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AN ADAPTIVE SCHEME TO HANDLE THE PHENOMENON OF QUENCHING FOR A LOCALIZED SEMILINEAR HEAT EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract — This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - f(u(1/2,t)), \quad (x,t) \in (0,1) \times (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in [0,1], \end{cases}$$

where $f : (0,\infty) \to (0,\infty)$ is a C^1 convex, nonincreasing function, $\int_0^\alpha \frac{d\sigma}{f(\sigma)} < \infty$ for any positive real α, $\lim\limits_{s\to 0^+} f(s) = \infty$. The initial datum $u_0 \in C^0([0,1])$, $u_0(x) > 0$, $x \in [0,1]$. Under some assumptions, we prove that the solution of a discrete form of the above problem quenches in a finite time and estimate its numerical quenching time. We also show that the numerical quenching time in certain cases converges to the real one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

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1. Introduction

In this paper, we address the following initial-boundary value problem for a semilinear heat equation of the form

$$u_t(x,t) = u_{xx}(x,t) - f(u(1/2,t)), \quad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

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where $f : (0,\infty) \to (0,\infty)$ is a C^1 convex, nonincreasing function, $\int_0^\alpha \frac{d\sigma}{f(\sigma)} < \infty$ for any positive real α, $\lim\limits_{s\to 0^+} f(s) = \infty$. The initial datum $u_0 \in C^0([0,1])$, $u_0(x) > 0$, $x \in [0,1]$,

$$u_0'(x) < 0, \quad x \in (0,1/2), \quad u_0''(x) - f(u_0(1/2)) < 0, \quad x \in (0,1), \tag{1.4}$$

$$u_0'(0) = 0, \quad u_0'(1) = 0. \tag{1.5}$$

The problem describes in (1.1)–(1.3) models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology, and its particularity is that it represents a model in physical phenomena where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological systems and in chemical reaction diffusion processes in which the reaction takes place only at some local sites. For instance, the above model is appropriate to describe:

(i) the influence of defect structures on a catalytic surface;

(ii) the temperature in a solid-fuel combustion scenario where the heat that is input into the system is localized, say as in a laser focused on one spot in the domain;

(iii) chemical reaction-diffusion processes in which, due to the effect of the catalyst, the reaction takes place only at a single site;

(iv) the ignition of a combustible medium with damping, where either a heated wire or a pair of small electrodes supplies a large amount of energy to every confined area.

For more physical motivation see [3, 4, 15]. Here, the interval $(0,T)$ is the maximal time interval on which $u_{\min}(t)$ is positive, where

$$u_{\min}(t) = \min_{0\leqslant x\leqslant 1} u(x,t).$$

The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T} u_{\min}(t) = 0.$$

In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u. In this paper, we are interested in the numerical study of the phenomenon of quenching for localized semilinear heat equations. We start with the construction of an adaptive scheme as follows. Let I be a positive integer, and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leqslant i \leqslant I$, and approximate the solution u of the problem (1.1)–(1.3) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^\top$ of the following discrete equations:

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - f(U_k^{(n)}), \quad 0 \leqslant i \leqslant I, \tag{1.6}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leqslant i \leqslant I, \tag{1.7}$$

where $n \geqslant 0$, k is the integer part of $I/2$,

$$\delta^2 U_i^{(n)} = \begin{cases} (2U_1^{(n)} - 2U_0^{(n)})/h^2, & i = 0, \\ (U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)})/h^2, & 1 \leqslant i \leqslant I-1, \\ (2U_{I-1}^{(n)} - 2U_I^{(n)})/h^2, & i = I, \end{cases}$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \leqslant i \leqslant I,$$

$$\varphi_i = \varphi_{I-i}, \quad 0 \leqslant i \leqslant I, \quad \delta^+\varphi_i \leqslant 0, \quad 0 \leqslant i \leqslant k-1,$$

$$\delta^+\varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\left\{\frac{(1-\tau)h^2}{3}, \frac{\tau U_{h\min}^{(n)}}{f(U_{h\min}^{(n)})}\right\}, \tag{1.8}$$

with $\tau \in (0,1)$, where $U_{h\min}^{(n)} = \min\limits_{0\leqslant i\leqslant I} U_i^{(n)}$.

Let us notice that the restriction on the time step ensures the positivity of the discrete solution.

To facilitate our discussion, we need to define the notion of numerical quenching.

Definition 1.1. We say that the solution $U_h^{(n)}$ of the explicit scheme quenches in a finite time if $\lim\limits_{n\to\infty} U_{h\min}^{(n)} = 0$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution.

The theoretical study of quenching of solutions for localized semilinear heat equations has been the subject of investigations of many authors (see [8, 9], and the references therein). Under the assumptions given in the introduction of the present paper, by standard methods, it is not hard to prove the local in time existence and uniqueness of the solution u of (1.1)–(1.3) (see [2, 8, 9, 16]). In addition, the authors proved that the solution u of (1.1)–(1.3) quenches in a finite time, and its quenching time is estimated (see, [8, 9]). An interesting question about the phenomenon of quenching is the determination of the quenching time. Theoretically, in most of the cases, it is impossible to get the real quenching time. However, numerically, it is possible to obtain good approximations of the real quenching time in certain situations. The objective of the numerical method is to propose an algorithm which permits the determination of an approximation of the real quenching time. In the present paper, we are interested in the numerical study using the discrete form of (1.1)–(1.3) defined in (1.6), (1.7). We give some assumptions under which the solution of the discrete problem quenches in a finite time and estimate its numerical quenching time. We also show that the numerical quenching time converges to the theoretical one when the mesh size goes to zero. Recently, in [13], Nabongo and Boni obtained an analogous result considering a semidiscrete scheme. Previously, some authors used discrete schemes to study the phenomenon of quenching. However, only the case where the reaction term $f(u(1/2,t))$ is replaced by $f(u(x,t))$ was taken into account (see [12]). Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle for localized parabolic problems. In the third section, we prove that the solution of the discrete problem quenches in a finite time and estimate its numerical quenching time. In the fourth section, we give a result about the convergence of numerical quenching times in some cases where the quenching occurs. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the discrete scheme

In this section, we give some lemmas about the discrete maximum principle for localized parabolic problems and reveal certain properties concerning the discrete solution.

