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NUMERICAL SOLUTION OF A NONLOCAL PROBLEM MODELLING OHMIC HEATING OF FOODS

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Abstract — An upwind and a Lax-Wendroff scheme are introduced for the solution of a one-dimensional non-local problem modelling ohmic heating of foods. The schemes are studied regarding their consistency, stability, and the rate of convergence for the cases that the problem attains a global solution in time. A high resolution scheme is also introduced and it is shown that it is total-variation-stable. Finally some numerical experiments are presented in support of the theoretical results.

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1. Introduction

We consider the nonlocal initial boundary value problem

$$u_t(x,t) + u_x(x,t) = \lambda f(u(x,t)) \left(\int_0^1 f(u(x,t))\, dx \right)^{-2}, \quad 0 < x < 1, \quad t > 0, \tag{1.1a}$$

$$u(0,t) = 0, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x) \geqslant 0, \quad 0 < x < 1, \tag{1.1c}$$

where $\lambda > 0$. The function $u(x,t)$ represents the dimensionless temperature when an electric current flows through a conductor (e.g., food) with a temperature dependent on electrical resistivity $f(u) > 0$, subject to a fixed potential difference $V > 0$. The (dimensionless) resistivity $f(u)$ may be either an increasing or a decreasing function of temperature depending strongly on the type of the material (food). Problem (1.1) models one of the main methods for sterilizing foods. Sterilization can occur by fast electrically heating of a food. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in ohmic heating of the food which quickly gets hot. This procedure can be modelled by problem (1.1). A detailed derivation of the model, (1.1), can be found in [15].

The problem was considered initially in [19] where the stability of models allowing for different types of flow was studied. More information on this type of the process can be found in [4, 5, 9, 20, 22, 24]. In [15], problem (1.1) was also studied and it was found that for f decreasing with $\int_0^\infty f(s)ds < \infty$, a blow-up occurs if the parameter $\lambda\,(\propto V^2)$ is too

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large for the steady state to exist or if the initial condition is too big. If f is increasing with $\int_0^\infty ds/f(s) < \infty$, then a blow-up is also possible. If f is increasing with $\int_0^\infty ds/f(s) = \infty$ or decreasing with $\int_0^\infty f(s)ds = \infty$, then the solution is global in time ([15]).

In the following, we assume f to satisfy

$$f(s) > 0, \quad f'(s) < 0, \quad s \geqslant 0, \tag{1.2a}$$

$$\int\limits_0^\infty f(s)\, ds < \infty \tag{1.2b}$$

, for instance, either $f(s) = e^{-s}$ or $f(s) = (1+s)^{-p}$, $p > 1$, satisfy (1.2). In addition, for the initial data it is required that $u_0(x)$, $u_0'(x)$ be bounded and $u_0(x) \geqslant 0$ in $[0,1]$ (the last requirement is a consequence of the fact that for any initial data the solution u becomes nonnegative over $(0,1]$ for some time t and thus, with an appropriate redefinition of t, we can always make this assumption [13, 15]).

The steady problem corresponding to (1.1) is

$$w' = \lambda \frac{f(w)}{\left(\int_0^1 f(w)\, dx\right)^2}, \quad 0 < x < 1, \quad w(0) = 0, \tag{1.3}$$

where $w = w(x) = w(x;\lambda)$, (see [6, 8, 13–15]). For example, in the case that $f(s) = e^{-s}$, $w(x) = \ln(\lambda x/\mu^2 + 1)$ for $\mu > 0$ being the root of the equation $\sqrt{\mu}\ln(\lambda/\mu^2 + 1) = \lambda$.

Under assumptions (1.2), problem (1.3) has at least one classical (regular) stable solution ([15, 17]) $w^* = w(x;\lambda^*)$, (more than one w^* may exist). In the following, we assume that w^* is unique and that the pair $(\underline{w}, \overline{w})$ at $\lambda < \lambda^*$ (λ close to λ^*) has the property: $\underline{w} = w_1$ is stable while $\overline{w} = w_2$ is unstable (since without loss of generality it may be required that there exists at least one w^* at λ^* and that $\underline{w}(x) < \overline{w}(x)$ for x in $(0,1]$ where $\overline{w}$ is the next steady solution greater than $\underline{w}(x)$ at $\lambda < \lambda^*$).

It is known that if (1.2b) holds, then there exists a critical value of the parameter λ, which can be identified, amongst other things, with the square of the applied potential difference V (actually λ is proportional to V^2), say $\lambda^* < \infty$, such that for $\lambda > \lambda^*$ the solution $u(x,t;\lambda)$ to problem (1.1) blows up globally in finite time t^* ($u \to \infty$ for all $x \in (0,1]$ as $t \to t^*-$, [15]) and problem (1.3) has no solutions (of any kind). For a fixed $\lambda \in (0,\lambda^*)$ there exist at least two solutions $w(x;\lambda)$ and a unique solution $u(x,t;\lambda)$; $u(x,t;\lambda)$ may either exist for all times or blow up globally depending on the initial data (for the blow-up, u_0 must be greater than the greatest stable solution $w(x;\lambda)$ and (1.2) holds) [13–15].

A numerical computation of problem (1.1) by using the upwind scheme has already been presented in [17]. Although it has not yet appeared in the literature, a theoretical analysis of finite difference schemes that can be applied for the numerical solution of non-local problems having a form similar to (1.1). In the present work we first study two explicit finite difference schemes: the upwind and Lax — Wendroff scheme, regarding their consistency, stability, and convergence. Initially the upwind scheme is applied to problem (1.1) and it is shown to be first-order convergent if we apply an appropriate discretization to the nonlocal source term. As a next step, we apply the Lax — Wendroff scheme in order to get a more accurate numerical approximation. This scheme is of second order accuracy. This accuracy is obtained by the addition of extra correction terms in the discretization of the nonlocal source term. The analysis for both methods holds in the case that f is a decreasing function, if (1.2b) holds for $\lambda < \lambda^*$, and for small initial data, or if $\int_0^\infty f(s)\,ds = \infty$, so that the solution to problem (1.1) exists for all times. In addition, it should be noted that the results presented in this work are valid, with minor modifications in the proofs, if we also consider an increasing f. Moreover, a high-resolution method combining the above methods is presented and it is shown that it is total-variation-stable.

In studying the numerical solution of such a nonlocal problem, it is interesting to investigate the effect of the non-local term in the numerical approximation of the solution. The present analysis for these two methods indicates that other standard methods applied for first-order homogeneous hyperbolic equations should result in the same order of convergence to this nonlocal problem. This can be done if the nonlocal source term is discretized appropriately, if necessary, with the addition of extra correction terms, and if the integration method used to approximate the nonlocal term is of the same order. Moreover, as is obvious from the application of the Lax-Wendroff method to this problem, additional correction terms associated with the derivatives of the nonlocal source term, must be included in the scheme in order to obtain a second-order accuracy in both space and time. Similar analysis for other nonlocal problems exists in (e.g, [1,7]). More specifically, in [1] an approach based on the method of characteristics is given. In principle, it can also be used for solving problem (1.1). For $s = x + t$, Eq. (1.1b) becomes

$$\frac{du(s)}{ds} = \lambda f(u(s)) \Big/ \Big[\int_t^{1+t} f(u(s))ds \Big]^2 .$$

Then this equation can be integrated along the characteristics resulting, e.g., in a numerical scheme of the form

$$U_{j+1}^{n+1} = U_j^n + h\lambda f(U_j^n)/I_h^2(U^n),$$

where h is a time step. The analysis in [1] indicates that in our case, due to the fact that the nonlocal term $\int_0^1 f(u)\,dx$ is a function of time, extra care should be needed to obtain and analyze a method of higher-order accuracy by the method of characteristics. In addition, the upwind and the Lax — Wendroff methods can be generalized in a more natural way to problems of the form $u_t + (g(u))_x = \lambda F(u)$. Thus, in the present work we will not consider further this method.

Note also that error estimates for a method approximating the solution of the problem in the case of blow-up are important for investigating the characteristics of the blow-up phenomenon, e.g., the blow-up time, useful from the point of view of applications. Similar investigations the convergence of the numerical solution during blow-up were carried out for a parabolic problem in [2,3,11]. In our case, for example, for f being an increasing function a discontinuity in the initial data may cause a blow-up of the solution [15]. In using a similar approach as in [2,3] for a non-local problem such as problem (1.1), a high-resolution method based on a combination of the upwind and the Lax-Wendroff method, which is also introduced and analyzed in this work, would be useful (see [16], Chapter 6).

In Section 2, we present appropriate notations and definitions and consider the upwind method regarding its consistency, stability, and convergence. In Section 3, we obtain similar results for the Lax — Wendroff method and introduce a high-resolution method which is shown to be TV-stable. Finally, in Section 4 we present some numerical experiments in support of the results obtained in the previous Sections, and in Section 5 we present the conclusions and some open problems for future studies.

2. The Upwind Scheme

2.1. Notations. We introduce a spatial grid $x_j = j\Delta x$, $j = 0, 1, \ldots, J$, where $\Delta x = 1/J$ is the mesh size and J is a positive integer. We also consider a fixed time interval $0 \leqslant t \leqslant T$. The step length in time is denoted by Δt and $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$ (with $N = [T/\Delta t]$) are discrete time levels. Also $r = \frac{\Delta t}{\Delta x}$.

We consider the set $H = \{\Delta x > 0 : \Delta x = 1/J, J \in \mathbb{N}\}$ and for $\Delta x \in H$ we define the vector spaces : $X = Y = (\mathbb{R}^{J+1})^{N+1}$. Also, if $V = (V_0, V_1, \ldots, V_J) \in \mathbb{R}^{J+1}$ we define $\|V\|_\infty := \max\limits_{0 \leqslant j \leqslant J} |V_j|$ and $\|V\|_1 := \sum_{j=0}^{J''} \Delta x |V_j|$, where the $''$ means that the first and last terms of the sum are halved, i.e., the trapezoidal rule is used. For $V = (V^0, V^1, \ldots, V^N) \in X$, with $V^n \in \mathbb{R}^{J+1}$ we define the following norm on X, $\|V\|_X := \max\limits_{0 \leqslant n \leqslant N} \|V^n\|_1$. In addition, if $V \in Y$, then we define the norm $\|V\|_Y := \|V^0\|_1 + \sum_{n=1}^{N} \Delta t \|V^n\|_1$. Let R be a fixed positive constant and let us denote by $B(u_h, R)$ an open ball with center u_h and radius R of the space X endowed with the norm of X as defined above.

