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SINGULARLY PERTURBED CONVECTION — DIFFUSION PROBLEMS IN ONE DIMENSION: BOUNDS ON DERIVATIVES

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Abstract — A convection-dominated singularly perturbed two-point boundary problem is considered. For the numerical analysis of such problems, it is necessary to prove certain a priori bounds on the derivatives of its solution. This paper provides a survey of the ways in which such bounds can be proved, while assessing the feasibility of extending such proofs to convection-dominated partial differential equations, and also introduces a new proof based on a classical finite-difference argument of Brandt.

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1. Introduction

For the rigorous numerical analysis of convection-diffusion problems, one needs bounds on derivatives of their solutions that *inter alia* specify the dependence on the singular perturbation parameter. Such bounds are also of interest in their own right from the point of view of understanding the behaviour of solutions to such problems.

In this paper we consider a convection-diffusion two-point boundary value problem. While solution derivative bounds for such problems were established as long ago as 1978 in [5], the method of proof of [5] does not extend easily to partial differential equations. Consequently in this paper we shall discuss alternative approaches, some of which have previously appeared in the literature and at least one of which is new.

The problem examined in this paper is the two-point boundary value problem

$$Lu(x) := -\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x) \quad \forall\, x \in (0,1),$$
$$u(0) = u_0, \quad u(1) = u_1, \tag{1.1}$$

where u_0 and u_1 are given constants. Here the diffusion coefficient $\varepsilon \in (0,1]$ is a fixed parameter that is taken to be sufficiently small in various calculations below, e.g., in Section 5 one needs $(\varepsilon/a)|\ln\varepsilon| < 1$. It is assumed that $p, q, f \in C[0,1]$ with $q(x) \geqslant 0$ for all $x \in [0,1]$. Then the operator L satisfies a maximum or comparison principle [9] as described in Lemma 2.1 below and it follows from the standard theory of ordinary differential equations that (1.1) has a unique solution $u \in C^2[0,1]$.

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Set $\underline{p} = \min_{x\in[0,1]} p(x)$. The convection coefficient p is assumed to satisfy

$$\underline{p} > a > 0 \quad \text{for some constant } a. \tag{1.2}$$

Then it is well known (see, e.g., [5, 10]) that u has a boundary layer at $x = 1$.

Our aim in the present paper is to show how various analytical techniques can be used to demonstrate this boundary layer behaviour. We shall confine our attention to proving a pointwise bound on $u'(x)$, since an inductive argument (see [5, Lemma 2.3]) can then be invoked to deduce analogous bounds on high-order derivatives. Pointwise bounds imply bounds in other norms such as H^1.

It is also of interest to observe how some analytical approaches demand more regularity of the data p, q and f. Thus in each section below we shall where necessary make additional assumptions on this data.

Notation. Throughout this paper C will denote a generic constant that is independent of ε and of all norms of u. It may take different values at different places. A subscripted C (such as C_0) indicates a fixed constant that is independent of ε and of all norms of u. Write $\|\cdot\|$ for the $L_\infty[0,1]$ norm.

Thus we have $\|p\| \leqslant C$ and $\|q\| \leqslant C$. To push through the inductive argument to bound derivatives of u of order greater than 1, instead of $f = f(x)$ one must work (see [5]) with the more general hypothesis that $f = f(x, \varepsilon)$ with

$$|f(x,\varepsilon)| \leqslant C_0 \left(1 + \varepsilon^{-1} e^{-a(1-x)/\varepsilon}\right) \quad \text{for } x \in [0,1]. \tag{1.3}$$

2. Preliminary results

In this section we gather a few basic results that are used in the subsequent sections.

Lemma 2.1. *(Comparison principle) Let $[c,d] \subset [0,1]$. Let $v, w \in C^2(c,d) \cap C[c,d]$ satisfy $Lv(x) \geqslant |Lw(x)|$ on (c,d) and $v(x) \geqslant |w(x)|$ for $x = c, d$. Then $v \geqslant |w|$ on $[c,d]$.*

Proof. See, e.g., [9]. □

In Lemma 2.1 we say that v is a *barrier function* for w on the interval $[c,d]$.