The following lemma is a discrete form of the maximum principle for localized parabolic problems.

Lemma 2.1. *Let $a^{(n)}$ and $V_h^{(n)}$ be two sequences such that $a^{(n)}$ is nonnegative, and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_k^{(n)} \geqslant 0, \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0, \tag{2.1}$$

$$V_i^{(0)} \geqslant 0, \quad 0 \leqslant i \leqslant I. \tag{2.2}$$

Then, we have $V_i^{(n)} \geqslant 0$, $0 \leqslant i \leqslant I$, $n > 0$, when $\Delta t_n \leqslant h^2/2$.

Proof. A straightforward computation shows that

$$V_0^{(n+1)} \geqslant 2\frac{\Delta t_n}{h^2} V_1^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_0^{(n)} + \Delta t_n a^{(n)} V_k^{(n)},$$

$$V_i^{(n+1)} \geqslant \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + \Delta t_n a^{(n)} V_k^{(n)}, \quad 1 \leqslant i \leqslant I-1,$$

$$V_I^{(n+1)} \geqslant 2\frac{\Delta t_n}{h^2} V_{I-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_I^{(n)} + \Delta t_n a^{(n)} V_k^{(n)}.$$

If $V_h^{(n)} \geqslant 0$, then using an argument of recursion, we easily see that $V_h^{(n+1)} \geqslant 0$. This completes the proof. ☐

An immediate consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 2.2. *Let $V_h^{(n)}$, $W_h^{(n)}$ and $a^{(n)}$ be three sequences such that $a^{(n)}$ is nonnegative, and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} - a^{(n)} V_k^{(n)} \leqslant \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} - a^{(n)} W_k^{(n)},$$

$$0 \leqslant i \leqslant I, \quad n \geqslant 0,$$

$$V_i^{(0)} \leqslant W_i^{(0)}, \quad 0 \leqslant i \leqslant I.$$

Then, we have $V_i^{(n)} \leqslant W_i^{(n)}$, $0 \leqslant i \leqslant I$, $n > 0$ when $\Delta t_n \leqslant h^2/2$.

The lemma below reveals some properties of the discrete solution.

Lemma 2.3. *The discrete solution $U_h^{(n)}$ of (1.6), (1.7) obeys the following relations:*

$$U_i^{(n)} = U_{I-i}^{(n)}, \quad 0 \leqslant i \leqslant I, \quad \delta^+ U_i^{(n)} \leqslant 0, \quad 0 \leqslant i \leqslant k-1, \quad n \geqslant 0. \tag{2.3}$$

Proof. Introduce the vector $V_h^{(n)}$ defined as follows:

$$V_i^{(n)} = U_i^{(n)} - U_{I-i}^{(n)}, \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0.$$

A routine calculation reveals that

$$V_0^{(n+1)} = 2\frac{\Delta t_n}{h^2} V_1^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_0^{(n)}, \quad n \geqslant 0,$$

$$V_i^{(n+1)} = \frac{\Delta t_n}{h^2} V_{i-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i+1}^{(n)}, \quad 1 \leqslant i \leqslant I-1, \quad n \geqslant 0,$$

$$V_I^{(n+1)} = 2\frac{\Delta t_n}{h^2} V_{I-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_I^{(n)}, \quad n \geqslant 0,$$

$$V_i^{(0)} = 0, \quad 0 \leqslant i \leqslant I.$$

Using an argument of recursion, we easily note that $V_h^{(n)} = 0$, $n > 0$, and the first part of the lemma is proved. In order to prove the second one, we proceed as follows. Set

$$W_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}, \quad 0 \leqslant i \leqslant k-1.$$

We remark that

$$\delta_t W_0^{(n)} = \frac{W_1^{(n)} - 3W_0^{(n)}}{h^2}, \quad n \geqslant 0. \tag{2.4}$$

On the other hand, it is easy to check that $U_{k+1}^{(n)} = U_k^{(n)}$ if I is odd, and $U_{k+1}^{(n)} = U_{k-1}^{(n)}$ if I is even. This implies that

$$\delta^2 W_{k-1}^{(n)} = \begin{cases} (-2W_{k-1}^{(n)} + W_{k-2}^{(n)})/h^2, & \text{if } I \text{ is odd}, \\ (-3W_{k-1}^{(n)} + W_{k-2}^{(n)})/h^2, & \text{if } I \text{ is even}. \end{cases}$$

Obviously

$$\delta_t W_i^{(n)} = \delta^2 W_i^{(n)}, \quad 1 \leqslant i \leqslant k-2, \quad n \geqslant 0. \tag{2.5}$$