For a time step Δt and $\Delta x \in H$ we consider the element $u_h \in X$, $u_h = (u^0, u^1, \ldots, u^N) \in X$, with $u^n = (u_0^n, u_1^n, \ldots, u_J^n) \in \mathbb{R}^{J+1}$ and $u_j^n = u(x_j, t_n)$, where u is the exact solution of problem (1.1). In a similar way we denote by $U_h \in X$ the approximate numerical solution of problem (1.1), with U_j^n being the approximation of the solution at the point (x_j, t_n).

We also use the notations $I(u^n) = \int_0^1 u(x, t_n)\, dx$ and $F(u_j^n) = f(u_j^n)/I^2(u^n)$. By I_h we denote the numerical approximation of I, i.e., $I_h(u^n) = \sum_{j=0}^{J''} \Delta x u_j^n$. In this case, we have

$$\left| I_h(u^n) - I(u^n) \right| = O(\Delta x^2).$$

Finally, we set $F_h(u_j^n) = f(u_j^n)/I_h^2(u^n)$.

Note that C, c, c_i, M_i, $i = 1, 2, \ldots$ will denote positive constants independent of $\Delta x, \Delta t, n$ $(0 \leqslant n \leqslant N)$ and j, $(0 \leqslant j \leqslant J)$ having possibly different values at different places.

2.2. Formulation and analysis of the numerical method. The upwind scheme applied to problem (1.1) gives

$$U_0^{n+1} = 0, \tag{2.1a}$$

$$U_j^{n+1} = U_j^n - r\left(U_j^n - U_{j-1}^n\right) + \lambda \Delta t F_h(U_j^n), \quad j = 1, \ldots, J, \tag{2.1b}$$

for $0 \leqslant n \leqslant N-1$ and with $U^0 = (U_0^0, U_1^0, \ldots, U_J^0)$ known.

Next we introduce the mapping $\phi_h : B(u_h, R) \subset X \to Y$ defined by the equations

$$\phi_h(V^0, V^1, \ldots, V^N) = (Z^0, Z^1, \ldots, Z^N), \quad Z^0 = V^0 - U^0, \quad Z_0^{n+1} = 0,$$

$$Z_j^{n+1} = \frac{1}{\Delta t}\left(V_j^n - \frac{\Delta t}{\Delta x}(V_j^n - V_{j-1}^n) + \Delta t \lambda \frac{f(V_j^n)}{I_h^2(V^n)} - V_j^{n+1}\right), \quad 1 \leqslant j \leqslant J,$$

for $0 \leqslant n \leqslant N-1$. Then $U_h = (U^0, U^1, \ldots, U^N)$ is a solution of scheme (2.1) if and only if $\phi_h(U_h) = (R^0, R^1, \ldots, R^N)$, with $R^n = \mathbf{0} \in \mathbb{R}^{J+1}$, $0 \leqslant n \leqslant N$.

In the following, we study the consistency, the stability, and the convergence of scheme (2.1).

2.3. Consistency. We define the local descretization error as $l_h = \phi_h(u_h) \in Y$ and we say that the descritization is consistent if, as Δx, $\Delta t \to 0$, $\lim\limits_{\Delta x,\, \Delta t \to 0} \|\phi_h(u_h)\|_Y = \lim\limits_{\Delta x,\, \Delta t \to 0} \|l_h\|_Y = 0$.

Proposition 2.1. *Assuming that f satisfies condition* (1.2*a*) *and u is a C^2 global bounded solution of problem* (1.1) *(i.e., the initial data are smooth enough and $\lambda \leqslant \lambda^*$, $u_0(x) < w_2(x)$ if (1.2b) holds or $\int_0^\infty ds/f(s) = \infty$), then, if for $u_0(x_j) = u_j^0$, $j = 0, 1, \ldots, J$, we have $\|u^0 - U^0\|_1 = o(1)$ and the local discretization error satisfies the condition*

$$\|\phi_h(u_h)\|_Y = O(\Delta t + \Delta x).$$

Proof. We set $\phi_h(u_h) = (u^0 - U^0, \tau^1, \tau^2, \ldots, \tau^N)$, where τ^n, $1 \leqslant n \leqslant N$, are the local truncation errors to be bounded. Indeed, for $j = 1, \ldots, J$ we have

$$|\tau_j^{n+1}| = \frac{1}{\Delta t}\left| u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) + \Delta t\, \lambda F_h(u_j^n) - u_j^{n+1}\right| =$$

$$\frac{1}{\Delta t}\left| u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n - u_j^n + \Delta x {u_x}_j^n - \frac{1}{2}\Delta x^2 {u_{xx}}_j^n + \cdots\right) + \Delta t \lambda F_h(u_j^n) - u_j^n - \Delta t {u_t}_j^n - \frac{1}{2}\Delta t^2 {u_{tt}}_j^n + \cdots \right| =$$

$$\left| - {u_x}_j^n - {u_t}_j^n + \lambda F_h(u_j^n) - \frac{1}{2}\Delta t\, {u_{tt}}_j^n - \frac{1}{2}\Delta x\, {u_{xx}}_j^n \right| + \cdots$$

or

$$|\tau_j^{n+1}| \leqslant \frac{1}{2}\Delta t |{u_{tt}}_j^n| + \frac{1}{2}\Delta x |{u_{xx}}_j^n| + \lambda f(u_j^n) \left| \frac{1}{I_h^2(u^n)} - \frac{1}{I^2(u^n)} \right|. \tag{2.2}$$

Regarding the third term in Eq. (2.2), we have that

$$\lambda f(u_j^n)\left| \frac{1}{I_h^2(u^n)} - \frac{1}{I^2(u^n)} \right| = \lambda f(u_j^n) \frac{[I_h(u^n) + I(u^n)]}{I_h^2(u^n)\, I^2(u^n)} \big| I_h(u^n) - I(u^n) \big|. \tag{2.3}$$

Since u is bounded in $[0,1] \times [0,T]$, there exists a constant M_u such that $u(x,t_n) \leqslant M_u$, $0 \leqslant n \leqslant N$. Therefore $f(u^n) \geqslant f(M_u) > 0$ and

$$\frac{1}{I_h^2(u^n)} = \left[\sum_{j=0}^{J''} \Delta x f(u_j^n) \right]^{-2} \leqslant \frac{1}{f(M_u)^2}.$$

Similarly

$$\frac{1}{I^2(u^n)} = \left[\int_0^1 f(u^n)\, dx \right]^{-2} \leqslant \frac{1}{f(M_u)^2}.$$

Also, for f decreasing and $f(0) > 0$ we have, $f(u_j^n) \leqslant f(0)$, $I_h(u^n) = \sum_{j=1}^{J''} \Delta x f(u_j^n) \leqslant f(0)$ and $I(u^n) = \int_0^1 f(u^n) dx \leqslant f(0)$. Therefore, due to the fact that $\big|I_h(u^n) - I(u^n)\big| = O(\Delta x^2)$ we deduce that

$$\lambda f(u_j^n) \left| \frac{1}{I_h^2(u^n)} - \frac{1}{I(u^n)^2} \right| \leqslant \frac{2\lambda f(0)^2}{f(M_u)^4} \big| I_h(u^n) - I(u^n) \big| = O(\Delta x^2). \tag{2.4}$$

We have $\tau_0^n = 0$, for $1 \leqslant n \leqslant N$. Therefore, combining also Eqs. (2.2) and (2.4), we have

$$|\tau_j^n| = O(\Delta t + \Delta x), \quad 0 \leqslant j \leqslant J, \quad 1 \leqslant n \leqslant N,$$

and hence, using the assumption that $\|u^0 - U^0\|_1 = o(1)$, we deduce that $\lim\limits_{\Delta x,\, \Delta t \to 0} \|\phi_h(u_h)\|_Y = \lim\limits_{\Delta x,\, \Delta t \to 0} \|l_h\|_Y = 0$ and that the scheme is consistent. $\square$

Remark 2.1. The previous proposition can easily be modified to include also the case that f is an increasing function. Indeed, if $f(s) > 0$, $f'(s) > 0$, for $s \geqslant 0$ and problem (1.1) attains a C^2 bounded global in time solution, $u(x,t)$, then we have $0 < f(0) < f(u(x,t)) < f(M_u)$, where M_u is an upper bound of u for $0 \leqslant x \leqslant 1$ and $t > 0$. Therefore

$$\frac{1}{I_h^2(u^n)} = \left[\sum_{j=0}^{J''} \Delta x f(u_j^n)\right]^{-2} \leqslant \frac{1}{f(0)^2}$$

and

$$\frac{1}{I^2(u^n)} = \left[\int_0^1 f(u^n)\,dx\right]^{-2} \leqslant \frac{1}{f(0)^2}.$$

Also, $f(u_j^n) \leqslant f(M_u)$, $I_h(u^n) = \sum_{j=1}^{J''} \Delta x f(u_j^n) \leqslant f(M_u)$ and $I(u^n) = \int_0^1 f(u^n)dx \leqslant f(M_u)$. Hence

$$\lambda f(u_j^n)\left|\frac{1}{I_h^2(u^n)} - \frac{1}{I(u^n)^2}\right| \leqslant \frac{2\lambda f(M_u)^2}{f(0)^4}\left|I_h(u^n) - I(u^n)\right| = O(\Delta x^2).$$

Likewise, the rest of the propositions in this work can be modified in a similar manner to include the case where f is increasing.

2.4. Stability. In the following, we show that the scheme is stable. For each Δx and Δt let $M_h > 0$ be a constant. We say that the discretization (2.1) is stable for u_h restricted to the thresholds M_h if there exist two positive constants r_0 and S such that for $r = \Delta t/\Delta x \leqslant r_0$, $B(u_h, M_h)$ is contained in the domain of ϕ_h and for every $V, W \in B(u_h, M_h)$, $\|V - W\|_X \leqslant S\|\phi_h(V) - \phi_h(W)\|_Y$.

Proposition 2.2. *Under the hypotheses of proposition* (2.1) *the discretization* (2.1) *is stable for* $r = \Delta t/\Delta x \leqslant 1$.

