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SOLVING NONLINEAR VOLTERRA — FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS USING THE MODIFIED ADOMIAN DECOMPOSITION METHOD

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Abstract — In this paper, a nonlinear Volterra — Fredholm integro-differential equation is solved by using the modified Adomian decomposition method (MADM). The approximate solution of this equation is calculated in the form of a series in which its components are computed easily. The accuracy of the proposed numerical scheme is examined by comparison with other analytical and numerical results. The existence, uniqueness and convergence and an error bound of the proposed method are proved. Some examples are presented to illustrate the efficiency and the performance of the modified decomposition method.

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1. Introduction

We consider a high-order nonlinear Volterra — Fredholm integro-differential equation given by

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \mu_1 \int_a^x k_1(x,t)g_1(t,u(t))dt + \mu_2 \int_a^b k_2(x,t)g_2(t,u(t))dt, \qquad (1.1)$$

with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0, 1, 2, \dots, k-1, \qquad (1.2)$$

where a, b, μ_1, μ_2, b_r are constant values, $f(x), k_1(x,t), k_2(x,t), g_1(t,u(t)), g_2(t,u(t))$ and $p_j(x)$, $j = 0, 1, \dots, k$ are functions that have suitable derivatives on an interval $a \leqslant t \leqslant x \leqslant b$ and $p_k(x) \neq 0$.

If we set $g_1(t,u(t)) = G_1(u(t))$, $g_2(t,u(t)) = G_2(u(t))$, where G_1 and G_2 are known smooth functions nonlinear in $u(t)$, then Eq. (1.1) reduces to the following equation:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \mu_1 \int_a^x k_1(x,t)G_1(u(t))dt + \mu_2 \int_a^b k_2(x,t)G_2(u(t))dt. \qquad (1.3)$$

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Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations such as the lineaziation method [1], the differential transform method [2], RF-pair method [3], and semianalytical-numerical techniques such as the Adomian decomposition method [5] and Taylor polynomials method [4, 6–8]. The modified decomposition method for solving nonlinear Volterra — Fredholm integral equations was presented by Bildik and Inc in [9]. In the present work, we apply the MADM for solving Eq. (1.3) and compare the results with the Taylor polynomial method. The paper is organized as follows. In Section 2, some preliminaries consisting of the standard Adomian decomposition method (ADM), the modified Adomian decomposition method (MADM) and the Adomian polynomials are briefly described. In Section 3, the MADM for solving nonlinear Volterra — Fredholm integro-differential equations is presented. Also, the existence, uniqueness and convergence and the error bound of the proposed method are proved. Finally, some numerical examples are solved using the MADM in Section 4.

2. Preliminaries

The Adomian decomposition method is applied to the following general nonlinear equation:

$$Lu + Ru + Nu = g(x), \tag{2.1}$$

where u is the unknown function, L is the highest-order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (2.1) and using the given conditions we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.2}$$

where the function $f(x)$ represents the terms arising from integrating the source term $g(x)$. The nonlinear operator $Nu = G(u)$ is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where A_n, $n \geqslant 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0}. \tag{2.4}$$

The Adomian polynomials were introduced in [10–12] as

$$A_0 = G(u_0), \quad A_1 = u_1 G'(u_0), \quad A_2 = u_2 G'(u_0) + \frac{1}{2!}u_1^2 G''(u_0),$$

$$A_3 = u_3 G'(u_0) + u_1 u_2 G''(u_0) + \frac{1}{3!}u_1^3 G'''(u_0), \quad \ldots$$

2.1. Adomian decomposition method. In recent years the Adomian decomposition method [12] has been applied to a wide class of functional equations and inverse problems such as integral equations [13, 14].

The standard decomposition technique represents the solution of u in (2.1) as the following series:

$$u = \sum_{i=0}^{\infty} u_i, \tag{2.5}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$u_0 = f(x), \quad u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geqslant 0. \tag{2.6}$$

Substituting (2.4) into (2.6) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (2.5) follows immediately.