Lemma 2.2. *[5, Lemma 2.1] There exists a constant C_1 such that $\|u\| \leqslant C_1$.*

Proof. A quick calculation shows that for all $x \in (0,1)$ one has

$$L(1+x) = p(x) + (1+x)q(x) > a$$

and

$$L\left(e^{-a(1-x)/\varepsilon}\right) = \left\{\frac{a[p(x)-a]}{\varepsilon} + q(x)\right\} e^{-a(1-x)/\varepsilon} \geqslant \frac{a\left[\underline{p}-a\right]}{\varepsilon} e^{-a(1-x)/\varepsilon}.$$

Set

$$v(x) = \left(\frac{C_0}{a} + u_0 + u_1\right)(1+x) + \frac{C_0}{a(\underline{p}-a)} e^{-a(1-x)/\varepsilon} \quad \text{for } x \in [0,1].$$

The above inequalities and (1.3) imply that $Lv(x) \geqslant |f(x,\varepsilon)|$ on $(0,1)$ with $v(x) \geqslant |u(x)|$ for $x = 0, 1$. By Lemma 2.1 we therefore have $|u(x)| \leqslant v(x)$ on $[0,1]$. Finally,

$$\|v\| \leqslant 2\left(\frac{C_0}{a} + u_0 + u_1\right) + \frac{C_0}{a(\underline{p}-a)} =: C_1.$$

□

Note how vital the strict inequality $\underline{p} > a$ of (1.2) is to the proof of Lemma 2.2.

Lemma 2.3. *There exists a constant C_2 such that $|u(x) - u_0| \leqslant C_2 x$ for $0 \leqslant x \leqslant 2/3$ and $|u'(0)| \leqslant C_2$.*

Proof. Set $w(x) = u(x) - u_0$ for $x \in [0, 2/3]$. Then $|w(0)| = 0$ and $|w(2/3)| \leqslant 2C_1$. From (1.3) and $q \in C[0,1]$ we get

$$|Lw(x)| = |f(x,\varepsilon) - q(x)u_0| \leqslant C_3 \text{ (say) for } 0 < x < 2/3.$$

Let $C_2 = \max\{3C_1, C_3/a\}$. Set $v(x) = C_2 x$. Then $v(0) = |w(0)|$ and $v(2/3) = 2C_2/3 \geqslant |w(2/3)|$, while $Lv(x) \geqslant |Lw(x)|$ on $(0, 2/3)$. Invoking Lemma 2.1 we get $|w(x)| \leqslant v(x)$ for $0 \leqslant x \leqslant 2/3$, as desired. The bound $|u'(0)| \leqslant C_2$ follows. □

Lemma 2.3 implies that u has no layer at $x = 0$, insofar as $u'(0)$ is bounded independently of ε. On the other hand there are points in $(0,1)$ where $|u'(x)|$ is large when ε is close to zero, as the next result implies.

Lemma 2.4. *There exists a constant C_4 such that $\|u'\| \leqslant C_4 \varepsilon^{-1}$.*

Proof. Choose $x \in [0,1]$ such that $|u'(x)| = \|u'\|$. Choose an interval $[x_1, x_2] \subset [0,1]$ such that $x \in [x_1, x_2]$ and $x_2 - x_1 = \varepsilon/(2\|p\|)$. By the mean value theorem and Lemma 2.2 there exists $\tilde{x} \in (x_1, x_2)$ such that

$$|u'(\tilde{x})| = \left|\frac{u(x_2) - u(x_1)}{x_2 - x_1}\right| \leqslant 4C_1\|p\|\varepsilon^{-1} = C\varepsilon^{-1}.$$

Integrating (1.1)from x to $\tilde{x}$ and rearranging gives

$$\|u'\| = |u'(x)| \leqslant |u'(\tilde{x})| + \varepsilon^{-1}\int_x^{\tilde{x}} [|p(s)u'(s)| + |f(s)| + |q(s)u(s)|]\, ds.$$

Hence, invoking (1.3) and Lemma 2.2 and observing that $|x - \tilde{x}| \leqslant \varepsilon/(2\|p\|)$, we get

$$\|u'\| \leqslant C\varepsilon^{-1} + \|u'\|/2.$$

The result follows. □

The statement of Lemma 2.4 is sharp but it does not reveal that $|u'(x)|$ is large only near $x = 1$. The proof of this layer property of u' is the subject of the rest of this paper.

Theorem 2.1. *There exists a constant C such that*

$$|u'(x)| \leqslant C\left[1 + \varepsilon^{-1}e^{-a(1-x)/\varepsilon}\right] \quad \text{for } 0 \leqslant x \leqslant 1. \tag{2.1}$$

Proof. In each subsequent section we shall provide a different proof of (2.1). □

3. Kellogg and Tsan technique

In [5] an integrating factor and some elementary manipulations are used to handle (1.1), as we now describe.