Proof. Let V, $W \in B(u_h, M_h)$ of X with $\phi_h(V) = Z$ and $\phi_h(W) = S$. We set $E^n = V^n - W^n \in \mathbb{R}^{J+1}$, $0 \leqslant n \leqslant N$. We have $E_0^n = 0$, for $1 \leqslant n \leqslant N$ and for $0 \leqslant n \leqslant N-1$, $1 \leqslant j \leqslant J$, that

$$\left|E_j^{n+1}\right| = \left|V_j^{n+1} - W_j^{n+1}\right| =$$

$$\left|V_j^n - \frac{\Delta t}{\Delta x}(V_j^n - V_{j-1}^n) + \Delta t\lambda F_h(V_j^n) - W_j^n + \frac{\Delta t}{\Delta x}(W_j^n - W_{j-1}^n) - \Delta t\lambda F_h(W_j^n) - \Delta t(Z_j^{n+1} - S_j^{n+1})\right| \leqslant$$

$$(1-r)|V_j^n - W_j^n| + r|V_{j-1}^n - W_{j-1}^n| + \lambda\Delta t\left|F_h(V_j^n) - F_h(W_j^n)\right| - \Delta t\left|Z_j^{n+1} - S_j^{n+1}\right|. \quad (2.5)$$

By the assumptions on f we have that f is locally Lipschitz, i.e., $|f(V_j^n) - f(W_j^n)| \leqslant L|V_j^n - W_j^n|$ for a constant $L > 0$. In addition, $0 < f(V_j^n) \leqslant f(0)$ for f decreasing and for $V, W \in B(u_h, M_h)$ we have that $1/I_h(V^n)^2 \leqslant 1/f(M_h)^2$, $1/I_h(W^n)^2 \leqslant 1/f(M_h)^2$ and $I_h(V^n) \leqslant f(0)$, $I_h(W^n) \leqslant f(0)$. Also, $|I_h(V^n) - I_h(W^n)| \leqslant \sum_{j=0}^{J''} \Delta x|f(V_j^n) - f(W_j^n)| \leqslant L\sum_{j=0}^{J''} \Delta x|V_j^n - W_j^n| = L\|E^n\|_1$. Thus,

$$\left|F_h(V_j^n) - F_h(W_j^n)\right| = \left|\frac{f(V_j^n)}{I_h(V^n)} - \frac{f(W_j^n)}{I_h(W^n)}\right| \leqslant \left|\frac{f(V_j^n)}{I_h(V^n)} - \frac{f(V_j^n)}{I_h(W^n)}\right| + \left|\frac{f(V_j^n)}{I_h(W^n)} - \frac{f(W_j^n)}{I_h(W^n)}\right| \leqslant$$

$$\left|f(V_j^n)\right|\frac{\left|I_h(V^n) + I_h(W^n)\right|}{I_h^2(V^n)\,I_h^2(W^n)}\left|I_h(V^n) - I_h(W^n)\right| + \frac{1}{I_h^2(W^n)}\left|f(V_j^n) - f(W_j^n)\right| \leqslant$$

$$\frac{2f^2(0)}{f^4(M_h)}\big|I_h(V^n) - I_h(W^n)\big| + \frac{L}{f^2(M_h)}\big|V_j^n - W_j^n\big|. \tag{2.6}$$

Therefore, for $0 < r < 1$, $c_1 = \lambda L/f^2(M_h)$ and $c_2 = 2\lambda L f^2(0)/f^4(M_h)$ we obtain

$$|E_j^{n+1}| \leqslant [(1-r) + c_1\Delta t]|E_j^n| + r|E_{j-1}^n| + c_2\|E^n\|_1 + \Delta t|Z_j^{n+1} - S_j^{n+1}|. \tag{2.7}$$

For $c = \max(c_1, c_2)$ we deduce that

$$\|E^{n+1}\|_1 \leqslant [(1-r) + c\Delta t]\|E^n\|_1 + r\|E^n\|_1 + \Delta t\|Z_j^{n+1} - S_j^{n+1}\|_1 \leqslant$$

$$(1 + c\Delta t)\|E^n\|_1 + \Delta t\|Z^{n+1} - S^{n+1}\|_1. \tag{2.8}$$

Applying the above relation recursively, we have

$$\|E^{n+1}\|_1 \leqslant C\left(\|E^0\|_1 + \Delta t\sum_{m=1}^{n+1}\|S^m - Z^m\|_1\right), \tag{2.9}$$

for some constant C. Therefore, by the discrete Gronwall lemma we get

$$\max_{0\leqslant n\leqslant N}\|E^n\|_1 \leqslant C\left(\|E^0\|_1 + \Delta t\sum_{n=1}^{N}\|S^n - Z^n\|_1\right) = C\|\phi_h(V) - \phi_h(W)\|_Y, \tag{2.10}$$

and thus $\|V - W\|_X \leqslant C\|\phi_h(V) - \phi_h(W)\|_Y$. □

2.5. Convergence. Regarding the convergence of the scheme, we have the following proposition:

Proposition 2.3. *Assuming that the hypotheses of proposition* (2.1) *hold and that* U^0 *is such that* $\|u^0 - U^0\|_1 = O(\Delta x)$, *as* $\Delta x \to 0$ *then the numerical solution of the scheme* U_h *satisfies*

$$\|U_h - u_h\|_X = O(\Delta t + \Delta x),$$

and

$$\|U_h - u_h\|_\infty = O(\Delta t + \Delta x),$$

as Δx, $\Delta t \to 0$.

Proof. We have that ϕ_h is continuous and stable on $B(u_h, M_h)$. Hence (see [1,21]) there exist inverse ϕ_h^{-1} defined on $B(u_h, M_h/S)$ for S being the stability constant. We consider the vector $R = (R^0, R^1, \ldots, R^N) \in X$ such that $\phi_h(U_h) = R$ with $R^n = 0 \in \mathbb{R}^{J+1}$, $0 \leqslant n \leqslant N$. Then U_h exists and is a unique solution of the scheme.

By the consistency property and by the fact that $\|U^0 - u^0\|_1 = O(\Delta x)$ we have that $\|\phi_h(u_h) - R\|_Y = \|\phi_h(u_h)\|_Y = O(\Delta t + \Delta x)$. Thus, for Δx, Δt small enough $U_h \in B(u_h, M_h)$ and by the stability property, i.e., $\|U_h - u_h\|_X \leqslant C\|\phi_h(u_h) - \phi_h(U_h)\|_Y$, we have that

$$\|U_h - u_h\|_X \leqslant C\|\phi_h(u_h) - \phi_h(U_h)\|_Y = C\|\phi_h(u_h) - R\|_Y = O(\Delta t + \Delta x).$$

It remains to prove that $\|U_h - u_h\|_\infty = O(\Delta t + \Delta x)$.

We set $e^n := U^n - u^n$ and $L(V_j^n) := V_j^n - \Delta t(V_j^n - V_{j-1}^n)/\Delta x + \Delta t\,\lambda F(V_j^n)$. Then $e_0^n = 0$ for $1 \leqslant n \leqslant N$ and for $1 \leqslant j \leqslant J$ we have

$$e_j^{n+1} = U_j^{n+1} - u_j^{n+1} = L(U_j^n) - u_j^{n+1} =$$

$$\big(L(U_j^n) - L(u_j^n)\big) + \big(L(u_j^n) - u_j^{n+1}\big) = L(U_j^n) - L(u_j^n) + \Delta t\, \tau_j^{n+1}, \tag{2.11}$$

where $|\tau_j^{n+1}| = O(\Delta t + \Delta x)$. Also

$$|L(U_j^n) - L(u_j^n)| \leqslant (1-r)|e_j^n| + r|e_{j-1}^n| + \lambda \Delta t |F_h(U_j^n) - F_h(u_j^n)|, \tag{2.12}$$

and for Δx, Δt small enough $U_h \in B(u_h, M_h)$, so as it is stated in relation (2.6) we have $|F_h(U_j^n) - F_h(u_j^n)| \leqslant c_1|e_j^n| + c_2\|e^n\|_1$. Therefore,

$$\big|L(V_j^n) - L(W_j^n)\big| \leqslant (1 + c\Delta t)\|e^n\|_1, \tag{2.13}$$

and

$$|e_j^{n+1}| \leqslant (1 + c\Delta t)\|e^n\|_1 + \Delta t\, |\tau_j^{n+1}|, \quad j = 1, \ldots, J. \tag{2.14}$$

Thus

$$\|e^n\|_\infty = \max_{0 \leqslant j \leqslant J} |e^n| \leqslant (1 + c\Delta t)\|e^n\|_1 + O(\Delta t + \Delta x) \tag{2.15}$$

and, in addition, from Eq. (2.14) we have

$$\|e^{n+1}\|_1 \leqslant (1 + c\Delta t)\|e^n\|_1 + O(\Delta t + \Delta x). \tag{2.16}$$

Therefore, by relations (2.15) and (2.16) recursively we obtain

$$\|e^n\|_\infty \leqslant C\|e^0\|_1 + O(\Delta t + \Delta x), \tag{2.17}$$

for some constant C and for every n, $1 \leqslant n \leqslant N$. Finally, we deduce that

$$\max_{0 \leqslant n \leqslant N} \|e^n\|_\infty = \max_{0 \leqslant n \leqslant N} \|U^n - u^n\|_\infty \leqslant C\|e^0\|_1 + O(\Delta t + \Delta x), \tag{2.18}$$

and taking also into account that $\|e^0\|_1 = \|U^0 - u^0\|_1 = O(\Delta x)$, we get the required result

$$\max_{0 \leqslant n \leqslant N} \|e^n\|_\infty = \max_{0 \leqslant n \leqslant N} \|U^n - u^n\|_\infty = O(\Delta t + \Delta x). \tag{2.19}$$

$\square$

3. The Lax — Wendroff Scheme

For the following analysis we will use the notations given in the previous section. Also, for convenience we will denote by $I_1(u) := \int_0^1 f'(u)f(u)dx$ and, accordingly, $I_{1h}(U^n) := \sum_{j=0}^{J''} f'(U_j^n)f(U_j^n)$. In the following, f' is assumed to be locally Lipschitz with constant L' and bounded, i.e., $|f'(s)| \leqslant M_1$ for $s \geqslant 0$ and some constant M_1.