2.2. The modified decomposition method. The modified decomposition method was introduced by Wazwaz [15]. This method is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{2.7}$$

We apply this decomposition when the function f consists of several parts and can be decomposed into two different parts. In this case, f is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part f_1 is important. For the method to be more efficient, we select f_1 as one term of f or at least a number of terms if possible and f_2 consists of the remaining terms of f. In comparison with the standard decomposition method, the MADM minimizes the size of calculations and the cost of computational operations in the algorithm. Both standard and modified decomposition methods are reliable for solving nonlinear problems such as Volterra — Fredholm integro-differential equations, but in order to decrease the complexity of the algorithm and simplify the calculations we prefer to use the MADM. The MADM will accelerate the rapid convergence of the series solution in comparison with the standard Adomian decomposition method. The modified technique may give the exact solution for nonlinear equations without the necessity to find the Adomian polynomials. We refer the reader to [15–17] for more details about the MADM.

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the component u_0, whereas the remaining part f_2 be combined with the other terms given in (2.6) to define u_1. Consequently, the following modified recursive relation was developed:

$$u_0 = f_1(x), \quad u_1 = f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \quad \ldots, \quad u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geqslant 1.$$

3. Description of the method

By using the MADM, from (2.7), we can write Eq. (1.3) in the form

$$p_k(x)u^{(k)}(x) + \sum_{j=0}^{k-1} p_j(x)u^{(j)}(x) = f_1(x) + f_2(x) + \mu_1 \int_a^x k_1(x,t)G_1(u(t))dt + \mu_2 \int_a^b k_2(x,t)G_2(u(t))dt. \tag{3.1}$$

Since $p_k(x) \neq 0$, then

$$u^{(k)}(x) = \frac{f_1(x)}{p_k(x)} + \frac{f_2(x)}{p_k(x)} + \mu_1 \int\limits_a^x \frac{k_1(x,t)}{p_k(x)} G_1(u(t))dt +$$

$$\mu_2 \int\limits_a^b \frac{k_2(x,t)}{p_k(x)} G_2(u(t))dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x), \tag{3.2}$$

where $f_1(x)/p_k(x)$, $f_2(x)/p_k(x)$, $k_1(x,t)/p_k(x)$, $k_2(x,t)/p_k(x)$ and $p_j(x)/p_k(x)$, $j = 0, 1, \ldots$ $\ldots, k-1$ are functions that have suitable derivatives on an interval $a \leqslant t \leqslant x \leqslant b$.

To obtain the approximate solution of Eq. (1.3), by integrating (k) times from Eq. (3.2) in the interval $[a, x]$ with respect to x we obtain,

$$u(x) = L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \mu_1 L^{-1}\left(\int\limits_a^x \frac{k_1(x,t)}{p_k(x)} G_1(u(t))dt\right) +$$

$$\mu_2 L^{-1}\left(\int\limits_a^b \frac{k_2(x,t)}{p_k(x)} G_2(u(t)dt)\right) - L^{-1}\left(\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x)\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \tag{3.3}$$

where L^{-1} is the multiple integration operator as follows:

$$L^{-1}(\cdot) = \int\limits_a^x \int\limits_a^x \ldots \int\limits_a^x (\cdot)\, dx\, dx \ldots dx, \quad (k \text{ times}).$$

We obtain the term $\sum_{r=0}^{k-1}(x-a)^r b_r/(r!)$ from the initial conditions.

The nonlinear operators $G_1(u)$ and $G_2(u)$ are usually represented by an infinite series of the so-called Adomian polynomials as follows:

$$G_1(u) = \sum_{i=0}^{\infty} A_i, \quad G_2(u) = \sum_{i=0}^{\infty} B_i. \tag{3.4}$$

The polynomials A_i are generated for all kinds of nonlinearity so that A_0 depends only on u_0, A_1 depends on u_0, u_1, and so on. Specific algorithms were set in [10-12] to formulate Adomian polynomials.

$$A_0 = G_1(u_0), \quad A_1 = u_1 G_1'(u_0), \quad A_2 = u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0),$$

$$A_3 = u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), \quad \ldots \tag{3.5}$$

