COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 10 (2010), No. 2, pp. 195–203
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RUNGE-KUTTA NYSTROM METHOD OF ORDER THREE FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

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Abstract — In this paper we present a numerical algorithm for solving fuzzy differential equations based on Seikkala's derivative of a fuzzy process. We discuss in detail a numerical method based on a Runge-Kutta Nystrom method of order three. The algorithm is illustrated by solving some fuzzy differential equations.

2000 Mathematics Subject Classification: 34A12, 34K28, 65L05.

Keywords: numerical solution, fuzzy differential equation, Runge-Kutta Nystrom method of order 3.

1. Introduction

The fuzzy set theory is a tool that makes it possible to describe vague and uncertain notions. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [4]. Later Dubois and Prade [5] defined and used the extension principle. Other methods have been discussed by Puri and Ralescu [12]. Fuzzy differential equations have been suggested as a way of modelling uncertain and incompletly specified systems and were studied by many researchers [7, 8, 9]. The existence of solutions of fuzzy differential equations has been studied by several authors [2, 3]. It is difficult to obtain an exact solution for fuzzy differential equations and, therefore, several numerical methods were proposed [10, 11]. Abbasbandy and Allahviranloo [1] developed numerical algorithms for solving fuzzy differential equations based on Seikkala's derivative of the fuzzy process introduced in [14]. In this paper, we apply the Runge-Kutta Nystrom method of order three to solve fuzzy differential equations and have established that this method is better than the Euler method. The structure of the paper is organized as follows:

In Section 2, we give some basic definitions and results. In Section 3, we define the initial value problem and discuss the Runge-Kutta Nystrom method of order three. In Section 4, we apply the third order Runge-Kutta Nystrom method to solve the initial value problem and give the convergence result. Finally, in Section 5, we give some examples to illustrate our results.

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2. Preliminaries

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); \quad a \leqslant t \leqslant b, \\ y(a) = \alpha. \end{cases} \tag{2.1}$$

The point of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i, \tag{2.2}$$

where w_i's are constants and for $i = 1, 2, \cdots m$,

$$k_i = hf\left(t_n + c_i h,\ y_n + h \sum_{j=1}^{i-1} a_{ij} k_j\right). \tag{2.3}$$

Equation (2.2) must be exact for powers of h through h^m, because it must be coincident with Taylor series of order m. Therefore, the truncation error T_m, can be writtern as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}).$$

The true value of γ_m will generally be much less than the bound of Theorem 2.1. Thus, if the $O(h^{m+2})$ term is small compared to $\gamma_m h^{m+1}$ for small h, then the bound on $\gamma_m h^{m+1}$ will usually be a bound on the error as a whole. The famous nonzero constants c_i, a_{ij} in the Runge-Kutta Nystrom method of order three are

$$c_1 = 0, \quad c_2 = 2/3, \quad c_3 = 2/3, \quad a_{21} = 2/3, \quad a_{32} = 2/3,$$

where $m = 3$. Hence we have (see [6])

$$\begin{aligned} k_1 &= hf\Big(t_i, y_i\Big), \\ k_2 &= hf\left(t_i + \tfrac{2h}{3}, y_i + \tfrac{2}{3}k_1\right), \\ k_3 &= hf\left(t_i + \tfrac{2h}{3}, y_i + \tfrac{2}{3}k_2\right), \\ y_{i+1} &= y_i + \tfrac{1}{8}(2k_1 + 3k_2 + 3k_3), \end{aligned} \tag{2.4}$$

where

$$a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{2.5}$$

Theorem 2.1. *Let $f(t, y)$ belong to $C^3[a, b]$ and its partial derivatives be bounded and let us assume that there exist positive constants L, M, such that*

$$|f(t, y)| < M, \quad \left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-1}}, \quad i + j \leqslant m,$$

then in the Runge-Kutta Nystrom method of order three, we have (see [13])

$$\begin{aligned} y(t_{i+1}) - y_{i+1} &\approx \gamma_3 h^4 + O(h^5), \\ y(t_{i+1}) - y_{i+1} &\approx \frac{25}{108} h^4 M L^3 + O(h^5). \end{aligned}$$

The triangular fuzzy number v is defined by three numbers $a_1 < a_2 < a_3$, where the graph of $v(x)$ (a membership function of the fuzzy number v) is a triangle with the base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify v as $(a_1/a_2/a_3)$. We will write (2.1) $v > 0$ if $a_1 > 0$; (2.2) $v \geqslant 0$ if $a_1 \geqslant 0$; (2.3) $v < 0$ if $a_3 < 0$; and (2.4) $v \leqslant 0$ if $a_3 \leqslant 0$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $r-$level intervals. This means that if $v \in E$, then the $r-$level set

$$[v]_r = \{s \mid v(s) \geqslant r\}, \quad 0 < r \leqslant 1,$$

is a closed bounded interval denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. The mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(t)]_r = [x_1(t;r),\ x_2(t;r)], \quad t \in I, \quad r \in (0,1].$$

The derivative $x'(t)$ of the fuzzy process $x(t)$ is defined by

$$[x'(t)]_r = [x'_1(t;r),\ x'_2(t;r)], \quad t \in I, \quad r \in (0,1],$$

provided that this equation defines a fuzzy number, as in [14].