In order to derive a Lax - Wendroff Scheme for problem (1.1) we note that $u_t = -u_x + \lambda F(u)$ and $u_{tt} = u_{xx} + \lambda(F_t(u) - F_x(u))$. Thus, by expanding $u(x, t + \Delta t)$ about the point (x, t) we obtain

$$u(x, t + \Delta t) = u(x,t) - \Delta t u_x(x,t) + \frac{\Delta t^2}{2} u_{xx}(x,t) + \lambda \Delta t F(u(x,t)) +$$

$$\lambda \frac{\Delta t^2}{2} \left(F_t(u(x,t)) - F_x(u(x,t))\right) + O(\Delta t^3). \tag{3.1}$$

Note also that by using the fact that $u_t = -u_x + \lambda F(u)$ we have $G(u) := F_t(u) - F_x(u) = G_1(u) + G_2(u)u_x + G_3(u)$ where the expressions for G_i, $i = 1, 2, 3$ are

$$G_1(u) = \lambda \frac{f'(u)f(u)}{I^4(u)}, \quad G_2(u) = -2\frac{f'(u)}{I^2(u)},$$

$$G_3(u) = -2\frac{f(u)}{I^3(u)}\left(\lambda \frac{I_1(u)}{I^2(u))} - (f(u(1,t)) - f(u(0,t))\right).$$

Taking the central differences for the approximation of u_x and u_{xx}, we can derive the following Lax — Wendroff scheme:

$$U_0^{n+1} = 0, \tag{3.2a}$$

$$U_j^{n+1} = \frac{r}{2}(1+r)U_{j-1}^n + (1-r^2)U_j^n - \frac{r}{2}(1-r)U_{j+1}^n + \lambda\Delta t F_h(U_j^n) + \lambda\frac{\Delta t^2}{2}G_h(U_j^n), \tag{3.2b}$$

$$U_J^{n+1} = \left(1-\frac{3r}{2}+\frac{r^2}{2}\right)U_J^n + (2r-r^2)U_{J-1}^n - \frac{r}{2}(1-r)U_{J-2}^n + \lambda\Delta t F_h(U_J^n) + \lambda\frac{\Delta t^2}{2}G_h(U_J^n), \tag{3.2c}$$

for $j = 1, \ldots, J-1$, $0 \leqslant n \leqslant N-1$ and U^0 known. Also

$$G_h(U_j^n) = G_{1h}(U_j^n) + G_{2h}(U_j^n)\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}\right) + G_{3h}(U_j^n), \quad j = 1, \ldots, J-1,$$

$$G_h(U_J^n) = G_{1h}(U_J^n) + G_{2h}(U_J^n)\left(\frac{3U_J^n - 4U_{J-1}^n + U_{J-2}^n}{2\Delta x}\right) + G_{3h}(U_J^n),$$

where

$$G_{1h}(U_j^n) = \lambda\frac{f'(U_j^n)f(U_j^n)}{I_h^4(U^n)}, \quad G_{2h}(U_j^n) = -2\frac{f'(U_j^n)}{I_h^2(U_j^n)},$$

$$G_{3h}(U_j^n) = -2\frac{f(U_j^n)}{I_h^3(U_j^n)}\left(\lambda\frac{I_{1h}(U^n)}{I_h^2(U^n))} - (f(U_J^n) - f(U_0^n)\right).$$

Note that for the approximation of the solution at the J-th point the one-sided, second-order approximation of the derivatives ${u_x}_J^n$ and ${u_{xx}}_J^n$, i.e., the Beam Warming method, is used to maintain the $O(\Delta x^2)$ accuracy of the scheme.

This scheme can give a more accurate approximation of the solution of order $O(\Delta t^2 + \Delta x^2)$, subject to the fact that for the integral of the source term a second order approximation rule is used.

3.1. Consistency. In the present case, we redefine the mapping $\phi_h : B(u_h, R) \subset X \to Y$ in the appropriate, for the Lax-Wendroff scheme, way

$$\phi_h(V^0, V^1, \ldots, V^N) = (Z^0, Z^1, \ldots, Z^N), \quad Z^0 = V^0 - U^0, \quad Z_0^{n+1} = 0,$$

$$Z_j^{n+1} = \frac{1}{\Delta t}\left(\frac{r}{2}(1+r)V_{j-1}^n + (1-r^2)V_j^n - \frac{r}{2}(1-r)V_{j+1}^n + \lambda\Delta t F_h(V_j^n) + \lambda\frac{\Delta t^2}{2}G_h(V_j^n) - V_j^{n+1}\right),$$

$$Z_J^{n+1} = \frac{1}{\Delta t}\left(\left(1-\frac{3r}{2}+\frac{r^2}{2}\right)V_J^n + (2r-r^2)V_{J-1}^n - \frac{r}{2}(1-r)V_{J-2}^n + \lambda\Delta t F_h(V_J^n) + \lambda\frac{\Delta t^2}{2}G_h(V_J^n) - V_J^{n+1}\right),$$

for $j = 1, \ldots, J-1$ and $0 \leqslant n \leqslant N-1$.

Proposition 3.1. *If we assume that f satisfies* (1.2a)*, with f' being locally Lipschitz, and that u is a C^3 global bounded solution of problem* (1.1) *(i.e. the initial data are smooth enough and $\lambda \leqslant \lambda^*$, $u_0(x) < w_2(x)$ if (1.2b) holds or $\int_0^\infty ds/f(s) = \infty$), then, if $\|u_0(x) - U^0\|_1 = o(1)$, the local discretization error for the scheme* (3.2b), (3.2c) *satisfies the relation*

$$\|\phi_h(u_h)\|_Y = O(\Delta t^2 + \Delta x^2).$$

Proof. We have $\tau_0^n = 0$, for $1 \leqslant n \leqslant N$ and for $0 \leqslant n \leqslant N-1$, $1 \leqslant j \leqslant J-1$ that

$$|\tau_j^{n+1}| = \frac{1}{\Delta t}\left|\frac{r}{2}(1+r)u_{j-1}^n + (1-r^2)u_j^n - \frac{r}{2}(1-r)u_{j+1}^n + \lambda\Delta t F_h(u_j^n) + \lambda\frac{\Delta t^2}{2}G_h(u_j^n) - u_j^{n+1}\right|.$$

Therefore,

$$|\tau_j^{n+1}| = \frac{1}{\Delta t}\left|\frac{r}{2}(1+r)\left(u_j^n - \Delta x u_{x}{}_j^n + \frac{\Delta x^2}{2}u_{xx}{}_j^n - \frac{\Delta x^3}{6}u_{xxx}{}_j^n + \cdots\right) + \right.$$

$$(1-r^2)u_j^n - \frac{r}{2}(1-r)\left(u_j^n + \Delta x u_{x}{}_j^n + \frac{\Delta x^2}{2}u_{xx}{}_j^n + \frac{\Delta x^3}{6}u_{xxx}{}_j^n + \cdots\right) + \lambda\Delta t F_h(u_j^n) +$$

$$\left.\lambda\frac{\Delta t^2}{2}G_h(u_j^n) - \left(u_j^n + \Delta t u_{t}{}_j^n + \frac{\Delta t^2}{2}u_{tt}{}_j^n + \frac{\Delta t^3}{6}u_{ttt}{}_j^n + \cdots\right)\right|$$

or

$$|\tau_j^{n+1}| = \frac{1}{\Delta t}\left| - \Delta t(u_{t}{}_j^n + u_{x}{}_j^n) + \lambda\Delta t F_h(u_j^n) + \frac{\Delta t^2}{2}(u_{xx}{}_j^n - u_{tt}{}_j^n + \lambda G_h(u_j^n)) - \right.$$

$$\left.\frac{\Delta t\Delta x^2}{6}u_{xxx}{}_j^n + \frac{\Delta t^3}{6}u_{ttt}{}_j^n + \cdots\right| \leqslant \left|\lambda\left(F_h(u_j^n) - F(u_j^n)\right)\right| + \frac{\Delta t}{2}\left(\left|u_{xx}{}_j^n - u_{tt}{}_j^n + \lambda G_h(u_j^n))\right|\right) +$$

$$\frac{\Delta x^2}{6}\left|u_{xxx}{}_j^n\right| + \frac{\Delta t^2}{6}\left|u_{ttt}{}_j^n\right| + \cdots$$

As is shown in Proposition 2.1, we have $|F_h(u_j^n) - F(u_j^n)| = O(\Delta x^2)$. Also, $u_{tt} = u_{xx} + \lambda(F_t(u) - F_x(u)) = u_{xx} + \lambda G(u)$, and $u_{xx}{}_j^n - u_{tt}{}_j^n + \lambda G_h(u_j^n) = \lambda G(u_j^n) - \lambda G_h(u_j^n)$ and in a similar way we have

$$\left|G_1(u_j^n) - G_{1h}(u_j^n)\right| = \lambda\left|\frac{f'(u_j^n)f(u_j^n)}{I^4(u^n)} - \frac{f'(u_j^n)f(u_j^n)}{I_h^4(u^n)}\right| \leqslant c_1\left|\frac{1}{I^4(u^n)} - \frac{1}{I_h^4(u^n)}\right| \leqslant O(\Delta x^2),$$

for some constant $c_1 = 4\lambda f^4(0)M_1/f^8(M_u)$. Also,

$$\left|G_2(u_j^n)u_{x}{}_j^n - G_{2h}(u_j^n)\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right| = \left|G_2(u_j^n)u_{x}{}_j^n - G_{2h}(u_j^n)\left(u_{x}{}_j^n + \frac{\Delta x^2}{6}u_{xxx}{}_j^n + \cdots\right)\right| =$$

$$\left|\frac{2f'(u_j^n)}{I^2(u^n)}u_{x}{}_j^n - \frac{2f'(u_j^n)}{I_h^2(u^n)}u_{x}{}_j^n\right| + O(\Delta x^2) \leqslant c_2\left|\frac{1}{I^2(u^n)} - \frac{1}{I_h^2(u^n)}\right| + O(\Delta x^2) \leqslant O(\Delta x^2),$$

for some constant $c_2 = 2M_1M_2$, with $M_2 = \sup u_x$, $x \in [0,1]$. Finally,

$$\left|G_3(u_j^n) - G_{3h}(u_j^n)\right| = \left|2\frac{f(u_j^n)}{I^3(U_j^n)}\left(\lambda\frac{I_1(u^n)}{I^2(u^n))} - (f(u_J^n) - f(u_0^n)\right) - 2\frac{f(u_j^n)}{I_h^3(u_j^n)}\left(\lambda\frac{I_{1h}(u^n)}{I_h^2(u^n))} - \right.\right.$$

$$\left.\left.(f(u_J^n) - f(u_0^n)\right)\right| \leqslant c_3\left(\left|\frac{1}{I_h^2(u^n)} - \frac{1}{I^2(u^n)}\right| + |I_1(u^n) - I_{1h}(u^n)| + |I(u^n) - I_h(u^n)|\right) \leqslant O(\Delta x^2)$$