Making use of the above relations, we arrive at

$$W_0^{(n+1)} = \frac{\Delta t_n}{h^2} W_1^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2}\right) W_0^{(n)}, \quad n \geqslant 0,$$

$$W_i^{(n+1)} = \frac{\Delta t_n}{h^2} W_{i-1}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) W_i^{(n)} + \frac{\Delta t_n}{h^2} W_{i+1}^{(n)}, \quad 1 \leqslant i \leqslant k-2, \quad n \geqslant 0,$$

$$W_{k-1}^{(n+1)} = \frac{\Delta t_n}{h^2} W_{k-2}^{(n)} + \left(1 - 3\frac{\Delta t_n}{h^2}\right) W_{k-1}^{(n)}, \quad n \geqslant 0, \quad \text{if } I \text{ is even},$$

$$W_{k-1}^{(n+1)} = \frac{\Delta t_n}{h^2} W_{k-2}^{(n)} + \left(1 - 2\frac{\Delta t_n}{h^2}\right) W_{k-1}^{(n)}, \quad n \geqslant 0, \quad \text{if } I \text{ is odd},$$

$$W_i^{(0)} \leqslant 0, \quad 1 \leqslant i \leqslant k-1.$$

We deduce by induction that

$$W_i^{(n)} \leqslant 0, \quad 1 \leqslant i \leqslant k-1, \quad n > 0.$$

This completes the proof. □

The above lemma says that if the initial datum of the discrete solution is symmetric in space, then the discrete solution also obeys this property. In addition, if the initial datum is nonincreasing in space, then the discrete solution also verifies this assertion. These properties imply that the discrete solution attains its minimum at the node x_k.

3. Numerical quenching time

In this section, under some assumptions, we show that the solution of the discrete problem quenches in a finite time and estimate its numerical quenching time.
We need the following lemmas.

Lemma 3.1. *Let a and b be two positive numbers such that $b < 1$. Then the following estimate holds:*

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leqslant \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}.$$

Proof. We observe that

$$\int_0^{\infty} \frac{ab^x dx}{f(ab^x)} = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ab^x dx}{f(ab^x)} \geqslant \sum_{n=0}^{\infty} \int_n^{n+1} \frac{ab^{n+1} dx}{f(ab^{n+1})},$$

because $f(s)$ is nonincreasing for $s > 0$. We deduce that

$$\int_0^{\infty} \frac{ab^x dx}{f(ab^x)} \geqslant \sum_{n=0}^{\infty} \frac{ab^{n+1}}{f(ab^{n+1})} = -\frac{a}{f(a)} + \sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)}.$$

On the other hand, by a change of variables, we see that

$$\int_0^{\infty} \frac{ab^x dx}{f(ab^x)} = -\frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)},$$

which implies that

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leqslant \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}.$$

This completes the proof. □

Lemma 3.2. *Let $a^{(n)}$ be a positive sequence. Then we have*

$$\delta_t f(a^{(n)}) \geqslant f'(a^{(n)}) \delta_t a^{(n)}, \quad n \geqslant 0.$$

Proof. Apply Taylor's expansion to obtain

$$\delta_t f(a^{(n)}) = f'(a^{(n)}) \delta_t a^{(n)} + \frac{(a^{(n+1)} - a^{(n)})^2}{2\Delta t_n} f''(c^{(n)}),$$

where $c^{(n)}$ is an intermediate value between $a^{(n)}$ and $a^{(n+1)}$. Use the fact that $a^{(n)} > 0$ and $f''(s) \geqslant 0$ for $s > 0$ to complete the rest of the proof. □

The statement of our first result on quenching is the following.

Theorem 3.1. *Assume that there exists a constant $A \in (0, 1)$ such that the initial datum at* (1.7) *satisfies*

$$\delta^2 \varphi_i - f(\varphi_{h\,\min}) \leqslant -A f(\varphi_{h\,\min}), \quad 0 \leqslant i \leqslant I. \tag{3.1}$$

Then the solution $U_h^{(n)}$ of (1.6), (1.7) *quenches in a finite time, and its numerical quenching time $T_h^{\Delta t}$ obeys the following estimate:*

$$T_h^{\Delta t} \leqslant \frac{\tau\varphi_{h\min}}{f(\varphi_{h\min})} - \frac{\tau}{\ln(1-\tau')} \int\limits_0^{\varphi_{h\min}} \frac{d\sigma}{f(\sigma)},$$

where

$$\tau' = \min\left\{\frac{(1-\tau)h^2 f(\varphi_{h\min})}{3\varphi_{h\min}}, \tau\right\}.$$

Proof. Introduce the vector $J_h^{(n)}$ defined as follows:

$$J_i^{(n)} = \delta_t U_i^{(n)} + Af(U_k^{(n)}), \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0.$$

A straightforward computation reveals that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}) + A\delta_t f(U_k^{(n)}), \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0.$$

Taking into account (1.6), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = -(1-A)\delta_t f(U_k^{(n)}), \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0. \tag{3.2}$$

The application of Lemma 3.2 renders

$$\delta_t f(U_k^{(n)}) \geqslant f'(U_k^{(n)})\delta_t U_k^{(n)}, \quad n \geqslant 0. \tag{3.3}$$

Using the expression of $J_h^{(n)}$, we see that

$$\delta_t U_k^{(n)} = J_k^{(n)} - Af(U_k^{(n)}), \quad n \geqslant 0. \tag{3.4}$$

We infer from (3.2), (3.3) and (3.4) that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \leqslant -(1-A)f'(U_k^{(n)})J_k^{(n)} + (1-A)Af'(U_k^{(n)})f(U_k^{(n)}), \quad 0 \leqslant i \leqslant I-1, \quad n \geqslant 0,$$

which implies that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \leqslant -(1-A)f'(U_k^{(n)})J_k^{(n)}, \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0, \tag{3.5}$$

because $f(s)$ is nonincreasing for $s > 0$. We know by virtue of the definition of the number k that $U_k^{(0)} = \varphi_{h\min}$, and according to (3.1), we easily see that

$$J_i^{(0)} = \delta^2\varphi_i - f(\varphi_{h\min}) + Af(\varphi_{h\min}) \leqslant 0, \quad 0 \leqslant i \leqslant I.$$