1st proof of Theorem 2.1. Set $h = f - qu$ and

$$P(x) = \int_0^x p(t)\, dt \quad \text{for } 0 \leqslant x \leqslant 1.$$

Then rewriting (1.1) as $-\varepsilon u'' + pu' = h$, multiplying by the integrating factor $\varepsilon^{-1}e^{-P(x)/\varepsilon}$ and rearranging, we get

$$u'(x) = e^{-[P(1)-P(x)]/\varepsilon}u'(1) + \varepsilon^{-1}\int_{t=x}^{1} e^{-[P(t)-P(x)]/\varepsilon}h(t)\,dt.$$

Invoking Lemma 2.4 to bound $u'(1)$, and noting that $P(s) - P(x) \geqslant \underline{p}(s-x)$ for $s \geqslant x$, it follows that

$$|u'(x)| = C\varepsilon^{-1}e^{-a(1-x)/\varepsilon} + C\varepsilon^{-1}\int_{t=x}^{1} e^{-\underline{p}(t-x)/\varepsilon}|h(t)|\,dt. \tag{3.1}$$

By (1.3) and Lemma 2.2,

$$\varepsilon^{-1}\int_{t=x}^{1} e^{-\underline{p}(t-x)/\varepsilon}|h(t)|\,dt \leqslant C\varepsilon^{-1}\int_{t=x}^{1} e^{-\underline{p}(t-x)/\varepsilon}\left[1+\varepsilon^{-1}e^{-a(1-t)/\varepsilon}\right]\,dt =$$

$$C\left[1-e^{-\underline{p}(1-x)/\varepsilon}\right] + C\varepsilon^{-2}e^{-a(1-x)/\varepsilon}\int_{t=x}^{1} e^{-(\underline{p}-a)(t-x)/\varepsilon}\,dt \leqslant C\left[1+\varepsilon^{-1}e^{-a(1-x)/\varepsilon}\right].$$

Recalling (3.1), we are done. □

Remark 3.1. While the analysis of this section is short and requires only that p, q and f lie in $C[0,1]$, it does not seem possible to generalize it to problems in higher dimensions such as

$$-\varepsilon\nabla u + p_1(x,y)u_x + p_2(x,y)u_y + q(x,y)u = f(x,y) \quad \text{on} \quad \Omega = (0,1)^2,$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

where $p_1 > 0$, $p_2 > 0$ and $q \geqslant 0$ on $\bar{\Omega}$.

4. Majorizing function approach

This elementary method generalizes the argument of Lemma 2.3. It has been used by many authors in many contexts but we are unaware of any published proof of Theorem 2.1 that is based on it.

2nd proof of Theorem 2.1. Let $x_0 \in [0,1]$ be arbitrary but fixed. We shall show that

$$|u'(x_0)| \leqslant C\left[1+\varepsilon^{-1}e^{-a(1-x_0)/\varepsilon}\right].$$

If $x_0 \geqslant 1-\varepsilon$ then the result is immediate from Lemma 2.4, so we can assume that $0 \leqslant x_0 \leqslant 1-\varepsilon$. For $x \in [x_0, 1]$, set $\psi(x) = u(x) - u(x_0)$,

$$C_5 = \frac{C_0 + C_1\|q\|}{\underline{p}}, \quad C_6 = \frac{C_0}{a(\underline{p}-a)} + \frac{2C_1}{1-e^{-a}},$$

and $\phi(x) = C_5(x-x_0) + C_6\left[e^{-a(1-x)/\varepsilon} - e^{-a(1-x_0)/\varepsilon}\right]$, where C_0 and C_1 are defined in (1.3) and Lemma 2.2. We shall show that ϕ is a barrier function for ψ on the interval $[x_0, 1]$.

Now $|\psi(x_0)| = 0 = \phi(x_0)$ and Lemma 2.2 implies that $|\psi(1)| = |u(1) - u(x_0)| \leqslant 2C_1 \leqslant \phi(1)$ owing to the definition of C_6 and $1 - x_0 \geqslant \varepsilon$. Furthermore, for $x \in (x_0, 1)$ one has

$$|L\psi(x)| = |L[u(x) - u(x_0)]| = |f(x,\varepsilon) - q(x)u(x_0)| \leqslant C_0\left(1 + \varepsilon^{-1}e^{-a(1-x)/\varepsilon}\right) + C_1\|q\| \quad (4.1)$$

by (1.3) and Lemma 2.2, while a short calculation shows that

$$L\phi(x) = C_6\varepsilon^{-1}e^{-a(1-x)/\varepsilon}a\,[p(x) - a] + C_5p(x) + q(x)\phi(x) \geqslant C_6\varepsilon^{-1}e^{-a(1-x)/\varepsilon}a\left[\underline{p} - a\right] + C_5\underline{p}\,.$$

Comparing this with (4.1), it is clear that the definitions of C_5 and C_6 imply that $L\phi(x) \geqslant |L\psi(x)|$. Thus ϕ is a barrier function for ψ on the interval $[x_0, 1]$ and Lemma 2.1 yields $\phi(x) \geqslant |\psi(x)|$ on $[x_0, 1]$.

Consequently

$$|u'(x_0)| = \left|\lim_{x\to x_0^+}\frac{\psi(x)}{x - x_0}\right| \leqslant \lim_{x\to x_0^+}\left|\frac{\phi(x)}{x - x_0}\right| = |\phi'(x_0)| = C_5 + C_6a\varepsilon^{-1}e^{-a(1-x_0)/\varepsilon}$$

and we are done. □

Remark 4.1. For the two-dimensional problem (3.2) it does not seem possible to generalize the above argument by finding a suitable barrier function that vanishes at the point (x_0, y_0) while satisfying all the inequalities required in the argument.