By using a similar manner in (3.5), the polynomials B_i in (3.4) can be written in the form

$$B_0 = G_2(u_0), \quad B_1 = u_1 G_2'(u_0), \quad B_2 = u_2 G_2'(u_0) + \frac{1}{2!} u_1^2 G_2''(u_0),$$

$$B_3 = u_3 G_2'(u_0) + u_1 u_2 G_2''(u_0) + \frac{1}{3!} u_1^3 G_2'''(u_0), \quad \ldots \tag{3.6}$$

By substituting (2.5) and (3.4) in Eq. (3.3) we have

$$\sum_{i=0}^{\infty} u_i(x) = L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \mu_1 \sum_{i=0}^{\infty} L^{-1}\left(\int_a^x \frac{k_1(x,t)}{p_k(x)} A_i(t)\,dt\right) +$$

$$\mu_2 \sum_{i=0}^{\infty} L^{-1}\left(\int_a^b \frac{k_2(x,t)}{p_k(x)} B_i(t)\,dt\right) - \sum_{i=0}^{\infty}\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_i^{(j)}(x)\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \tag{3.7}$$

where the components $u_0, u_1, u_2, \dots$ are usually determined recursively by

$$u_0 = L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r,$$

$$u_1 = L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \mu_1 L^{-1}\left(\int_a^x \frac{k_1(x,t)}{p_k(x)} A_0(t)\,dt\right) +$$

$$\mu_2 L^{-1}\left(\int_a^b \frac{k_2(x,t)}{p_k(x)} B_0(t)\,dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right), \quad \dots$$

$$u_{n+1} = \mu_1 L^{-1}(\int_a^x \frac{k_1(x,t)}{p_k(x)} A_n(t)\,dt\Big) + \mu_2 L^{-1}\left(\int_a^b \frac{k_2(x,t)}{p_k(x)} B_n(t)\,dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right),$$

$$n \geqslant 1. \tag{3.8}$$

Relation (3.8) will enable us to determine the components $u_n(x)$ recursively for $n \geqslant 0$, and as a result, the series solution of $u(x)$ is readily obtained. When the kernel of the integral equation is complicated or the terms of the series $\sum_{i=0}^{\infty} u_i(x)$ are difficult or impossible to calculate analytically, then the Adomian decomposition method needs some modifications.

In Eq. (3.3), if we consider $k_2(x,t)$ is a separable kernel, then we can write

$$k_2(x,t) = g_2(x) \cdot h_2(t). \tag{3.9}$$

By using the following relation mentioned in [15]:

$$\int_a^x \int_a^{x_1} \int_a^{x_2} \dots \int_a^{x_{n-1}} f(x_n)\,dx_n \dots dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\,dt,$$

we can write

$$L^{-1}\left(\int_a^x \frac{k_1(x,t)}{p_k(x)} G_1(u(t))\,dt\right) = \frac{1}{k!} \int_a^x (x-t)^k \frac{k_1(x,t)}{p_k(x)} G_1(u(t))\,dt,$$

$$\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(t)}{p_k(t)}\right) u^{(j)}(t) = \sum_{j=0}^{k-1} \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} \frac{p_j(t)}{p_k(t)} u^{(j)}(t)\,dt. \tag{3.10}$$

By substituting (3.9) and (3.10) in Eq. (3.3) we obtain

$$u(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \mu_2 \int_a^b L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) G_2(u(t))\, dt +$$

$$\frac{\mu_1}{k!} \int_a^x (x-t)^k \frac{k_1(x,t)}{p_k(x)} G_1(u(t))\, dt - \sum_{j=0}^{k-1} \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} \frac{p_j(t)}{p_k(t)} u^{(j)}(t)\, dt. \tag{3.11}$$

In Eq. (3.11), we set

$$L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r = F(x),$$

where $F(x)$ is assumed to be bounded for all t in $J=[a,b]$ and

$$\left|\frac{\mu_1 k_1(x,t)(x-t)^k}{k!\; p_k(x)}\right| \leqslant M', \quad \left|\mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t)\right| \leqslant M'',$$

$$\left|\frac{(x-t)^{k-1} p_j(t)}{p_k(t)\; (k-1)!}\right| \leqslant M_j, \quad j=0,1,\ldots,k-1, \quad \forall a \leqslant t \leqslant x \leqslant b.$$