It follows from Lemma 2.1 that $J_h^{(n)} \leqslant 0$, $n \geqslant 0$, or equivalently

$$U_i^{(n+1)} \leqslant U_i^{(n)} - A\Delta t_n f(U_k^{(n)}), \quad 0 \leqslant i \leqslant I, \quad n \geqslant 0. \tag{3.6}$$

Invoking Lemma 2.3, we know that $U_k^{(n)} = U_{h\min}^{(n)}$. Replace i by k in (3.6) to obtain

$$U_k^{(n+1)} \leqslant U_{h\min}^{(n)} - A\Delta t_n f(U_{h\min}^{(n)}), \quad n \geqslant 0,$$

which implies that

$$U_{h\min}^{(n+1)} \leqslant U_{h\min}^{(n)} - A\Delta t_n f(U_{h\min}^{(n)}), \quad n \geqslant 0. \tag{3.7}$$

We note that

$$A\Delta t_n \frac{f(U_{h\min}^{(n)})}{U_{h\min}^{(n)}} = A\min\{\frac{(1-\tau)h^2 f(U_{h\min}^{(n)}}{3U_{h\min}^{(n)}}), \tau\}. \tag{3.8}$$

Thanks to (3.7), we get $U_{h\min}^{(n+1)} \leqslant U_{h\min}^{(n)}$, $n \geqslant 0$, and by induction, we see that $U_{h\min}^{(n)} \leqslant \varphi_{h\min}$, $n \geqslant 0$. We infer from (3.8) that

$$A\Delta t_n \frac{f(U_{h\min}^{(n)})}{U_{h\min}^{(n)}} = A\min\left\{\frac{(1-\tau)h^2 f(\varphi_{h\min})}{3\varphi_{h\min}}, \tau\right\} = \tau^{'}.$$

Consequently, making use of (3.7), we derive the following estimate:

$$U_{h\min}^{(n+1)} \leqslant U_{h\min}^{(n)}(1-\tau^{'}), \quad n \geqslant 0. \tag{3.9}$$

Using an argument of recursion, we obtain

$$U_{h\min}^{(n)} \leqslant U_{h\min}^{(0)}(1-\tau^{'})^n = \varphi_{h\min}(1-\tau^{'})^n, \quad n \geqslant 0. \tag{3.10}$$

This implies that $U_{h\min}^{(n)}$ goes to zero as n approaches infinity. Now, let us estimate the numerical quenching time. The restriction on the time step and (3.10) allow us to write

$$\sum_{n=0}^{\infty} \Delta t_n \leqslant \tau \sum_{n=0}^{\infty} \frac{\varphi_{h\min}(1-\tau^{'})^n}{f(\varphi_{h\min}(1-\tau^{'})^n)}.$$

Invoking Lemma 3.1, the above estimate becomes

$$\sum_{n=0}^{\infty} \Delta t_n \leqslant \frac{\tau\varphi_{h\min}}{f(\varphi_{h\min})} - \frac{\tau}{\ln(1-\tau^{'})} \int_0^{\varphi_{h\min}} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

Remark 3.1. Using (3.9), we deduce by induction that

$$U_{h\min}^{(n)} \leqslant U_{h\min}^{(q)}(1-\tau^{'})^{n-q}, \quad n \geqslant q. \tag{3.11}$$

Thanks to (3.11), the restriction on the time step leads us to

$$T_h^{\Delta t} - t_q = \sum_{n=q}^{\infty} \Delta t_n \leqslant \sum_{n=q}^{\infty} \frac{\tau U_{h\min}^{(q)}(1-\tau^{'})^{n-q}}{f(U_{h\min}^{(q)}(1-\tau^{'})^{n-q})}.$$

It follows from Lemma 3.1 that

$$T_h^{\Delta t} - t_q \leqslant \frac{\tau U_{h\min}^{(q)}}{f(U_{h\min}^{(q)})} - \frac{\tau}{\ln(1-\tau^{'})} \int_0^{U_{h\min}^{(q)}} \frac{d\sigma}{f(\sigma)}.$$

Apply Taylor's expression to obtain $\ln(1-\tau') = -\tau' + o(\tau')$, which implies that

$$-\frac{\tau}{\ln(1-\tau')} = \frac{\tau}{\tau'(1+o(1))}.$$

If we pick $\tau = h^2$, then we note that

$$\frac{\tau'}{\tau} = \min\left\{\frac{(1-h^2)f(\varphi_{h\min})}{3\varphi_{h\min}}, 1\right\}.$$

This implies $-\tau/\ln(1-\tau') = O(1)$ with the choice $\tau = h^2$.

In the sequel, we choose $\tau = h^2$.

4. Convergence of the numerical quenching time

In this section, under some conditions, we show that the discrete solution quenches in a finite time and, its numerical quenching time converges to the real one when the mesh size goes to zero. In order to prove this result, we firstly show that the discrete solution approaches the continuous one on any interval $[0,1]\times[0,T-\tau]$ with $\tau\in(0,T)$ as the parameter h goes to zero. We denote by

$$u_h(t) = (u(x_0,t),\dots,u(x_I,t))^\top.$$

The result on the convergence of the discrete solution to the theoretical one is stated in the following theorem.