5. Using the Green's function

Andreev [1] derives various weighted estimates of the Green's function $G(x,\xi)$ associated with (1.1) (with $u_0 = u_1 = 0$) by considering G as a perturbation of the Green's function for the case where $q \equiv 0$. (The latter Green's function can be written down explicitly.) He is thereby able to prove the inequalities

$$|u'(x)| \leqslant C\left[1 + \varepsilon^{-1}e^{-r(1-x)/\varepsilon}\right]\|f\| \quad \forall x \in [0,1], \tag{5.1}$$

$$\max_{0\leqslant x\leqslant 1}\left|(|u(x) + \varepsilon|u'(x)|)e^{r(1-x)/\varepsilon}\right| \leqslant C\varepsilon\max_{0\leqslant x\leqslant 1}\left|f(x,\varepsilon)e^{r(1-x)/\varepsilon}\right| \tag{5.2}$$

for any constant $r \in (0, \underline{p})$ and $C = C(r)$. Since $\|f\| = O(\varepsilon^{-1})$ by (1.3), inequality (5.1) does not provide an immediate proof of Theorem 2.1. The proof of this theorem that we now present is new.

3rd proof of Theorem 2.1. By a change of variable we can assume that $u_0 = u_1 = 0$ without disturbing any of our hypotheses (the value of C_0 in (1.3) will then change but we ignore this detail here). First we decompose f into two components of distinct types: from (1.3) one sees that $|f(x)| \leqslant 2C_0$ for $0 \leqslant x \leqslant 1 - (\varepsilon/a)|\ln\varepsilon|$. Choose $f_0 \in C[0,1]$ to agree with f on the interval $[0, 1 - (\varepsilon/a)|\ln\varepsilon|]$ and to satisfy $\|f_0\| \leqslant 2C_0$. Set $f_1 = f - f_0$. Then $f_1 \equiv 0$ on $[0, 1 - (\varepsilon/a)|\ln\varepsilon|]$, while for $x \geqslant 1 - (\varepsilon/a)|\ln\varepsilon|$ one has

$$|f_1(x)| \leqslant |f(x)| + |f_0(x)| \leqslant C_0\left(3 + \varepsilon^{-1}e^{-a(1-x)/\varepsilon}\right) \leqslant 4C_0\varepsilon^{-1}e^{-a(1-x)/\varepsilon}.$$

For $i = 0, 1$ define $v_i \in C^2[0,1]$ to be the solution of $Lv_i = f_i$ on $(0,1)$ with $v_i(0) = v_i(1) = 0$. Applying (5.1) to v_0 with $r = a$ yields

$$|v_0'(x)| \leqslant C\left[1 + \varepsilon^{-1}e^{-a(1-x)/\varepsilon}\right],$$

while applying (5.2) to v_1 with $r = a$ yields a similar result. But $u = v_0 + v_1$ so the proof is complete. □

Remark 5.1. As the Green's function for (3.2) is more complicated and less well behaved than the Green's function for (1.1), it is uncertain whether an argument like this could work in the two-dimensional case.

6. Applying L to $u'(x)$ directly

The idea of this section is the most obvious one of all — one uses the barrier function technique of Lemma 2.1 to bound $u'(x)$ for $x \in [0,1]$. This technique has been used by many authors. To push through the argument one needs the following extension of Lemma 2.1 to more general operators.

Lemma 6.1 (Comparison principle without $q \geqslant 0$). *Define the operator* $M : C^2(0,1) \to C(0,1)$ *by*

$$Mv(x) := -\varepsilon v''(x) + p(x)v'(x) + \tilde{q}(x)v(x) \quad \forall\, x \in (0,1),$$

where $\tilde{q} \in C[0,1]$ *satisfies* $\underline{p}^2 + 4\varepsilon\tilde{q}(x) \geqslant 0$ *for all* x. *Let* $v, w \in C^2(0,1) \cap C[0,1]$ *satisfy* $Mv(x) \geqslant |Mw(x)|$ *on* $(0,1)$ *and* $v(x) \geqslant |w(x)|$ *for* $x = 0,1$. *Then* $v \geqslant |w|$ *on* $[0,1]$.