Also, we assume that the nonlinear terms $G_1(u(t))$, $G_2(u(t))$ and $D^j(u(t)) = d^j(u(t))/dt^j$, ($D^j$ is the derivative operator, $j=0,1,\ldots,k-1$) are Lipschitz continuous with $|G_1(u) - G_1(z)| \leqslant L'|u-z|$, $|G_2(u)-G_2(z)| \leqslant L''|u-z|$, $|D^j(u) - D^j(z)| \leqslant L_j|u-z|$ for $j = 0,1,\ldots,k-1$, and have the following Adomian polynomials representation:

$$G_1(u) = \sum_{i=0}^{\infty} A_i, \quad G_2(u) = \sum_{i=0}^{\infty} B_i, \quad D^j(u) = \sum_{i=0}^{\infty} L_{i_j}, \quad j=0,1,\ldots,k-1.$$

Furthermore, we can write the following formula for the Adomian polynomials [18]:

$$A_n = G_1(s_n) - \sum_{i=0}^{n-1} A_i, \quad B_n = G_2(s_n) - \sum_{i=0}^{n-1} B_i, \quad L_{n_j} = D^j(s_n) - \sum_{i=0}^{n-1} L_{i_j}, \quad j=0,1,\ldots,k-1, \tag{3.12}$$

where $s_n = \sum_{i=0}^{n} u_i(t)$ is the partial sum. Consequently, by applying the ADM to (3.11), the following recursive formula is obtained:

$$u(t) = \sum_{i=0}^{\infty} u_i(t),$$

where

$$u_0(t) = F(x), \quad u_i(t) = \int_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{k!\, p_k(x)} A_{i-1}\, dt +$$

$$\int_a^b \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) B_{i-1}\, dt - \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}\; p_j(t)}{(k-1)!\, p_k(t)} L_{i-1_j}\, dt, \quad i \geqslant 1. \tag{3.13}$$

The following theorems are proved under the above assumptions. Theorems of uniqueness of the solution, convergence and the truncation error of the Adomian method for a class of nonlinear integral equations was proposed in [18]. In order to extend these theorems to nonlinear Volterra — Fredholm integro-differential equations, we prove the following theorems for the solution (2.5) of problem (3.11).

Theorem 3.1. *Problem* (3.11) *has a unique solution whenever* $0 < \alpha < 1$, *where*

$$\alpha = (L^{'}M^{'} + L^{''}M^{''} + kLM)(b-a), \quad M = \max|M_j|, \quad L = \max|L_j|, \quad j = 0, 1, \dots, k-1.$$

Proof. Let u and u^* be two different solutions of (3.11). Then

$$|u - u^*| = \Bigg| \int\limits_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{p_k(x)\, k!}[G_1(u) - G_1(u^*)]\, dt + \int\limits_a^b \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) \times$$

$$[G_2(u) - G_2(u^*)]\, dt - \sum_{j=0}^{k-1} \int\limits_a^x \frac{(x-t)^{k-1}\; p_j(t)}{p_k(t)\; (k-1)!}\; [D^j(u) - D^j(u^*)]\, dt \Bigg| \leqslant$$

$$\int\limits_a^x \left|\frac{\mu_1 k_1(x,t)(x-t)^k}{p_k(x)\; k!}\right| |G_1(u) - G_1(u^*)|\, dt + \int\limits_a^b \left|\mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t)\right| |G_2(u) - G_2(u^*)|\, dt +$$

$$\sum_{j=0}^{k-1} \int\limits_a^x \left|\frac{(x-t)^{k-1}\; p_j(t)}{p_k(t)\; (k-1)!}\right| |D^j(u) - D^j(u^*)|\, dt \leqslant (b-a)(L^{'}M^{'} + L^{''}M^{''} + kLM)|u - u^*|,$$

from which we get $(1-\alpha)|u - u^*| \leqslant 0$. Since $0 < \alpha < 1$, then $|u - u^*| = 0$ implies $u = u^*$ and this completes the proof. $\square$

Theorem 3.2. *The series solution* (2.5) *of problem* (3.11) *using the ADM converges if* $0 < \alpha < 1$ *and* $|u_1(t)| < \infty$.