for $c_3 = \max\{2\lambda f(0)(1+M_1 f(0)f^2(M_u))/f^5(M_u),\ 6\lambda f^4(0)M_1(1+f(0)f^2(M_u))/f^8(M_u)\}$. The last inequality is obtained by adding and subtracting the term $2f(u_j^n)(\lambda I_{1h}(u^n)/I_h^2(u^n))-(f(u_J^n)-f(u_0^n))/I^3(u_j^n)$, taking into account that $\left|f(u_j^n)/I^3(U_j^n)-f(u_j^n)/I_h^3(U_j^n)\right| \leqslant O(\Delta x^2)$ and that $\left|I_1(u^n)/I^2(u^n))-I_{1h}(u^n)/I_h^2(u^n))\right| \leqslant O(\Delta x^2)$. Therefore, $\left|G(u_j^n)-G_h(u_j^n)\right| \leqslant O(\Delta x^2)$. Thus, we obtain that for $1 \leqslant n \leqslant N$, $|\tau_j^n| = O(\Delta t^2+\Delta x^2)$. In addition

$$|\tau_J^{n+1}| = \frac{1}{\Delta t}\left|(1-\frac{3r}{2}+\frac{r^2}{2})u_J^n+(2r-r^2)u_{J-1}^n-\frac{r}{2}(1-r)u_{J-2}^n+\lambda\Delta t F_h(u_J^n)+\lambda\frac{\Delta t^2}{2}G(u_J^n)-u_J^{n+1}\right| =$$

$$\frac{1}{\Delta t}\left|(1-\frac{3r}{2}+\frac{r^2}{2})u_J^n+(2r-r^2)\left(u_J^n-\Delta x\, u_{xJ}^{\ n}+\frac{\Delta x^2}{2}u_{xxJ}^{\ n}-\frac{\Delta x^3}{6}u_{xxxj}^{\ n}+\cdots\right)-\right.$$

$$\frac{r}{2}(1-r)\left(u_J^n-2\Delta x\, u_{xJ}^{\ n}+2\Delta x^2 u_{xxJ}^{\ n}-\frac{4\Delta x^3}{3}u_{xxxJ}^{\ n}+\cdots\right)+\lambda\Delta t F_h(u_J^n)+$$

$$\left.\lambda\frac{\Delta t^2}{2}G(u_J^n)-\left(u_J^n+\Delta t\, u_{tJ}^{\ n}+\frac{\Delta t^2}{2}u_{ttJ}^{\ n}+\frac{\Delta t^3}{6}u_{tttJ}^{\ n}+\cdots\right)\right|$$

or in a similar way as before

$$|\tau_J^{n+1}| = \frac{1}{\Delta t}\Big|-\Delta t(u_{tJ}^{\ n}+u_{xJ}^{\ n})+\lambda\Delta t F_h(u_J^n)+\lambda\frac{\Delta t^2}{2}G(u_J^n)-\frac{\Delta t^2}{2}u_{ttJ}^{\ n}+\frac{\Delta t^2}{2}u_{xxJ}^{\ n}+$$

$$\frac{\Delta t\Delta x^2}{3}u_{xxxJ}^{\ n}+\frac{\Delta t^3}{6}u_{tttJ}^{\ n}+\cdots\Big| \leqslant \lambda\left|F_h(u_J^n)-F(u_J^n)\right|+\lambda\left|G(u_J^n)-G_h(u_J^n)\right|+$$

$$\frac{\Delta x^2}{3}\left|u_{xxxJ}^{\ n}\right|+\frac{\Delta t^2}{6}\left|u_{tttJ}^{\ n}\right|+\cdots = O(\Delta t^2+\Delta x^2).$$

Therefore, we have $|\tau_j^n| = O(\Delta t^2+\Delta x^2)$, for $j=0,\dots,J,\quad n=1,\dots,N$ and hence, using also the assumption on the initial condition, the scheme is consistent and $\lim\limits_{\Delta x,\,\Delta t\to 0}\|\phi_h(u_h)\|_Y = \lim\limits_{\Delta x,\,\Delta t\to 0}\|l_h\|_Y = 0.$ □

3.2. Stability.

Proposition 3.2. *Under the hypotheses of proposition* (3.1) *the discretization* (2.1) *is stable for $r = \Delta t/\Delta x \leqslant 1$.*

Proof. Let $V, W \in B(u_h, M_h)$ of X with $\phi_h(V) = Z$ and $\phi_h(W) = S$. We set $E^n = V^n - W^n \in \mathbb{R}^{J+1}$, $0 \leqslant n \leqslant N$. We have for $j = 1,\dots,J-1$ that

$$|E_j^{n+1}| = \left|\frac{r}{2}(1+r)V_{j-1}^n+(1-r^2)V_j^n-\frac{r}{2}(1-r)V_{j+1}^n+\lambda\Delta t F_h(V_j^n)+\lambda\frac{\Delta t^2}{2}G(V_j^n)-\frac{r}{2}(1+r)W_{j-1}^n-\right.$$

$$\left.(1-r^2)W_j^n+\frac{r}{2}(1-r)W_{j+1}^n-\lambda\Delta t F_h(W_j^n)-\lambda\frac{\Delta t^2}{2}G(W_j^n)-\Delta t(Z_j^{n+1}-S_j^{n+1})\right| \leqslant$$

$$\frac{r}{2}(1+r)|E_{j-1}^n|+(1-r^2)|E_j^n|+\frac{r}{2}(1-r)|E_{j+1}^n|+\lambda\Delta t\left|F_h(V_j^n)-F_h(W_j^n)\right|+$$

$$\lambda\frac{\Delta t^2}{2}\left|G_h(V_j^n)-G_h(W_j^n)\right|+\Delta t\left|(Z_j^{n+1}-S_j^{n+1})\right|.$$

We have, as is shown in proposition (2.2), that $\big|F_h(V_j^n) - F_h(W_j^n)\big| \leqslant c_1|E_j^n| + c_2\|E^n\|_1$, $j = 1, \ldots, J$. Then, regarding the term $\big|G_h(V_j^n) - G_h(W_j^n)\big|$ we obtain

$$|G_{1h}(V_j^n) - G_{1h}(W_j^n)| = \lambda\left|\frac{f'(V_j^n)f(V_j^n)}{I_h^4(V^n)} - \frac{f'(W_j^n)f(W_j^n)}{I_h^4(W^n)}\right| \leqslant \lambda\left|\frac{f'(V_j^n)}{I_h^2(V^n)}(F_h(V_j^n) - F_h(W_j^n))\right| +$$

$$\lambda\left|\frac{f'(W_j^n)}{I_h^2(W_j^n)}\left[\frac{1}{I_h(V^n)}(f'(V_j^n) - f'(W_j^n)) + f'(W_j^n)\left(\frac{1}{I_h^2(V_j^n)} - \frac{1}{I_h^2(W_j^n)}\right)\right]\right| \leqslant c_3|E_j^n| + c_4\|E^n\|_1,$$

where $c_3 = \max\{\lambda M_1 L/f^4(M_h), \lambda f(0)L'/f^4(M_h)\}$ and $c_4 = 2\lambda M_1 f^2(0)L/f^6(M_h)$. Also

$$\left|G_{2h}(V_j^n)\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta x} - G_{2h}(W_j^n)\frac{W_{j+1}^n - W_{j-1}^n}{2\Delta x}\right| = \frac{1}{2\Delta x}\big|G_{2h}(V_j^n)(E_{j+1}^n - E_{j-1}^n) +$$

$$(W_{j+1}^n - W_{j-1}^n)\left(G_2(V_j^n) - V_2(W_j^n)\right)\big| \leqslant \frac{1}{2\Delta x}\left[c_5\left(\big|E_{j+1}^n\big| + \big|E_j^n\big| + \big|E_{j-1}^n\big|\right) + c_6\|E^n\|_1\right],$$

where $c_5 = \max\{2M_1/f^2(M_h),\ 4L'M_h/f^2(M_h),\ c_6 = 8M_hM_1f(0)L/f^4(M_h)$. Finally,

$$\big|G_{3h}(V_j^n) - G_{3h}(W_j^n)\big| = \left| -2\frac{f(V_j^n)}{I_h^3(V^n)}\left(\lambda\frac{I_{1h}(V^n)}{I_h^2(V^n)} - f(V_J^n) + f(V_0^n)\right) +\right.$$

$$\left. 2\frac{f(W_j^n)}{I_h^3(W^n)}\left(\lambda\frac{I_{1h}(W^n)}{I_h^2(W^n)} - f(W_J^n) + f(W_0^n)\right)\right| \leqslant c_7|E_j^n| + c_8\|E^n\|_1) \leqslant c_7|E_j^n| + c_8\|E^n\|_1.$$

In this case,

$$c_8 = \max\left\{\frac{4\lambda f^3(0)M_1L}{f^7(M_h)}, \frac{2\lambda f(0)L}{f^3(M_h)}, \frac{2\lambda f(0)L(M_1L + f(0)L')}{f^5(M_h)}, \frac{6\lambda f^4(0)M_1L}{f^8(M_h)}, \frac{6f^4(0)L}{f^6(M_h)}\right\},$$

$$c_7 = \frac{2Lf(0)}{f^3(M_h)}\left(\frac{\lambda M_1}{f^2(M_h)} + 1\right).$$

Note also that here the fact that $\big|I_{1h}(V^n)) - I_{1h}(W^n)\big| \leqslant (M_1L + f(0)L')\big|\big|E^n\big|\big|_1$ and that $|E_J^n| \leqslant \|E^n\|_1$ was used.