Proof. Set $w(x) = e^{\sigma x}\tilde{w}(x)$ for $x \in [0,1]$, where σ is independent of x and will be specified in a moment. Then a calculation gives

$$Mw(x) = e^{\sigma x}\left\{-\varepsilon\tilde{w}''(x) + [p(x) - 2\varepsilon\sigma]\,\tilde{w}'(x) + \left[\tilde{q}(x) + p(x)\sigma - \varepsilon\sigma^2\right]\tilde{w}(x)\right\} = e^{\sigma x}\tilde{M}\tilde{w}(x),$$

say. Similarly setting $v(x) = e^{\sigma x}\tilde{v}(x)$, one gets $Mv(x) = e^{\sigma x}\tilde{M}\tilde{v}(x)$, so we now have $\tilde{M}\tilde{v}(x) \geqslant |\tilde{M}\tilde{w}(x)|$ on $(0,1)$. Moreover $\tilde{v}(x) \geqslant |\tilde{w}(x)|$ for $x = 0,1$. Set $\underline{\tilde{q}} = \min_{0\leqslant x\leqslant 1}\tilde{q}(x)$. Choose $\sigma = [\underline{p} + (\underline{p}^2 + 4\varepsilon\underline{\tilde{q}})^{1/2}]/(2\varepsilon)$. Then $0 < \sigma$ and $-\varepsilon\sigma^2 + \underline{p}\sigma + \underline{\tilde{q}} = 0$. Thus $\tilde{q}(x) + p(x)\sigma - \varepsilon\sigma^2 \geqslant 0$ and $\tilde{M}$ satisfies the comparison principle of Lemma 2.1. Hence $\tilde{v}(x) \geqslant |\tilde{w}(x)|$ on $[0,1]$, which gives $v(x) \geqslant |w(x)|$ on $[0,1]$, as desired. $\square$

Variants of this lemma have been used by various authors; the earliest example seems to be Lorenz [7].

Assume that $p, q \in C^1[0,1]$ and $f_x \in C[0,1]$ with $|f_x(x,\varepsilon)| \leqslant C_7\left(1 + \varepsilon^{-2}e^{-a(1-x)/\varepsilon}\right)$ for all x and some constant C_7.

4th proof of Theorem 2.1. From Lemmas 2.3 and 2.4 one has $|u'(0)| \leqslant C_2$ and $|u'(1)| \leqslant C_4\varepsilon^{-1}$. Now

$$L(u') = -\varepsilon u''' + pu'' + qu' = (-\varepsilon u'' + pu' + qu)' - p'u' - q'u = f_x - p'u' - q'u. \qquad (6.1)$$

Define the operator $\hat{L} : C^2[0,1] \to C(0,1)$ by $\hat{L}v = Lv + p'v$. Then $\hat{L}(u') = f_x - q'u$. Hence

$$|\hat{L}(u'(x))| \leqslant C_8\left(1 + \varepsilon^{-2}e^{-a(1-x)/\varepsilon}\right) \quad \text{for all} \quad x \in (0,1),$$

where C_8 is some constant. We shall apply the comparison principle of Lemma 6.1 to $\hat{L}$ and the function u'. To do this we must construct a barrier function. For any constant k one has $\hat{L}(e^{kx}) = e^{kx}(-\varepsilon k^2 + pk + q + p')$; choosing $k = 2(\|q\| + \|p'\|)/a$ and taking ε sufficiently small yields $\hat{L}(e^{kx}) \geqslant C_9 := \|q\| + \|p'\|$ for all $x \in [0,1]$. One also has

$$\hat{L}(e^{-a(1-x)/\varepsilon}) = \left\{\frac{a[p(x) - a]}{\varepsilon} + q(x) + p'(x)\right\}e^{-a(1-x)/\varepsilon} \geqslant \left\{\frac{a[\underline{p} - a]}{\varepsilon} - \|q\| - \|p'\|\right\}e^{-a(1-x)/\varepsilon}.$$

Thus for ε sufficiently small one obtains $\hat{L}\big(e^{-a(1-x)/\varepsilon}\big) \geqslant C_{10}\varepsilon^{-1}e^{-a(1-x)/\varepsilon}$ for some constant C_{10} and all x. These inequalities together yield

$$\hat{L}(C_8(e^{kx}/C_9 + \varepsilon^{-1}e^{-a(1-x)/\varepsilon}/C_{10})) \geqslant |\hat{L}u'(x)| \quad \text{for all} \quad x \in (0,1).$$

After modifying the barrier function to handle the boundary data for u', Lemma 6.1 then gives

$$|u'(x)| \leqslant \left(C_2 + \frac{C_8}{C_9}\right) e^{kx} + \left(C_4 + \frac{C_8}{C_{10}}\right) \varepsilon^{-1}e^{-a(1-x)/\varepsilon} \quad \text{for all} \quad x \in (0,1).$$

□

Remark 6.1. This technique can be used to bound the derivatives in the two-dimensional example (3.2), but as we have seen it does require extra regularity of the data, viz., that p, q and f be differentiable.