Theorem 4.1. *Suppose that the problem* (1.1)–(1.3) *has a solution* $u\in C^{4,2}([0,1]\times[0,T-\tau])$ *with* $\tau\in(0,T)$ *such that* $\min_{t\in[0,T-\tau]} u_{\min}(t) = \rho > 0$. *Assume that the initial datum at* (1.7) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as} \quad h\to 0. \tag{4.1}$$

Then the problem (1.6), (1.7) *admits a unique solution* $U_h^{(n)}$ *for* h *sufficiently small,* $0\leqslant n\leqslant J$, *and the following relation holds:*

$$\sup_{0\leqslant n\leqslant J}\|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad \text{as} \quad h\to 0,$$

where J *is any quantity satisfying the inequality* $\sum_{j=0}^{J-1}\Delta t_j \leqslant T-\tau$ *and* $t_n = \sum_{j=0}^{n-1}\Delta t_j$.

Proof. For each h, the problem (1.6), (1.7) has a solution $U_h^{(n)}$. Let $N\leqslant J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad n<N. \tag{4.2}$$

In view of the condition (4.1), we note that $N\geqslant 1$ when h is small enough. The application of the triangle inequality gives

$$U_{h\min}^{(n)} \geqslant u_{h\min}(t_n) - \|U_h^{(n)} - u_h(t_n)\|_\infty \geqslant \rho - \frac{\rho}{2} = \frac{\rho}{2} \quad \text{for} \quad n<N. \tag{4.3}$$

Use Taylor's expansion to obtain

$$\delta_t u(x_i,t_n) - \delta^2 u(x_i,t_n) + f(u(x_k,t_n)) = -\frac{h^2}{12}u_{xxxx}(\tilde{x}_i,t_n) + \frac{\Delta t_n}{2}u_{tt}(x_i,\tilde{t}_n), \quad 0\leqslant i\leqslant I, \quad n<N.$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem we get

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + f'(\xi_k^{(n)}) e_k^{(n)} = \frac{h^2}{12} u_{xxxx}(\tilde{x}_i, t_n) - \frac{\Delta t_n}{2} u_{tt}(x_i, \tilde{t}_n), \quad 0 \leqslant i \leqslant I, \quad n < N,$$

where $\xi_k^{(n)}$ is an intermediate value between $u(x_k, t_n)$ and $U_k^{(n)}$. Since $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, then there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + f'(\xi_k^{(n)}) e_k^{(n)} \leqslant M h^2, \quad 0 \leqslant i \leqslant I, \quad n < N. \tag{4.4}$$

Set $L = -f'(\rho/2)$ and introduce the vector $V_h^{(n)}$ defined as follows:

$$V_i^{(n)} = e^{(L+1)t_n} (\|\varphi_h - u_h(0)\|_\infty + M h^2), \quad 0 \leqslant i \leqslant I, \quad n < N.$$

A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + f'(\xi_k^{(n)}) V_k^{(n)} > M h^2, \quad 0 \leqslant i \leqslant I, \quad n < N, \tag{4.5}$$

$$V_i^{(0)} > e_i^{(0)}, \quad 0 \leqslant i \leqslant I. \tag{4.6}$$

It follows from Lemma 2.2 that $V_h^{(n)} \geqslant e_h^{(n)}$. In the same way, we also prove that $V_h^{(n)} \geqslant -e_h^{(n)}$, which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \leqslant e^{(L+1)t_n} (\|\varphi_h - u_h(0)\|_\infty + M h^2), \quad n < N. \tag{4.7}$$

Let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (32) and use (4.2), we find that

$$\frac{\rho}{2} \leqslant \|U_h^{(N)} - u_h(t_N)\|_\infty \leqslant e^{(L+1)T} (\|\varphi_h - u_h(0)\|_\infty + M h^2).$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $\frac{\rho}{2} \leqslant 0$, which is a contradiction and the proof is complete. □

Now we are in a position to prove the main result of this section.

Theorem 4.2. *Suppose that the problem* (1.1)–(1.3) *has a solution* u *which quenches in a finite time* T *such that* $u \in C^{4,2}([0,1] \times [0,T))$. *Assume that the initial datum at* (1.7) *satisfies the condition* (4.1).

Under the assumption of Theorem 3.1, *the problem* (1.6), (1.7) *admits a unique solution* $U_h^{(n)}$ *which quenches in a finite time* $T_h^{\Delta t}$, *and the following relation holds:*

$$\lim_{h \to 0} T_h^{\Delta t} = T.$$

Proof. We know from Remark 3.1 that $-\tau/\ln(1-\tau')$ is bounded. Letting $0 < \varepsilon < T/2$, there exists a positive constant R such that

$$\frac{\tau R}{f(R)} - \frac{\tau}{\ln(1-\tau')} \int_0^R \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{4.8}$$

Since u quenches at the time T, then we observe that there exist $T_0 \in (T - \varepsilon/2, T)$ and $h_0(\varepsilon) > 0$ such that $0 < u_{h\,\min}(t) < R/2$ for $t \in [T_0, T)$, $h \leqslant h_0(\varepsilon)$. Let q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_0, T)$. Invoking Theorem 4.1, we see that the problem (1.6), (1.7) has a unique solution $U_h^{(n)}$ which obeys $\|U_h^{(n)} - u_h(t_n)\|_\infty < R/2$ for $n \leqslant q$, $h \leqslant h_0(\varepsilon)$. This implies that

$$U_{h\min}^{(q)} \leqslant u_{h\min}(t_q) + \|U_h^{(q)} - u_h(t_q)\|_\infty \leqslant \frac{R}{2} + \frac{R}{2} = R, \quad h \leqslant h_0(\varepsilon). \tag{4.9}$$