Proof. Let $(C[J], \|.\|)$ be the Banach space of all continuous functions on J with the norm $\|f(t)\| = \max|f(t)|$ for all t in $J = [a,b]$. We suppose the sequence of partial sums s_n and let s_n and s_m be arbitrary partial sums with $n \geqslant m$. We will prove that s_n is a Cauchy sequence in this Banach space.

$$\|s_n - s_m\| = \max_{\forall t \epsilon J} |s_n - s_m| = \max_{\forall t \epsilon J} \left| \sum_{i=m+1}^{n} u_i(t) \right| =$$

$$\max_{\forall t \epsilon J} \Bigg| \sum_{i=m+1}^{n} \int\limits_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{k!\; p_k(x)} A_{i-1}\, dt + \sum_{i=m+1}^{n} \int\limits_a^b \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) B_{i-1}\, dt -$$

$$\sum_{j=0}^{k-1} \int\limits_a^x \frac{p_j(t)(x-t)^{k-1}}{p_k(t)\; (k-1)!} L_{i-1_j} dt \Bigg| = \max_{\forall t \epsilon J} \Bigg| \int\limits_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{k!\; p_k(x)} \left(\sum_{i=m}^{n-1} A_i\right) dt +$$

$$\int\limits_a^b \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) \left(\sum_{i=m}^{n-1} B_i\right) dt - \sum_{j=0}^{k-1} \int\limits_a^x \frac{p_j(t)(x-t)^{k-1}}{p_k(t)\; (k-1)!} \left(\sum_{i=0}^{n-1} L_{i_j}\right) dt \Bigg|. \qquad (3.14)$$

From (3.12) we have

$$\sum_{i=m}^{n-1} A_i = G_1(s_{n-1} - s_{m-1}), \quad \sum_{i=m}^{n-1} B_i = G_2(s_{n-1} - s_{m-1}), \quad \sum_{i=m}^{n-1} L_{i_j} = D^j(s_{n-1} - s_{m-1}),$$

where $j = 0, 1, 2, \ldots k-1$. So,

$$\|s_n - s_m\| = \max_{\forall t \epsilon J} \left| \int_a^x \frac{\mu_1 k_1(x,t)(x-t)^k}{k!\ p_k(x)} \ [G_1(s_{n-1}) - G_1(s_{m-1})]\, dt + \right.$$

$$\left. \int_a^b \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t)[G_2(s_{n-1} - s_{m-1})]dt - \sum_{j=0}^{k-1} \int_a^x \frac{p_j(t)(x-t)^{k-1}}{p_k(t)(k-1)!}[D^j(s_{n-1} - s_{m-1})]dt \right| \leqslant$$

Thus,

$$\max_{\forall t \epsilon J} \int_a^x \left| \frac{\mu_1 k_1(x,t)(x-t)^k}{k!\ p_k(x)} \right| |G_1(s_{n-1}) - G_1(s_{m-1})| dt + \int_a^b \left| \mu_2 L^{-1}\left(\frac{g_2(x)}{p_k(x)}\right) h_2(t) \right| \times$$

$$|G_2(s_{n-1} - s_{m-1})| dt + \sum_{j=0}^{k-1} \int_a^x \left| \frac{p_j(t)(x-t)^{k-1}}{p_k(t)(k-1)!} \right| |D^j(s_{n-1} - s_{m-1})| dt \leqslant \alpha \|s_{n-1} - s_{m-1}\|.$$

Let $n = m+1$, then

$$\|s_{m+1} - s_m\| \leqslant \alpha\ \|s_m - s_{m-1}\| \leqslant \alpha^2\ \|s_{m-1} - s_{m-2}\| \leqslant \ldots \leqslant \alpha^m\ \|s_1 - s_0\|.$$

By using the triangle inequality, we have

$$\|s_n - s_m\| \leqslant \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\| \leqslant$$

$$[\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}]\|s_1 - s_0\| \leqslant$$

$$\alpha^m\ [1 + \alpha + \alpha^2 + \ldots + \alpha^{n-m-1}]\|s_1 - s_0\| \leqslant \alpha^m \left[\frac{1 - \alpha^{n-m}}{1 - \alpha}\right] \|u_1(t)\|.$$