7. Brandt's finite difference method

Finally we come to the method of Brandt [2], who applied it only to problems that are not singularly perturbed; the extension of the method to singularly perturbed problems is non-trivial and the analysis of this section is new. In this method a second-order elliptic operator such as L is transformed into an elliptic operator that acts on differences of functions in a higher-dimensional setting. Applying the maximum principle to the modified operator yields pointwise bounds on difference quotients of solutions to the problems, from which bounds on the derivatives follow.

Assume as in Section 6 that $|f_x(x,\varepsilon)| \leqslant C_7\left(1 + \varepsilon^{-2}e^{-a(1-x)/\varepsilon}\right)$ for all x and some constant C_7. Assume that $p, q \in C^1[0,1]$. Set $P = \|p'\|$ for convenience.

Set $k = \min\{1/3, (\underline{p}-a)(1-e^{-2a})/(4P), a(\underline{p}-a)/(2Pe^a)\}$ (if $P = 0$ then choose $k = 1/3$). Let $\eta \in [0,k]$ be a parameter. Although (1.1) is a two-point boundary value problem, nevertheless the analysis of this section takes place in a two-dimensional trapezoidal domain. Set $\Omega_1 = \Omega_2 \cup \Omega_3 \cup \Omega_4$, where

$$\Omega_2 = \{(x,\eta) : 0 < x \leqslant k,\ 0 < \eta < x\}, \quad \Omega_3 = \{(x,\eta) : k < x \leqslant 1-k,\ 0 < \eta < k\},$$

$$\Omega_4 = \{(x,\eta) : 1-k < x < 1,\ 0 < \eta < 1-x\}.$$

Given a function F defined on $[0,1]$, define the finite difference operator δ and the finite mean operator μ by

$$\delta(\eta)F(x) = [F(x+\eta) - F(x-\eta)]/2, \quad \mu(\eta)F(x) = [F(x+\eta) + F(x-\eta)]/2 \quad \text{for} \quad (x,\eta) \in \bar{\Omega}_1.$$

The construction of Ω_1 guarantees that these functions are well defined. It is easy to verify the following product rule [2, Lemma 2.1] for the operator δ: if F and G are defined on $[0,1]$ then

$$\delta(\eta)\{F(x)G(x)\} = [\mu(\eta)F(x)][\delta(\eta)G(x)] + [\delta(\eta)F(x)][\mu(\eta)G(x)].$$

Define the difference function $\psi(x,\eta) = \delta(\eta)u(x)$. Clearly

$$\frac{\partial\psi(x,\eta)}{\partial x} = \frac{1}{2}\left[\frac{\partial u(x+\eta)}{\partial(x+\eta)}\frac{\partial(x+\eta)}{\partial x} - \frac{\partial u(x-\eta)}{\partial(x-\eta)}\frac{\partial(x-\eta)}{\partial x}\right] = \delta(\eta)u'(x),$$

$$\frac{\partial\psi(x,\eta)}{\partial\eta} = \mu(\eta)u'(x) \quad \text{and} \quad \frac{\partial^2\psi(x,\eta)}{\partial x^2} = \frac{\partial^2\psi(x,\eta)}{\partial\eta^2} = \delta(\eta)u''(x).$$

Using these identities we have

$$\delta(\eta)Lu(x) = -\frac{\varepsilon}{2}\frac{\partial^2\psi(x,\eta)}{\partial x^2} - \frac{\varepsilon}{2}\frac{\partial^2\psi(x,\eta)}{\partial \eta^2} + [\mu(\eta)p(x)]\frac{\partial\psi(x,\eta)}{\partial x} + [\delta(\eta)p(x)]\frac{\partial\psi(x,\eta)}{\partial \eta} +$$

$$[\mu(\eta)q(x)]\psi(x,\eta) + [\delta(\eta)q(x)][\mu(\eta)u(x)]; \tag{7.1}$$

here the $-\varepsilon u''$ term has been split into two to give an elliptic operator in the variables (x,η).

Define $L_1 : C^2(\Omega_1) \to C(\Omega_1)$ by

$$L_1 w = -\frac{\varepsilon}{2}\frac{\partial^2 w}{\partial x^2} - \frac{\varepsilon}{2}\frac{\partial^2 w}{\partial \eta^2} + [\mu(\eta)p(x)]\frac{\partial w}{\partial x} + [\delta(\eta)p(x)]\frac{\partial w}{\partial \eta} + [\mu(\eta)q(x)]w. \tag{7.2}$$

Then rearranging (7.1) and recalling that $Lu(x) = f(x)$ yields

$$L_1[\delta(\eta)u(x)] = f_1(x,\eta), \tag{7.3}$$

where $f_1(x,\eta) := \delta(\eta)f(x) - [\delta(\eta)q(x)][\mu(\eta)u(x)]$. This identity is a discrete analogue of (6.1). For the subsequent analysis it is convenient to work with the closely-related but simpler operator $L_2 : C^2(\Omega_1) \to C(\Omega_1)$ defined by

$$L_2 w = -\frac{\varepsilon}{2}\frac{\partial^2 w}{\partial x^2} - \frac{\varepsilon}{2}\frac{\partial^2 w}{\partial \eta^2} + \underline{p}\,\frac{\partial w}{\partial x} - P\eta\frac{\partial w}{\partial \eta}\,.$$

Observe that if $w \geqslant 0$, $\partial w/\partial x \geqslant 0$ and $\partial w/\partial \eta \geqslant 0$, then $L_1 w(x,\eta) \geqslant L_2 w(x,\eta)$ for all $(x,\eta) \in \Omega_1$.