Thus, by denoting $C_1' = \max\{c_1, c_2, \lambda r c_5/4, \lambda r c_6/4\}$, $C_2 = (\lambda/2)\max\{c_3, c_4, c_7, c_8\}$ and given that $r \leqslant 1$ we obtain

$$|E_j^{n+1}| \leqslant \left[\frac{r}{2}(1 + r) + C_1'\Delta t\right]|E_{j-1}^n| + [(1 - r^2) + C_1'\Delta t + C_2\Delta t^2]|E_j^n| +$$

$$\left[\frac{r}{2}(1 - r) + C_1'\Delta t\right]|E_{j+1}^n| + [C_1'\Delta t + C_2\Delta t^2]\|E^n\|_1 + \Delta t\big|(Z_j^{n+1} - S_j^{n+1})\big|. \qquad (3.3)$$

for $1 \leqslant j \leqslant J - 1$. In a similar way we deduce that

$$|E_J^{n+1}| \leqslant \left(1 - \frac{3r}{2} + \frac{r^2}{2}\right)\big|E_J^n\big| + (2r - r^2)\big|E_{J-1}^n\big| + \frac{r}{2}(1 - r)|E_{J-2}^n| +$$

$$\lambda\Delta t\big|F_h(V_J^n) - F_h(W_J^n)\big| + \lambda\frac{\Delta t^2}{2}\big|G_h(V_J^n) - G_h(W_J^n)\big| + \Delta t\big|(Z_J^{n+1} - S_J^{n+1})\big|,$$

or

$$|E_J^{n+1}| \leqslant \left[\left(1 - \frac{3r}{2} + \frac{r^2}{2}\right) + C_1''\Delta t + C_2\Delta t^2\right]|E_J^n| + [(2r - r^2) + C_1''\Delta t]|E_{J-1}^n| +$$

$$\left[\frac{r}{2}(1-r)+C_1''\Delta t\right]|E^n_{J-2}|+[C_1''\Delta t+C_2\Delta t^2]\|E^n\|_1+\Delta t\big|(Z_J^{n+1}-S_J^{n+1})\big|, \qquad (3.4)$$

where

$$C_1''=\max\left\{c_1,c_2,c_9,c_{10}\right\},\quad c_9=\max\left\{\frac{8M_1}{f^2(M_h)},\frac{64M_1L'}{f^2(M_h)}\right\},\quad c_{10}=\frac{32f(0)M_hM_1L}{f^4(M_h)}.$$

Note that the terms $|E^n_j|$, $j=1,\dots,J-1$ can be bounded in the following way:

$$|E_j^{n+1}|\leqslant 2r|E^n_{j-1}|+[(1-r^2)+C_1'\Delta t+C_2\Delta t^2]|E^n_j|+\left[\frac{r}{2}(1-r)+C_1'\Delta t\right]|E^n_{j+1}|+$$

$$[C_1'\Delta t+C_2\Delta t^2]\|E^n\|_1+\Delta t\big|(Z_j^{n+1}-S_j^{n+1})\big|. \qquad (3.5)$$

Then, observing that

$$\frac{r}{2}(1+r)|E^n_{J-2}|+(2r-r^2)|E^n_{J-1}|+\frac{r}{2}(1-r^2)|E^n_{J-2}|=$$

$$r|E^n_{J-2}|+(2r-r^2)|E^n_{J-1}|\leqslant 2r(|E^n_{J-2}|+|E^n_{J-1}|+|E^n_J|)$$

and that

$$(1-r^2)|E^n_{J-1}|+\left(1-\frac{3r}{2}+\frac{r^2}{2}\right)|E^n_J|\leqslant(1-r^2)(|E^n_{J-1}|+|E^n_J|),$$

we can combine Eqs. (3.4) and (3.5), with $C_1=\max\{C_1',C_1''\}$, to obtain

$$\|E^{n+1}\|_1\leqslant[2r+C_1\Delta t]\|E^n\|_1+[(1-r^2)+C_1\Delta t+C_2\Delta t^2]\|E^n\|_1+$$

$$\left[\frac{r}{2}(1-r)+C_1\Delta t\right]\|E^n\|_1+\Delta t\big|\big|(Z^{n+1}-S^{n+1})\big|\big|_1$$

or

$$\|E^{n+1}\|_1\leqslant\left[\left(1+\frac{5r}{2}-\frac{3r^2}{2}\right)+C_1\Delta t+C_2\Delta t^2\right]\|E^n\|_1+\Delta t\big|\big|(Z^{n+1}-S^{n+1})\big|\big|_1.$$

Therefore by a standard recursive argument we obtain

$$\max_{0\leqslant n\leqslant N}\|E^n\|_1\leqslant C\left(\|E^0\|_1+\Delta t\sum_{n=1}^{N}\|(Z^n-S^n)\|_1\right)=C\|\phi_h(V)-\phi_h(W)\|_Y, \qquad (3.6)$$

for some constant C. Thus $\|V-W\|_X\leqslant C\|\phi_h(V)-\phi_h(W)\|_Y$, and the scheme is stable. □

3.3. Convergence. Regarding the convergence of the scheme, we have the following proposition:

Proposition 3.3. *Assuming that the hypotheses of proposition* (3.1) *hold and that* U^0 *is such that* $\|U^0-u^0\|_1=O(\Delta x^2)$ *as* $\Delta x\to 0$*, then the numerical solution of the scheme* U_h *satisfies*

$$\|U_h-u_h\|_X=O(\Delta t^2+\Delta x^2),$$

and

$$\|U_h-u_h\|_\infty=O(\Delta t^2+\Delta x^2),$$

as Δx, $\Delta t\to 0$.

Proof. Given that $\|U^0 - u^0\|_1 = O(\Delta x^2)$ and using the same arguments as in proposition (2.3), we have that $\|U_h - u_h\|_X = O(\Delta t^2 + \Delta x^2)$.

The relation
$$\|U_h - u_h\|_\infty = \max_{0\leqslant n\leqslant N} \|e^n\|_\infty = O(\Delta t^2 + \Delta x^2),$$
also holds. Indeed, for $e^n = {U_h}^n - {u_h}^n$ we have $|e_0^n| = 0$, for $1 \leqslant n \leqslant N$ and
$$e_j^{n+1} = U_j^{n+1} - u_j^{n+1} = L(U_j^n) - L(u_j^n) + L(u_j^n) - u_j^{n+1},$$
for $0 \leqslant n \leqslant N-1$, where
$$L(U_j^n) := \frac{r}{2}(1+r)U_{j-1}^n + (1-r^2)U_j^n - \frac{r}{2}(1-r)U_{j+1}^n + \lambda\Delta t F_h(U_j^n) + \lambda\frac{\Delta t^2}{2}G_h(U_j^n), \quad j=1,\ldots,J-1,$$
$$L(U_J^n) := \left(1 - \frac{3r}{2} + \frac{r^2}{2}\right)U_J^n + (2r - r^2)U_{J-1}^n - \frac{r}{2}(1-r)U_{J-2}^n + \lambda\Delta t F_h(U_J^n) + \lambda\frac{\Delta t^2}{2}G_h(U_J^n).$$
Hence
$$|e_j^{n+1}| \leqslant |L(U_j^n) - L(u_j^n)| + \Delta t|\tau_j^{n+1}| \tag{3.7}$$
for $j = 1, \ldots, J$. Then, following a similar procedure as for the derivation of Eqs. (3.4), (3.5), we obtain
$$\big|L(U_j^n) - L(u_j^n)\big| \leqslant \frac{r}{2}(1+r)|e_{j-1}^n| + [(1-r^2)]|e_j^n| - \frac{r}{2}(1-r)|e_{j+1}^n| +$$
$$\lambda\Delta t\big|F_h(U_j^n) - F_h(u_j^n)\big| + \lambda\frac{\Delta t^2}{2}\big|G_h(U_j^n) - G_h(u_j^n)\big|, \quad j = 1, \ldots, J-1,$$
$$\big|L(U_J^n) - L(u_J^n)\big| \leqslant \left(1 - \frac{3r}{2} + \frac{r^2}{2}\right)|e_J^n| + (2r - r^2)|e_{J-1}^n| - \frac{r}{2}(1-r)|e_{J-2}^n| +$$
$$\lambda\Delta t\big|F_h(U_J^n) - F_h(u_J^n)\big| + \lambda\frac{\Delta t^2}{2}\big|G_h(U_J^n) - G_h(u_J^n)\big|.$$
Thus, for constant $C_0 > (2 + 5r - 3r^2)/2$ we have
$$\big|L(U_j^n) - L(u_j^n)\big| \leqslant (C_0 + C_1\Delta t + C_2\Delta t^2)\|e^n\|_1, \quad j = 1, \ldots, J.$$
Hence, by Eq. (3.7) we obtain
$$\|e^n\|_\infty \leqslant (C_0 + C_1\Delta t + C_2\Delta t^2)\|e^n\|_1 + O(\Delta t^2 + \Delta x^2). \tag{3.8}$$
On the other hand, we can also derive the relation
$$\|e^{n+1}\|_1 \leqslant (C_0 + C_1\Delta t + C_2\Delta t^2)\|e^n\|_1 + O(\Delta t^2 + \Delta x^2). \tag{3.9}$$
By relations (3.8) and (3.9) we deduce that, for some constant C,
$$\|e^n\|_\infty \leqslant C\|e^0\|_1 + O(\Delta t^2 + \Delta x^2),$$
for every n, $1 \leqslant n \leqslant N$ and
$$\max_{O\leqslant n\leqslant N} \|e^n\|_\infty \leqslant C\|e^0\|_1 + O(\Delta t^2 + \Delta x^2).$$
Finally, provided that $\|e^0\|_1 = O(\Delta x^2)$, we have
$$\max_{O\leqslant n\leqslant N} \|e^n\|_\infty = O(\Delta t^2 + \Delta x^2).$$
□

Remark 3.1. Note that in the above Lax — Wendroff scheme omitting the term $\lambda\Delta t^2 \times G(u)/2$ will result in a scheme that is of order $O(\Delta t+\Delta x^2)$. This can easily be seen if in the relevant proofs $G(u)=0$ is set. Moreover, a modification of scheme (3.2) can be obtained by using the Beam Warming approximation for $j=2,\dots,J$ and the Lax — Wendroff approximation for $j=1$. Such a scheme will have the same stability properties as (3.2).

3.4. High-resolution scheme. In the following, motivated by the analysis in [18], we will introduce a high-resolution method. This method will allow for better behaviour of the numerical solution near the discontinuities with the use of the upwind method and a higher order of accuracy in the smooth parts of the solution with the use of the Lax — Wendroff or the Beam-Warming method. For simplicity, we will consider the Lax — Wendroff approach only for the linear part of the equation (i.e., setting $G=0$ in (3.2)) which gives accuracy $O(\Delta t+\Delta x^2)$ for the smooth parts of the solution.

We can construct a finite volume scheme by integrating Eq. (1.1a) over the set $[x_{j-1/2},x_{j+1/2}]\times[t_n,t_{n+1}]$, where $x_{j+a}:=x_j+a\Delta x$. Indeed in such a way we obtain

$$\int\limits_{x_{j-1/2}}^{x_{j+1/2}}(u(x,t_{n+1})-u(x,t_n))\,dx=-\int\limits_{t_n}^{t_{n+1}}(u(x_{j+1/2},t)-u(x_{j-1/2},t))\,dt+\lambda\int\limits_{t_n}^{t_{n+1}}\int\limits_{x_{j-1/2}}^{x_{j+1/2}}F(u(x,t))\,dx\,dt. \tag{3.10}$$

Equation (3.10) defines a weak solution for problem (1.1). As a next step, we consider numerical methods of the form

$$U_j^{n+1}=U_j^n-r[\Phi(U^n;j)-\Phi(U^n;j-1)]+\lambda\Delta tF_h(U_j^n), \tag{3.11}$$

where $\Phi(u^n;j):=\Phi(U^n_{j-2},U^n_{j-2},U^n_{j-1},U^n_j,U^n_{j+1})$. $\Phi(U^n;j)=U^n_j$ for the upwind method,

$$\Phi(U^n;j)=\frac{1}{2}(U^n_{j+1}+U^n_j)-\frac{r}{2}(U^n_{j+1}-U^n_j)$$

for the Lax — Wendroff method and for the Beam Warming method

$$\Phi(U^n;j)=U^n_j+\frac{1}{2}(1-r)(U^n_j-U^n_{j-1}).$$

Then method (3.11) is consistent with Eq. (3.10) if Φ reduces to the true homogeneous flux for constant flow.