Since $0 < \alpha < 1$, we have $(1 - \alpha^{n-m}) < 1$, then

$$\|s_n - s_m\| \leqslant \frac{\alpha^m}{1-\alpha} \max_{\forall t \epsilon J} |u_1(t)|. \tag{3.15}$$

Since $F(x)$ is bounded and $|u_1(t)| < \infty$. Therefore, as $m \to \infty$, then $\|s_n - s_m\| \to 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, hence the series is convergent and the proof is completed. □

Theorem 3.3. *The maximum absolute truncation error of the series solution* (2.5) *to problem* (3.11) *is estimated to be*

$$\max_{\forall t \epsilon J} \left| u(t) - \sum_{i=0}^{m} u_i(t) \right| \leqslant \frac{k\ \alpha^m}{1-\alpha}, \tag{3.16}$$

where

$$k = b\,(M' \max_{\forall t \epsilon J} |G_1(u_0)| + M'' \max_{\forall t \epsilon J} |G_2(u_0)| + M_j \max_{\forall t \epsilon J} |D^j(u_0)|), \quad j = 0, 1, \ldots, k-1.$$

Proof. From inequality (3.15), when $n \to \infty$, then $s_n \to u(t)$ and

$$\max_{\forall t \epsilon J} |u_1(t)| \leqslant b\,(M^{'} \max_{\forall t \epsilon J} |G_1(u_0)| + M^{''} \max_{\forall t \epsilon J} |G_2(u_0)| + M_j \max_{\forall t \epsilon J} |D^j(u_0)|), \quad j = 0, 1, \dots, k-1.$$

Therefore,

$$\|u(t) - s_m\| \leqslant \frac{\alpha^m}{1-\alpha} b\,(M^{'} \max_{\forall t \epsilon J} |G_1(u_0)| + M^{''} \max_{\forall t \epsilon J} |G_2(u_0)| + M_j \max_{\forall t \epsilon J} |D^j(u_0)|),$$

$$j = 0, 1, \dots, k-1.$$

Finally, the maximum absolute truncation error in the interval J is obtained by (3.16). $\square$

4. Numerical examples

In this Section, we compute two numerical examples which are solved by the method proposed in this article. The programs have been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm:

Step 1. Set $n \leftarrow 0$.

Step 2. Consider the Adomian polynomials as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[G_1\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\right]_{\lambda=0}, \quad B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[G_2\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\right]_{\lambda=0}.$$

Step 3. Calculate the recursive relation as follows:

$$u_0 = L^{-1}\left(\frac{f_1(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r,$$

$$u_1 = L^{-1}\left(\frac{f_2(x)}{p_k(x)}\right) + \mu_1 L^{-1}\left(\int_a^x \frac{k_1(x,t)}{p_k(x)} A_0(t)dt\right) +$$

$$\mu_2 L^{-1}\left(\int_a^b \frac{g_2(x)h_2(t)}{p_k(x)} B_0(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}\right) u_0^{(j)}(x), \quad \dots,$$

$$u_{n+1} = \mu_1 L^{-1}\left(\int_a^x \frac{k_1(x,t)}{p_k(x)} A_n(t)dt\right) +$$

$$\mu_2 L^{-1}\left(\int_a^b \frac{g_2(x)h_2(t)}{p_k(x)} B_n(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right), \quad n \geqslant 1.$$

Step 4. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 5, else $n \leftarrow n+1$ and go to step 2.

Step 5. Print $u(x) = \sum_{i=0}^{n} u_i$ as the approximation of the exact solution.