Our analysis uses a barrier function $\sigma(x,\eta)$ that will be constructed to have the properties described in the next lemma.

Lemma 7.1. *Suppose that there exists a function $\sigma(x,\eta) \in C^2(\Omega_1) \cap C(\bar{\Omega}_1)$ for which*

$$L_2\sigma(x,\eta) \geqslant |f_1(x,\eta)| \quad \forall (x,\eta) \in \Omega_1, \tag{7.4}$$

$$\sigma(x,\eta) \geqslant 0, \quad \frac{\partial\sigma(x,\eta)}{\partial x} \geqslant 0 \ \ \textit{and} \ \ \frac{\partial\sigma(x,\eta)}{\partial \eta} \geqslant 0 \quad \forall (x,\eta) \in \Omega_1, \tag{7.5}$$

$$\sigma(x,\eta) \geqslant |\delta(\eta)u(x)| \quad \forall (x,\eta) \in \partial\Omega_1. \tag{7.6}$$

Then $\sigma(x,\eta) \geqslant |\delta(\eta)u(x)|$ $\forall (x,\eta) \in \bar{\Omega}_1$.

Proof. The hypotheses (7.5), (7.4) and (7.3) imply that $L_1\sigma(x,\eta) \geqslant L_2\sigma(x,\eta) \geqslant |f_1(x,\eta)| = |L_1[\delta(\eta)u(x)]|$ for all $(x,\eta) \in \Omega_1$. This inequality, (7.6) and $\mu(\eta)q(x) \geqslant 0$ in (7.2) enable us to invoke a standard comparison principle [9] for the elliptic operator L_1 in Ω_1 to get the result. □

Our aim now is to construct a function σ that enjoys the properties (7.4)–(7.6). First, consider (7.6) and the four line segments that comprise $\partial\Omega_1$. Along the line $\eta = x$ with $0 \leqslant x \leqslant k$, Lemma 2.3 yields $|\delta(\eta)u(x)| = |u(2x) - u(0)|/2 \leqslant C_2 x = C_2\eta$. When $\eta = 1 - x$ on $\partial\Omega_1$, then Lemma 2.4 and the mean value theorem give $|\delta(\eta)u(x)| \leqslant 2C_4\varepsilon^{-1}(1-x)$ for $1 - k \leqslant x \leqslant 1$. On the upper horizontal boundary where $\eta = k$ and $k \leqslant x \leqslant 1 - k$, by Lemma 2.2 we have $|\delta(k)u(x)| \leqslant C_1$. Finally, when $\eta = 0$ on $\partial\Omega_1$ this clearly gives $\delta(\eta)u(x) = 0$.

Next we move on to (7.4). For $(x,\eta) \in \Omega_1$, by Lemma 2.2 one has

$$|f_1(x,\eta)| \leqslant |\delta(\eta) f(x)| + |[\delta(\eta) q(x)][\mu(\eta) u(x)]| \leqslant \frac{1}{2}\left| \int\limits_{x-\eta}^{x+\eta} f_x(t,\varepsilon)\, dt \right| + \eta \|q'\| C_1 \leqslant$$

$$\frac{C_7}{2} \int\limits_{x-\eta}^{x+\eta} \left[1 + \varepsilon^{-2} e^{-a(1-t)/\varepsilon}\right] dt + \eta \|q'\| C_1 = (C_7 + \|q'\| C_1)\eta + \frac{C_7}{2a\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right]. \tag{7.7}$$

A calculation shows that for $(x,\eta) \in \Omega_1$ one has

$$L_2\left[\eta e^{(1+P)x/a}\right] = \left[-\frac{\varepsilon}{2}\frac{(1+P)^2}{a^2} + \frac{\underline{p}(1+P)}{a}\right] \eta e^{(1+P)x/a} - P\eta e^{(1+P)x/a} \geqslant \eta e^{(1+P)x/a} \tag{7.8}$$

provided ε is sufficiently small (independently of η), and

$$L_2\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] = \frac{a(\underline{p}-a)}{\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] -$$

$$\frac{Pa\eta}{\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} + e^{-a(1-x+\eta)/\varepsilon}\right]. \tag{7.9}$$