The application of Theorem 3.1 shows that $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark 3.1 and (4.8) that

$$|T_h^{\Delta t} - t_q| \leqslant \frac{\tau U_{h\min}^{(q)}}{f(U_{h\min}^{(q)})} - \frac{\tau}{\ln(1-\tau')} \int_0^{U_{h\min}^{(q)}} \frac{d\sigma}{f(\sigma)} \leqslant \frac{\varepsilon}{2}, \tag{4.10}$$

because $U_{h\min}^{(q)} \leqslant R$ for $h \leqslant h_0(\varepsilon)$. We deduce that for $h \leqslant h_0(\varepsilon)$,

$$|T - T_h^{\Delta t}| \leqslant |T - t_q| + |t_q - T_h^{\Delta t}| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. □

5. Numerical results

In this section, we give some computational experiments to illustrate our analysis. Firstly, we take the explicit scheme defined in (1.6), (1.7). Secondly, we use the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - \frac{f(U_k^{(n)})U_i^{(n+1)}}{U_i^{(n)}}, \quad 0 \leqslant i \leqslant I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leqslant i \leqslant I,$$

where $n \geqslant 0$. In both cases, we choose $\varphi_i = [2 + \varepsilon(\sin(i\pi h))^2]/4$ with $\varepsilon \in (0,1)$. As in the case of the explicit scheme, here we pick

$$\Delta t_n = \frac{\tau U_{h\min}^{(n)}}{f(U_{h\min}^{(n)})}.$$

Let us notice that for the above implicit scheme, the existence and positivity of the discrete solution are also guaranteed using standard methods (see [1]).

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$|t_{n+1} - t_n| \leqslant 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

The results of numerical experiments for $f(U_k^{(n)}) = (U_k^{(n)})^{-1}$ are given in the Tables 5.1–5.6.

Remark 5.1. If we consider the problem (1.1)–(1.3) in the case where $u_0(x) = 2/4$, then it is not hard to see that the quenching time of the solution u is the same as the one of the solution $\alpha(t)$ of the following differential equation $\alpha'(t) = \alpha^{-1}(t)$, $t > 0$, $\alpha(0) = 2/4$. A routine computation reveals that the value of the quenching time of $\alpha(t)$ is equal to 0.125. We observe from Tables 5.1–5.6 that when ε diminishes, then the numerical quenching time goes to 0.125.

Table 5.1. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method, $\varepsilon = 1$**

I	t_n	n	CPU time	s
16	0.200341	100853	61	—
32	0.200303	216090	185	—
64	0.200294	538064	852	2.08
128	0.200292	1280592	5995	2.17

Table 5.2. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method, $\varepsilon = 1$**

I	t_n	n	CPU time	s
16	0.201264	119774	101	—
32	0.200537	225439	383	—
64	0.200356	561343	1765	2.01
128	0.200311	1335996	12420	2.01

Table 5.3. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method, $\varepsilon = 1/100$**

I	t_n	n	CPU time	s
16	0.125939	22501	13.6	—
32	0.125758	52704	44	—
64	0.125713	131511	203	2.01
128	0.125702	315627	1336	2.03

Table 5.4. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method, $\varepsilon = 1/100$**

I	t_n	n	CPU time	s
16	0.126431	23638	21.8	—
32	0.125882	53293	88	—
64	0.125744	131814	846	1.99
128	0.125709	316354	17356	1.98

Table 5.5. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method, $\varepsilon = 1/10000$**

I	t_n	n	CPU time	s
16	0.125251	3920	3.4	—
32	0.125068	14618	12.1	—
64	0.125022	54865	207	1.99
128	0.125011	203000	11921	2.06

Table 5.6. **Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method, $\varepsilon = 1/10000$**

I	t_n	n	CPU time	s
16	0.125739	3943	4.2	—
32	0.125190	14636	24.3	—
64	0.125052	54881	344	1.99
128	0.125018	203059	15492	2.02

In the following, we also give some plots to illustrate our analysis. In Figs. 5.1–5.6, we can appreciate that the discrete solution quenches globally. This implies that in this case the numerical quenching set is the whole domain $[0, 1]$. It is worth noting that the fact that the discrete solution quenches globally in a finite time can be remarked if one plots the approximation of $u_{\min}(t)$ against t and the approximation of $\|u(\cdot, t)\|_\infty$ against t. For instance, in the case where $\varepsilon = 1/100$, we observe from Figs. 5.7–5.10 that the trajectories of the above curves are the same. In the case where $\varepsilon = 1/100$, it is also important to notice that the trajectories of the approximation of $u_{\min}(t)$ against t, the approximation of

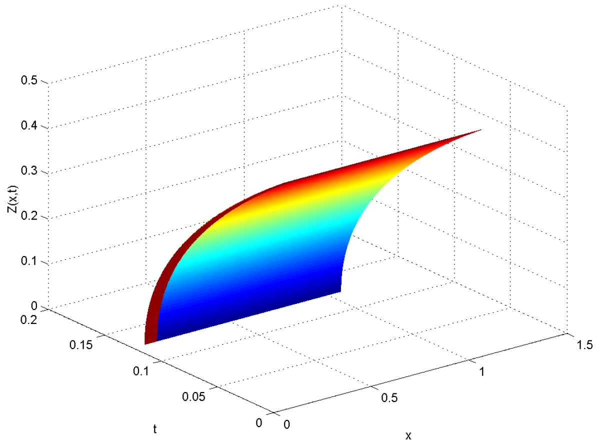

Fig. 5.1. Evolution of the discrete solution with the Euler explicit scheme: $I = 16$, $\varepsilon = 0$