Example 4.1. Let us first consider the nonlinear integro-differential equation

$$(x^3-1)u^{(4)}(x)+(x^2+1)u''(x)=e^{-x}(x^2+x^3)-\frac{x^2}{2e^2}-(0.130639)x+\int_0^x[u(t)]^2dt+\int_0^{0.5}xt(1+u(t)^2)dt,$$

with initial conditions $u(0)=1$, $u'(0)=-1$, $u''(0)=1$, $u'''(0)=-1$, $p_0(x)=p_1(x)=p_3(x)=0$, $p_2(x)=(x^2+1)$, $p_4(x)=(x^3-1)$. The exact solution is $u(x)=e^{-x}$. Also, $\alpha=0.88068$ and $\varepsilon=10^{-4}$.

Table 4.1 shows that the modified Adomian decomposition method (MADM) has a more rapid convergence with a smaller number of iterations than the number of terms (N) in Taylor polynomials method (TPM). Comparing the results of Table 4.1, we can observe that the increase in the error in the modified Adomian decomposition method is smaller than the increase in the error in the Taylor polynomials method.

Table 4.1. **Numerical results for Example 4.1, Comparison between (TPM) and (MADM)**

x	Exact solution	Errors (TPM)	Errors (MADM)
0.05	0.951229	$2.65643\times10^{-4}(N=2)$	2.6223×10^{-6} $(n=1)$
0.1	0.904837	$4.33795\times10^{-4}(N=2)$	7.94562×10^{-5} $(n=1)$
0.2	0.818731	0.00007242 $(N=4)$	2.5539×10^{-5} $(n=2)$
0.3	0.740818	0.00038350 $(N=4)$	0.00001492 $(n=2)$
0.4	0.670320	0.00012700 $(N=4)$	0.00005300 $(n=2)$
0.5	0.606531	0.00032650 $(N=4)$	0.00001330 $(n=2)$

Example 4.2. (See [6]). Let us consider the following linear integro-differential equation:

$$u''(x)+xu(x)=f(x)+\int_0^x x^2e^tu(t)\,dt,$$

where

$$p_0(x)=x,\quad p_1(x)=1,\quad \mu_1=1,\quad \mu_2=0,\quad 0\leqslant t\leqslant x\leqslant 0.8,\quad k_1(x,t)=x^2e^t,$$

$$f(x)=-(1+x)\cos(x)-\frac{1}{2}(e^x(\cos(x)+\sin(x))-1)x^2,$$

with the initial conditions $u(0)=1$, $u'(0)=0$. The exact solution is $u(x)=\cos(x)$. Also, $\alpha=0.776533$ and $\varepsilon=10^{-7}$.

Table 4.2 shows that the modified Adomian decomposition method (MADM) in [6] has a more rapid convergence than the Taylor polynomials method (TPM) in different iterations. So, we can observe that the result of the example for the modified Adomian decomposition method can be better than the result of the Taylor polynomials method.

Table 4.2. **Numerical results for Example 4.2, Comparison between (TPM) and (MADM)**

x	Exact solution	Errors (TPM)	Errors (MADM)
0.2	0.98006658	$6.34639\times10^{-7}(N=2)$	4.6178×10^{-8} $(n=2)$
0.4	0.92106099	$1.62251\times10^{-7}(N=3)$	1.4486×10^{-8} $(n=2)$
0.6	0.82533561	4.1491×10^{-7} $(N=4)$	1.3984×10^{-7} $(n=3)$
0.8	0.69670671	$4.13157\times10^{-8}(N=7)$	4.8341×10^{-8} $(n=3)$

5. Conclusions

The Adomian decomposition method isknown as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations, and so on. In this work, we have calculated the approximate solutions of high-order nonlinear Volterra — Fredholm integro-differential equations by using the modified Adomian decomposition method (MADM). The method can be developed to solve the integro-differential equation in the form of

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \mu_1 \int_a^x \sum_{i=0}^{p} k_1(x,t)G_1(u^{(i)})(t)\,dt + \mu_2 \int_a^b \sum_{l=0}^{s} k_2(x,t)G_2(u^{(l)})(t))\,dt,$$

with $G_1(u^{(i)})(t) = \sum_{n=0}^{\infty} A_n$, $G_2(u^{(l)})(t) = \sum_{n=0}^{\infty} B_n$, where A_n and B_n are Adomian polynomials.

We compared the MADM with the TPM and observed that the modified Adomian decomposition method has a more rapid convergence than the Taylor polynomials method.

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