We need a suitable lower bound for the right-hand side of (7.9). There are two cases: first, if $\eta \leqslant \varepsilon$ then

$$e^{-a(1-x-\eta)/\varepsilon} + e^{-a(1-x+\eta)/\varepsilon} \leqslant 2e^a e^{-a(1-x)/\varepsilon}$$

and

$$e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon} = e^{-a(1-x)/\varepsilon}\left(e^{a\eta/\varepsilon} - e^{-a\eta/\varepsilon}\right) \geqslant \frac{2a\eta}{\varepsilon} e^{-a(1-x)/\varepsilon}$$

since $e^t - e^{-t} \geqslant 2t$ for all $t \geqslant 0$. Consequently

$$\frac{a(\underline{p}-a)}{\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] - \frac{Pa\eta}{\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} + e^{-a(1-x+\eta)/\varepsilon}\right] \geqslant$$

$$\frac{a(\underline{p}-a)}{2\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] \tag{7.10}$$

provided $\varepsilon \leqslant a(\underline{p}-a)/(2Pe^a)$; the definition of k ensures that this condition on ε can be satisfied without violating $\eta \leqslant \varepsilon$. In the case where $\eta > \varepsilon$ one has

$$e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon} \geqslant e^{-a(1-x-\eta)/\varepsilon}\left(1 - e^{-2a}\right)$$

while

$$\frac{Pa\eta}{\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} + e^{-a(1-x+\eta)/\varepsilon}\right] \leqslant \frac{2Pak}{\varepsilon} e^{-a(1-x-\eta)/\varepsilon} \leqslant \frac{a(\underline{p}-a)(1-e^{-2a})}{2\varepsilon} e^{-a(1-x-\eta)/\varepsilon}$$

from the definition of k; thus (7.10) holds true. Combining (7.9) and (7.10) yields

$$L_2\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] \geqslant \frac{a(\underline{p}-a)}{2\varepsilon}\left[e^{-a(1-x-\eta)/\varepsilon} - e^{-a(1-x+\eta)/\varepsilon}\right] \quad \text{for all} \quad (x,\eta) \in \Omega_1 \tag{7.11}$$

provided that ε is sufficiently small (independently of η), which is not a restriction.

Define

$$\sigma(x,\eta)=(C_7+\|q'\|C_1)\eta e^{(1+P)x/a}+\frac{C_7}{a^2(\underline{p}-a)}[e^{-a(1-x-\eta)/\varepsilon}-e^{-a(1-x+\eta)/\varepsilon}] \text{ for all } (x,\eta)\in\bar{\Omega}_1.$$

Then $\sigma(x,\eta)\in C^2(\Omega_1)\cap C(\bar{\Omega}_1)$. It follows from (7.7), (7.8) and (7.11) that

$$L_2(\sigma(x,\eta))\geqslant|f_1(x,\eta)| \text{ for all } (x,\eta)\in\Omega_1.$$

It is easily seen that our selected σ satisfies all the conditions (7.4)–(7.6); in particular (7.6) follows from the comments immediately after the proof of Lemma 7.1. Consequently Lemma 7.1 gives

$$\sigma(x,\eta)\geqslant|\delta(\eta)u(x)| \quad \text{for all } (x,\eta)\in\bar{\Omega}_1. \tag{7.12}$$

5th proof of Theorem 2.1. By (7.12) and the definition of $\delta(\eta)$, for each $x\in(0,1)$ we have

$$|u'(x)|=\lim_{\eta\to 0^+}\left|\frac{\delta(\eta)u(x)}{\eta}\right|\leqslant\lim_{\eta\to 0^+}\frac{\sigma(x,\eta)}{\eta}=(C_7+\|q'\|C_1)e^{(1+P)x/a}+\frac{2C_7}{a(\underline{p}-a)\varepsilon}e^{-a(1-x)/\varepsilon},$$

from which the desired result follows. □

Remark 7.1. The technique of this section is applied to classical second-order elliptic partial differential operators in $n\geqslant 1$ variables in [2] and to parabolic operators in [3, 6]. The analysis described above should likewise be capable of extension to problems posed in higher dimensions and we shall pursue this topic in a forthcoming paper [8].

Furthermore, one can analyse elliptic and parabolic *difference* operators in the same framework—see [2, 3, 6].

8. Conclusions

The preceding sections have given five different proofs of the sharp pointwise bound on u' stated in Theorem 2.1. Some of these proofs can be generalized to the two-dimensional problem (3.2), but these arguments require more regularity of the data in the one-dimensional case. For (3.2), even when the norm used is slightly weaker than the standard L_∞ norm, increased regularity of the data seems to be needed if one is to show that certain first-order derivatives are bounded on most or all of the domain [4, Theorem 4.1]. It is unclear whether increased regularity of the data is a necessary condition for proving satisfactory pointwise bounds on derivatives in higher-dimensional singularly perturbed problems and we defer investigation of this question to a later paper.

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