In order to investigate the stability of this method, we need the following definitions. For a grid function $U^n\in\mathbb{R}^{J+1}$ we define the total variation by $TV(U^n):=\sum_{j=1}^J|U^n_j-U^n_{j-1}|$. We can also extend the grid function $U_h\in X$ by defining a piecewise constant function $u_h(x,t):=U^n_j$ for $(x,t)\in[x_{j-1/2},x_{j+1/2})\times[t_n,t_{n+1})$, for r fixed. In this case, for $u_h(x,t_n)=u^n_h$, $TV(u^n_h)=TV(U^n)$. Also we define the total variation $TV_T(U_h)$, of $U_h\in X$ in both space and time in the following way : $TV_T(U_h):=\sum_{n=0}^{[T/\Delta t]}\sum_{j=1}^J\left[\Delta t|U^n_j-U^n_{j-1}|+\Delta x|U^{n+1}_j-U^n_j|\right]=\sum_{n=0}^{[T/\Delta t]}(\Delta t\,TV(U^n)+\|U^{n+1}-U^n\|_1)$, and $TV_T(u_h)=TV_T(U)$. Note that the set $\mathcal{K}_T:=\{v\in L_{1,T}([0,1])\ :TV_T(v)\leqslant R,\ R>0\}$ is a compact subset of $L_{1\,T}([0,1]):=\{v,\ [0,1]\to\mathbb{R},:\ \|v\|_{1\,T}:=\int_0^T\int_0^1 v(x,t)dxdt<\infty\}$.

Proposition 3.4. *For a method of the general form* (3.11), *if the numerical homogeneous flux Φ is Lipschitz continuous, $r<1$ and for the initial data U^0 we have $TV(U^0)<\infty$, then the method is TV-stable.*

Proof. We know ([16, Theorem 12.2]) that for a numerical method with a Lipschitz continuous numerical flux, if for any initial data u^0 there exist some Δt_0 and $R > 0$ such that $TV(U^n) < R$ for every n and Δt with $\Delta t < \Delta t_0$, $n\Delta t \leqslant T$, then the method is TV-stable.

In our case, first, we have to show that the relation $TV(U^n) \leqslant R$ (and consequently $|U^n| \leqslant R/2$) implies that $\|U^{n+1} - U^n\|_1 \leqslant c\Delta t$. By Eq. (3.11) we have

$$\|U^{n+1} - U^n\|_1 = \Delta t \left[\sum_{j=1}^{J} |\Phi^n_{j+1/2} - \Phi^n_{j-1/2}| + \lambda \Delta x F_h(U^n_j) \right] \leqslant$$

$$\Delta t \left[K \sum_{j=1}^{J} \sum_{i=-2}^{1} |\Phi^n_{j+1/2} - \Phi^n_{j-1/2}| + \lambda \sum_{j=1}^{J} \Delta x F_h(U^n_j) \right] \leqslant c\Delta t,$$

for $c = 4KR + \lambda f(0)/f^2(R/2)$.

Thus, now it is sufficient to show that $TV(U^n) \leqslant R$ which can be implied by the relation $TV(U^{n+1}) < (c_1 + c_2\Delta t)TV(U^n)$ for some constants c_1, c_2 independent of Δt. Note also that according to [15], the discontinuities in the initial condition simply propagate along the characteristics and even in this case the solution remains bounded if u^0 is bounded. Thus, we may assume, by consistency, that U^n is also bounded by some constant M. We have

$$\sum_{j=1}^{J} |U^{n+1}_j - U^{n+1}_{j-1}| = \sum_{j=1}^{J} \big| U^n_j - r[\Phi(U^n; j) - \Phi(U^n; j-1)] + \lambda \Delta t F_h(U^n_j) - U^n_{j-1} +$$

$$r[\Phi(U^n; j-1) - \Phi(U^n; j-2)] - \lambda \Delta t F_h(U^n_{j-1}) \big| \leqslant \sum_{j=1}^{J} \Bigg[|U^n_j - U^n_{j-1}| +$$

$$r c_1 \sum_{i=-2}^{1} |U^n_{j+i} - U^n_{j+i-1}| + c_2 |U^n_j - U^n_{j-1}| \Bigg] \leqslant (c_3 + c_2 \Delta t) \sum_{j=1}^{J} |U^n_j - U^n_{j-1}|,$$

for some constant c_1, $c_2 = \lambda L/f^2(M)$, and $c_3 = 1 + 4rc_2$. Hence method (3.11) is TV-stable. □

Therefore, a method of the form (3.11), which generates a numerical solution U_h, consistent with the conservation law (3.10) is convergent to an element $w \in \mathcal{K}_T$. From the Lax-Wendroff theorem we know that w is also a weak solution of (3.10) ([16]). The method converges in the sense that $dist(U, \mathcal{W}) \to 0$ as $\Delta t \to 0$ for $\mathcal{W} = \{w,\ w \text{ is a weak solution of (3.10)}\}$, $\operatorname{dist}(U, \mathcal{W}) := \inf_{w \in \mathcal{W}} \|U - w\|$ (Theorem 12.3 in [16]).

In the following, we can introduce a high-resolution method by specifying the form of Φ with the use of appropriate limiters. We consider a specific form of (3.11)

$$U^{n+1}_j = U^n_j - r(U^n_j - U^n_{j-1}) + \lambda \Delta t f_h(U^n_j) - \frac{1}{2} r(1-r) \left[\phi(\theta_{j+1/2})(U^n_{j+1} - U^n_j) + \phi(\theta_{j-1/2})(U^n_j - U^n_{j-1}) \right]. \tag{3.12}$$

The limiter $\phi(\theta_j)$ is defined for the minmod method in the following way:

$$\phi(\theta) = \operatorname{minmod}(1, \theta) = \begin{cases} 1, & \text{for } 1 < |\theta|, \quad \theta > 0, \\ \theta, & \text{for } |\theta| < 1, \quad \theta > 0, \\ 0, & \text{for } \theta \leqslant 0, \end{cases}$$

for $j = 2, \ldots, J-2$ and $\theta_j^n := (U_{j-1}^n - U_{j-2}^n)/(U_j^n - U_{j-1}^n)$. Note that for $\phi(\theta) = 1$ we use the Lax — Wendroff method, for $\phi(\theta) = \theta$ the Beam Warming method, and for $\phi(\theta) = 0$ the upwind method. To ensure that for $j = 2$ we have the Lax — Wendroff approximation and that for $j = J$ we have the Beam Warming approximation, we set $\phi(\theta_{1+1/2}) = \phi(\theta_{1/2}) = \max\{0, \operatorname{sgn}(\theta)\}$ and $\phi(\theta_{J-1/2}) = \phi(\theta_{J-1-1/2}) = \max\{0, \theta\}$. Different choices of limiters ([16, 18]) can be treated in a similar way. For simplicity, here we consider only the minimod method.

Such a method, as is stated by *Harten's* theorem [10], is TV-stable if $r < 1$ and $0 \leqslant \phi(\theta) \leqslant \operatorname{minmod}(2, 2\theta)$. These conditions are clearly satisfied by scheme (3.12).

4. Numerical Results and Comparison

In this section we present the results of numerical experiments obtained by the upwind (UWM), the Lax — Wendroff (LWM) and the high-resolution method. All the methods were implemented in MATLAB programs using double precision arithmetic.

We present the numerical solution of the problem for $f(s) = e^{-s}$. The problem was solved numerically on a uniform grid consisting of $J = 20, 40, 80$, or 160 subintervals, for $r = 1/2$, $\lambda = 0.5476 = \lambda^* - 0.1 < \lambda^* = 0.6476$ (the value of λ^* is for this specific form of f, [17]), and in a time interval $[0, T]$ with $T = 10$. The time T is chosen as to ensure, in all of the following simulations, $\|U^N - U^{N-1}\|_\infty < 10^{-7}$, that the numerical solutions reach the steady state. Also the initial condition was taken to be $u_0(x) = u(x, 0) = 0$.

We compare the solution at the time level $t_N = T$ with the steady state solution $w(x)$ which is known. More specifically, by the smaller positive root, μ of the equation $\sqrt{\mu} \times \ln(\lambda/\mu^2 + 1) = \lambda$ we can obtain the steady solution $w(x) = \underline{w}(x) = \ln(\lambda x/\mu^2 + 1)$, i.e., the lower stable solution of the steady problem in which, starting with zero initial data, we know that the solution of problem (1.1) converges [15]. We set $\|e^N\|_\infty = \|U^N - w(x)\|_\infty$ where $w(x)$ is evaluated at the points $0 = x_0, x_1, \ldots, x_J = 1$.

We present in Table 4.1 the values of the calculated numerical solution at the time T with both methods together with the exact solution and their error. In this experiment, $J = 160$, $r = 1/2$ and $\Delta t = 0.0031$.

Table 4.1. **Calculated values of U_h together with the exact solution for $J = 160$**

x	Exact	Upwind	UWM error	Lax — Wendroff	LWM error
0.1	0.13112200724761	0.12982943696673	1.2926(−3)	0.13111091697485	1.1090(−5)
0.2	0.24702705481565	0.24477407267762	2.2530(−3)	0.24700780161837	1.9253(−5)
0.3	0.35088263653835	0.34789418183696	2.9885(−3)	0.35085713563715	2.5501(−5)
0.4	0.44495981994690	0.44139433305167	3.5655(−3)	0.44492938370692	3.0436(−5)
0.5	0.53094253734067	0.52691523080510	4.0273(−3)	0.53090809842761	3.4439(−5)
0.6	0.61011401350040	0.60571090144046	4.4031(−3)	0.61007625426600	3.7759(−5)
0.7	0.68347461117119	0.67876134637587	4.7133(−3)	0.68343404251149	4.0569(−5)
0.8	0.75181930597843	0.74684695435876	4.9724(−3)	0.75177631742049	4.2989(−5)
0.9	0.81579032767599	0.81059926081556	5.1911(−3)	0.81574522030464	4.5107(−5)
1.0	0.87591394710144	0.87053653743357	5.3774(−3)	0.87586696953442	4.6978(−5)

Note that the error of both methods increases as x increases and attains its maximum value at the point $x = 1$.