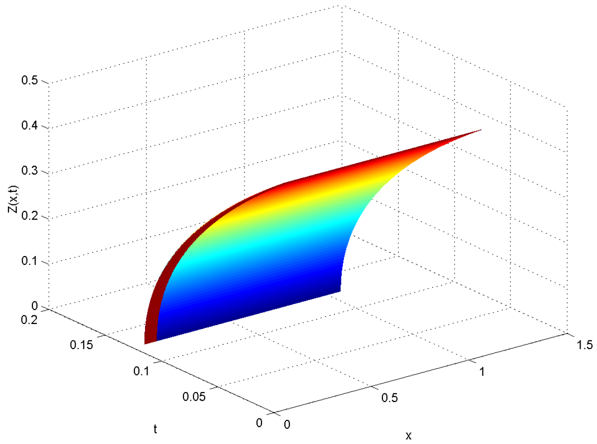

Fig. 5.2. Evolution of the discrete solution with the Euler implicit scheme: $I = 16$, $\varepsilon = 0$

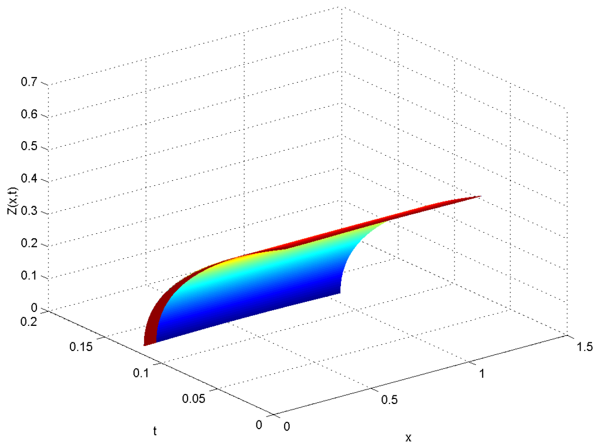

Fig. 5.3. Evolution of the discrete solution with the Euler explicit scheme: $I = 16$, $\varepsilon = 1/100$

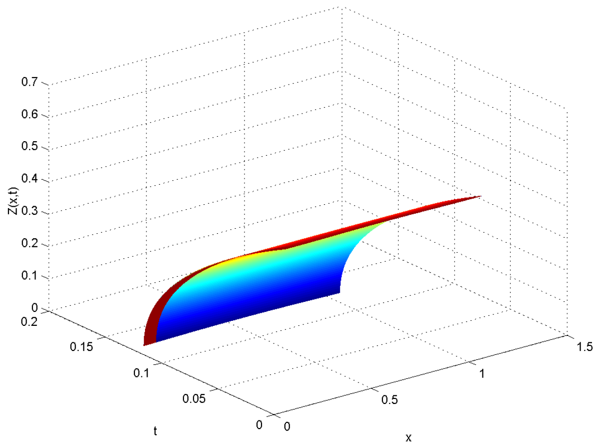

Fig. 5.4. Evolution of the discrete solution with the Euler implicit scheme: $I = 16$, $\varepsilon = 1/100$

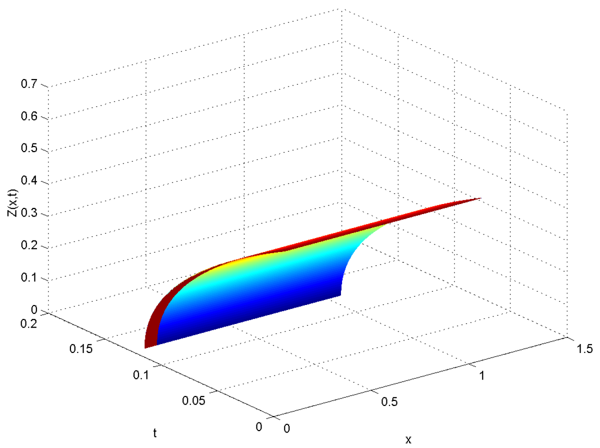

Fig. 5.5. Evolution of the discrete solution with the Euler explicit scheme: $I = 16$, $\varepsilon = 1/10000$

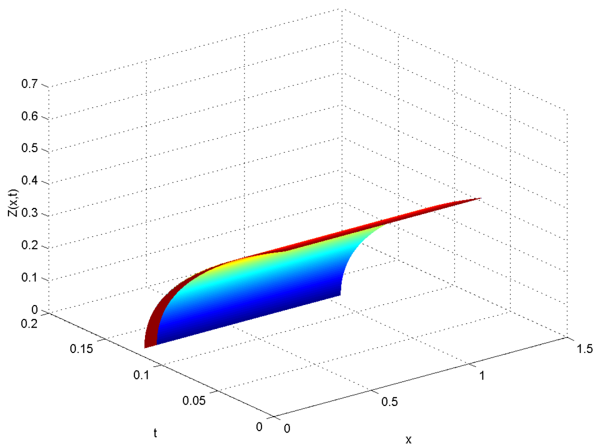

Fig. 5.6. Evolution of the discrete solution with the Euler implicit scheme: $I = 16$, $\varepsilon = 1/10000$

