell)}\|_{L^p(\Omega)}$, $\ell = 0, 1, 2$, (according to Remark 1.2 in [11]). Consequently, all error bounds remain valid after similar modifications.

Moreover, nonhomogeneous boundary conditions other than (5.17) could be considered as well.

Finally we note that an application to the time-dependent case can be found in [13, 15].

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References

1. J. Berg and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.
2. O. V. Besov, V. P. Il'in, and S. M. Nikolskii, *Integral Representations of Functions and Embedding Theorems*, Halsted Press Books, 1979.
3. J. Douglas, jr., T. Dupont, and L. Wahlbin, *The stability in L^q of the L^2–projection into finite element function spaces*, Numer. Math., **23** (1975), pp. 193–197.
4. J. Douglas, jr., T. Dupont, and L. Wahlbin, *Optimal L^∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems*, Math. Comp., **29** (1975), no. 130, pp. 475–483.
5. G. I. Marchuk and V. I. Agoshkov, *Introduction to Projective-Grid Methods*, Nauka, 1981 (in Russian).
6. A. A. Samarskii and V. B. Andreev, *Méthodes aux Différences pour Equations Elliptiques*, Mir, 1978 (translated from Russian).
7. M. H. Schultz, *Elliptic spline functions and the Rayleigh — Ritz — Galerkin method*, Math. Comp., **24** (1970), no. 109, pp. 65–80.
8. A. Yu. Shadrin, *On the approximation of functions by interpolating splines defined on nonuniform nets*, Math. USSR Sbornik, **71** (1992), no. 1, pp. 81–99.
9. G. Strang and G. Fix, *An Analysis of Finite Element Methods*, Prentice-Hall Inc, 1980.
10. R. Varga, *Functional Analysis and Approximation Theory in Numerical Analysis*, SIAM, 1971.
11. A. A. Zlotnik and O. I. Kireeva, *On error of certain projective-grid methods for an ordinary differential equation of 4-th order with non-smooth data*, Russian Mathematics (Iz. VUZ), **39** (1995), no. 4, pp. 49–61.

12. A. A. Zlotnik and O. I. Kireeva, *On error of methods with generalized subic splines for 4-th order ordinary differential equation with nonsmooth data*, Moscow Power Engineering Inst. Bulletin, **2** (1995), no. 6, pp. 59–71 (in Russian).

13. A. A. Zlotnik and O. I. Kireeva, *Finite element methods for the problem of dynamic vibrations of an inhomogeneous bar with nonsmooth data*, Math. Notes, **60** (1996), no. 1, pp. 105–109.

14. A. A. Zlotnik and O. I. Kireeva, *On some properties of generalized cubic splines*, Russ. J. Numer. Anal. Math. Modelling, **15** (2000), no. 6, pp. 539–551.

15. O. I. Kireeva, *Projective-grid methods for stationary and nonstationary 4-th order equation with nonsmooth data*, Ph. D. thesis, Moscow Power Engineering Institute (Technical University), Moscow, 2002, 106 pp. (in Russian).