In addition, Fig. 4.1 shows the numerical solution of the problem with the Lax — Wendroff method, $u(x, t)$, $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, versus the space and time. In this figure, $J = 20$.

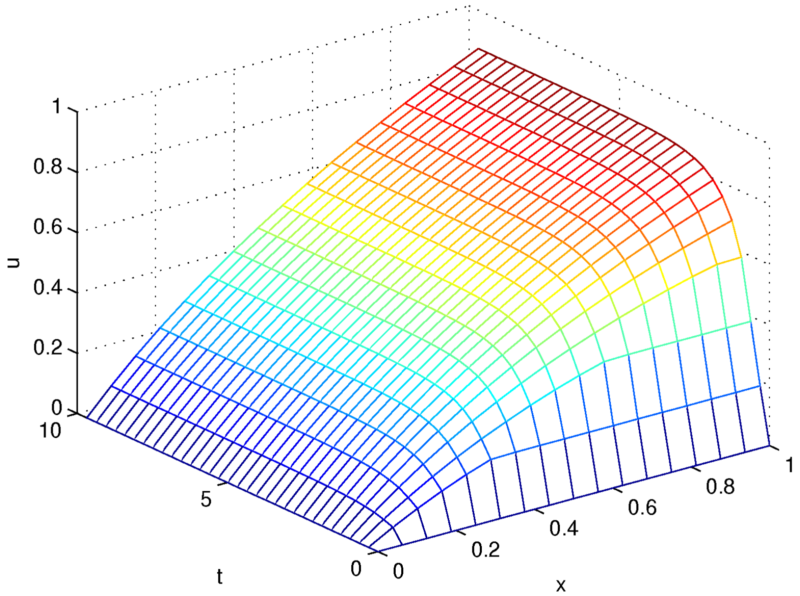

Fig. 4.1. Numerical solution of problem (1.1) with the Lax — Wendroff method, versus the space and time

Figure 4.2,a shows also the maximum in the space of the numerical solution, i.e. $u(1,t)$, with both the upwind and Lax — Wendroff methods, versus the time, Fig. 4.2,b gives the profile of the numerical solution at time T, again with both methods, is plotted against space, together with the steady state $w(x)$, $0 \leqslant x \leqslant 1$.

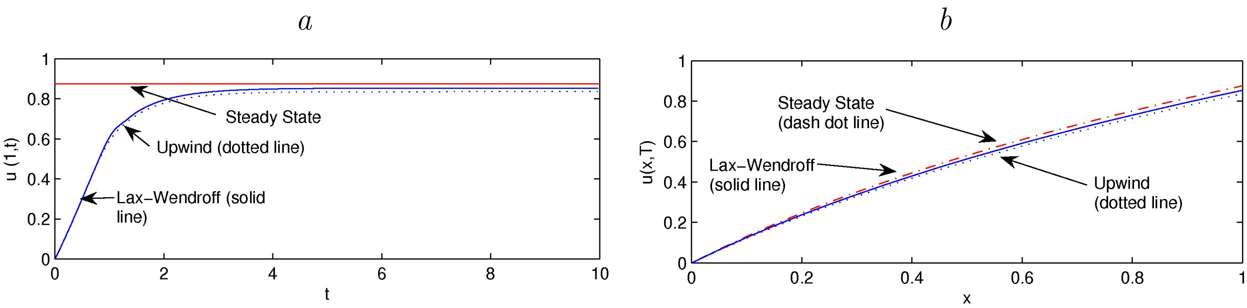

Fig. 4.2. Profiles of the numerical solution of problem (1.1) by the upwind and Lax — Wendroff methods plotted together with the steady state solution: $u(1,t)$, $0 \leqslant t \leqslant T$ (a) and $u(x,T)$, $0 \leqslant x \leqslant 1$ (b)

In Table 4.2, we present the error, the CPU time, the relative CPU time (in brackets), and the rate of convergence for these two methods. We see that the rate converges to 1 for the upwind method and to 2 for the Lax — Wendrof method. The maximum error of the Lax-Wendroff method is much smaller and the CPU time for the upwind method is shorter.

Table 4.2. **Rates of convergence of the Upwind and Lax — Wendroff methods**

J	$\|U^n - w_h\|_\infty^{UP}$	Rate	CPU time, sec	$\|U^n - w_h\|_\infty^{LW}$	Rate	CPU time, sec
20	3.9975(−2)		0.0492 (1)	2.6903(−3)		0.1886 (3.8338)
40	2.0816(−2)	0.9414	0.0739 (1.5009)	6.9895(−4)	1.9445	0.5815 (11.8187)
80	1.0635(−2)	0.9689	0.1370 (2.7827)	1.7930(−4)	1.9628	3.9824 (80.9441)
160	5.3774(−3)	0.9838	0.4757 (9.6614)	4.6978(−5)	1.9324	27.5828 (560.6251)

In Fig. 4.3 the problem is solved by both the high-resolution (with a minimod limiter) method defined by Eq. (3.11) in 4.3,a and the Lax — Wendroff method defined in 4.3,b.

The values of the parameters are the same as for the previous simulations, but for the initial condition we took $u(x, 0) = 0$ for $0 < x < 1/4$ and $x \geqslant 1/2$ and $u(x, 0) = 1$ for $1/4 \leqslant x < 1/2$, $r = 0.8$ and $\Delta t = 0.005$. As is stated also in [15], the discontinuities of the initial condition propagate along the characteristics. This can be seen in both Figs 4.3,*a* and 4.3,*b*. More specifically, in Fig. 4.3,*a* produced by the Lax — Wendroff method it can be seen that oscillations appear at the discontinuities in the direction of the characteristics and the high-resolution method has a much better behaviour with no oscillations.

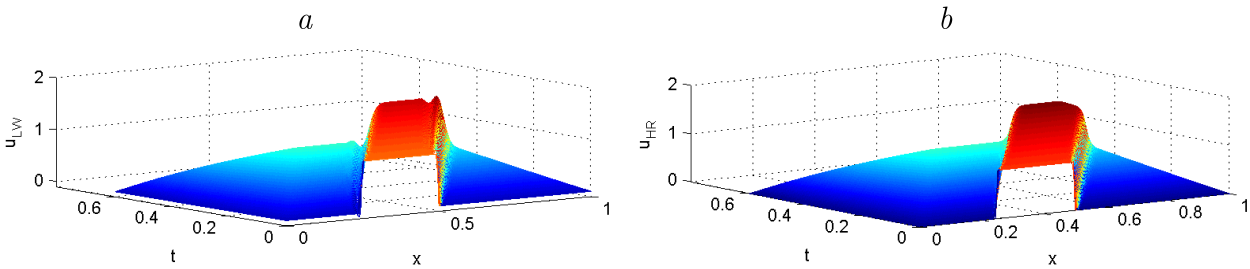

Fig. 4.3. Numerical solution of problem (1.1) by the Lax — Wendroff method (*a*) and the high-resolution method (*b*) versus the space and time for a discontinuous initial condition

This is more clear in Fig. 4.4 where the profiles of the numerical solutions presented in Fig. 4.3 obtained by both methods are given versus the space at time $t_0 = 0.045$.

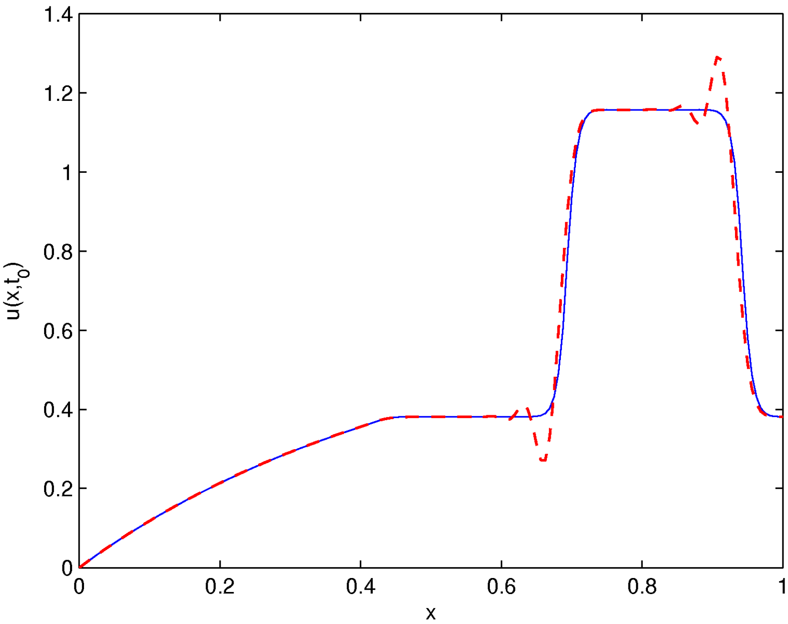

Fig. 4.4. Profiles of the numerical solution of problem (1.1) obtained by both the high-resolution method (solid line) and the Lax — Wendroff method (dashed line) for $t_0 = 0.045$

5. Conclusions

In the present work, an upwind and a Lax — Wendroff schemes have been introduced for the solution of a one-dimensional nonlocal problem modelling the ohmic heating of foods. These numerical schemes are studied regarding their consistency, stability and rate of convergence for the cases in which the problem attains a global solution in time. It has been found that the upwind scheme is of order $O(\Delta t + \Delta x)$ and the Lax — Wendroff scheme is of order

$O(\Delta t^2 + \Delta x^2)$. Also, a high resolution method has been introduced and shown to be total-variation-stable, as well as data of some numerical experiments are presented to verify the theoretical results.

The results of this work have shown that other finite difference methods can be adapted to this problem having the same order of convergence as for the relevant linear problem, in the absence of the source term, as far as an appropriate discretization is used for the non-local term. Moreover, in order to obtain a higher-order accuracy, higher-order terms associated with the derivatives of the non-local source term should be included. These numerical methods can serve as a tool for investigating the behaviour of the solution of the problem during the blow-up which is characteristic of many non-local problems like problem (1.1). It is possible that a high-resolution method like the one introduced here, together with the relevant theoretical analysis, will give more accurate results in the cases that singularities can be developed during the blow-up. It is also interesting to investigate similar numerical schemes for generalizations of problem (1.1) such as $u_t + (G(u))_x = \lambda F(u)$, where the function G may depend on u also in a non-local way ([12, 23]) as well as in parabolic problems of the form $u_t = u_{xx} + \lambda F(u)$ [13, 14].

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