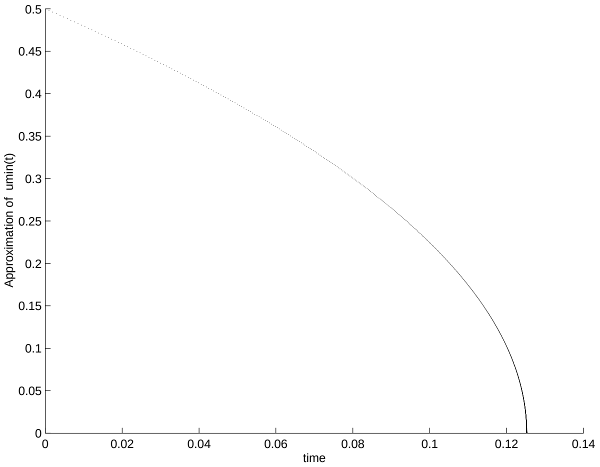

Fig. 5.7. Approximation of $u_{\min}(t)$ against t with the explicit scheme: $I = 16$, $\varepsilon = 1/100$

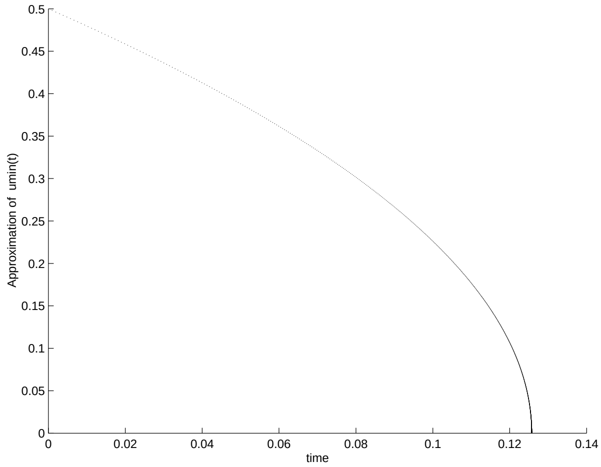

Fig. 5.8. Approximation of $u_{\min}(t)$ against t with the implicit scheme: $I = 16$, $\varepsilon = 1/100$

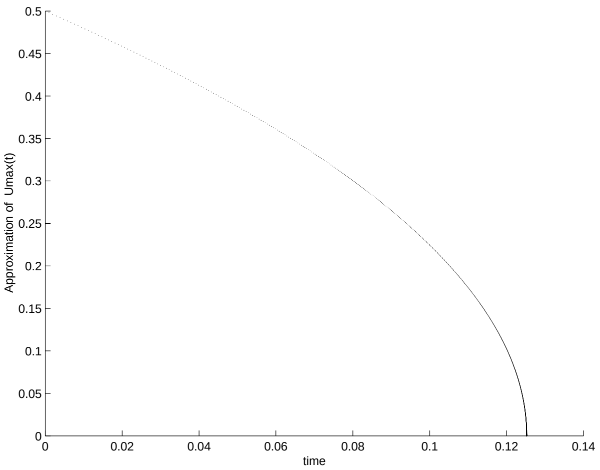

Fig. 5.9. Approximation of $\|u(\cdot, t)\|_\infty$ against t with the implicit scheme: $I = 16$, $\varepsilon = 1/100$

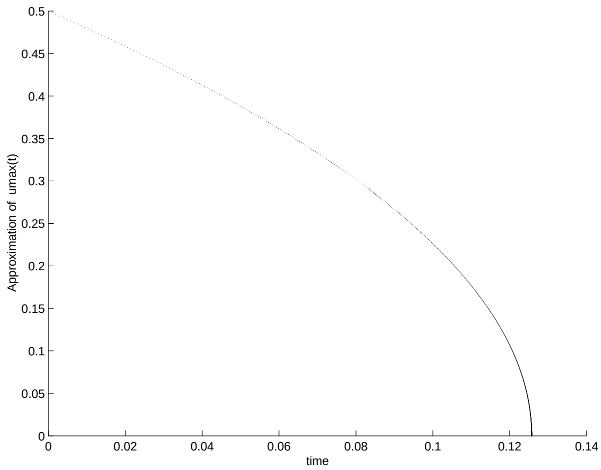

Fig. 5.10. Approximation of $\|u(\cdot, t)\|_\infty$ against t with the implicit scheme: $I = 16$, $\varepsilon = 1/100$

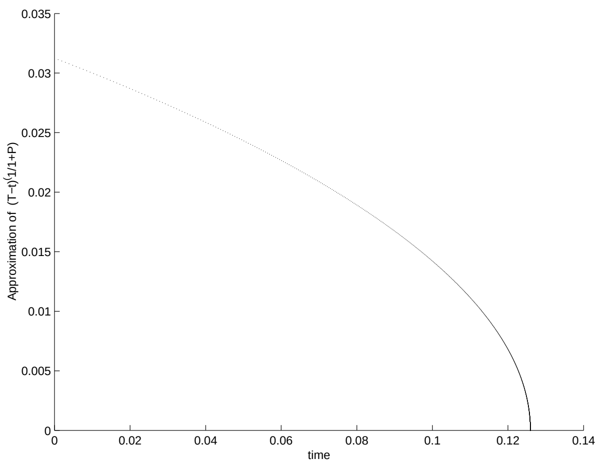

Fig. 5.11. $(T - t)^{1/2}$ against t with $T = 0.125$

$\|u(\cdot,t)\|_\infty$ against t and $(T-t)^{1/2}$ against t are practically the same, here $T = 0.125$ (see Fig. 5.11). This allows us to conclude that $U_i^{(n)} \sim (T-t)^{1/2}$, $0 \leqslant i \leqslant I$, which gives us some information about the quenching rate. Let us remark that, with our hypotheses, it is well known that the theoretical solution quenches globally in a finite time, and the quenching is complete (see [10, 16, 17]).

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