Lemma 2.1. *Let $v, w \in E$ and s be a scalar, then for $r \in (0,1]$*

$$\begin{aligned}
[v+w]_r &= [v_1(r)+w_1(r), v_2(r)+w_2(r)],\\
[v-w]_r &= [v_1(r)-w_1(r), v_2(r)-w_2(r)],\\
[v \cdot w]_r &= [\min\{v_1(r)\cdot w_1(r), v_1(r)\cdot w_2(r), v_2(r)\cdot w_1(r), v_2(r)\cdot w_2(r)\},\\
&\quad \max\{v_1(r)\cdot w_1(r), v_1(r)\cdot w_2(r), v_2(r)\cdot w_1(r), v_2(r)\cdot w_2(r)\}],\\
[sv]_r &= s[v]_r.
\end{aligned}$$

3. Fuzzy Cauchy Problem

Consder the fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); \quad t \in I = [0, T], \\ y(a) = y_0, \end{cases} \tag{3.1}$$

where f is a continuous mapping from $R_+ \times R$ onto R and $y_0 \in E$ with r-level sets

$$[y_0]_r = [y_1(0;r), y_2(0;r)], \quad r \in (0,1].$$

The extension principle of Zadeh leads to the following definition of $f(t,y)$ when $y = y(t)$ is a fuzzy number:

$$f(t,y)(s) = \sup\{y(\tau) | s = f(t,r)\}, \quad s \in R.$$

It follows that

$$[f(t,y)]_r = [f_1(t,y;r),\ f_2(t,y;r)], \quad r \in (0,1],$$

where

$$\begin{aligned}
f_1(t,y;r) &= \min\{f(t,u)| \quad u \in [y_1(r), y_2(r)]\},\\
f_2(t,y;r) &= \max\{f(t,u)| \quad u \in [y_1(r), y_2(r)]\}.
\end{aligned} \tag{3.2}$$

Theorem 3.1. *[14] Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \leqslant g(t, |v-\overline{v}|), \quad t \geqslant 0, \quad v, \overline{v} \in R,$$

where $g : R_+ \times R_+$ is a continuous mapping such that $r \to g(t,r)$ is nondecreasing and the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{3.3}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (3.3) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique solution.

4. Third-order Runge-Kutta Nystrom method

Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ be approximated by some $[y(t)]_r = [y_1(t;r), y_2(t,r)]$. From (2.2),(2.3) we define

$$\begin{aligned} y_1(t_{n+1};r) - y_1(t_n;r) &= \sum_{i=1}^{3} w_i k_{i,1}(t_n, y(t_n;r)), \\ y_2(t_{n+1};r) - y_2(t_n;r) &= \sum_{i=1}^{3} w_i k_{i,2}(t_n, y(t_n;r)), \end{aligned} \tag{4.1}$$

where w_i's are constants and

$$[k_i(t, y(t;r))]_r = [k_{i,1}(t, y(t;r), k_{i,2}(t, y(t;r))], \quad i = 1,2,3$$

$$\begin{aligned} k_{i,1}(t_n, y(t_n;r)) &= hf\left(t_n + c_i h \ , \ y_1(t_n) + \sum_{j=1}^{i-1} a_{ij} k_{j,1}(t_n, y(t_n;r))\right), \\ k_{i,2}(t_n, y(t_n;r)) &= hf\left(t_n + c_i h \ , \ y_2(t_n) + \sum_{j=1}^{i-1} a_{ij} k_{j,2}(t_n, y(t_n;r))\right), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} k_{1,1}(t, y(t;r)) &= \min\Big\{hf\,(t,u)\,|u \in [y_1(t;r), y_2(t;r)]\Big\}, \\ k_{1,2}(t, y(t;r)) &= \max\Big\{hf\,(t,u)\,|u \in [y_1(t;r), y_2(t;r)]\Big\}, \\ k_{2,1}(t, y(t;r)) &= \min\left\{hf\left(t + \tfrac{2}{3}h, u\right) \Big| u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r)]\right\}, \\ k_{2,2}(t, y(t;r)) &= \max\left\{hf\left(t + \tfrac{2}{3}h, u\right) \Big| u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r)]\right\}, \\ k_{3,1}(t, y(t;r)) &= \min\left\{hf\left(t + \tfrac{2}{3}h, u\right) \Big| u \in [z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r)]\right\}, \\ k_{3,2}(t, y(t;r)) &= \max\left\{hf\left(t + \tfrac{2}{3}h, u\right) \Big| u \in [z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r)]\right\}, \end{aligned} \tag{4.3}$$

where in the third-order Runge-Kutta method

$$\begin{aligned} z_{1,1}(t,y(t;r)) &= y_1(t;r)+\tfrac{2}{3}k_{1,1}(t,y(t;r)),\\ z_{1,2}(t,y(t;r)) &= y_2(t;r)+\tfrac{2}{3}k_{1,2}(t,y(t;r)),\\ z_{2,1}(t,y(t;r)) &= y_1(t;r)+\tfrac{2}{3}k_{2,1}(t,y(t;r)),\\ z_{2,2}(t,y(t;r)) &= y_2(t;r)+\tfrac{2}{3}k_{2,2}(t,y(t;r)). \end{aligned} \tag{4.4}$$

Define

$$\begin{aligned} F[t,y(t;r)] &= 2k_{1,1}(t,y(t;r)+3k_{3,1}(t,y(t;r))+3k_{3,1}(t,y(t;r)),\\ G[t,y(t;r)] &= 2k_{1,2}(t,y(t;r)+3k_{3,2}(t,y(t;r))+3k_{3,1}(t,y(t;r)). \end{aligned} \tag{4.5}$$

The exact and approximate solutions at t_n, $0 \leqslant n \leqslant N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solution is calculated by the grid points (2.5). By (4.1),(4.5) we have

$$\begin{aligned} Y_1(t_{n+1};r) &\approx Y_1(t_n;r)+\frac{1}{8}F\left[t_n, Y(t_n;r)\right],\\ Y_2(t_{n+1};r) &\approx Y_2(t_n;r)+\frac{1}{8}G\left[t_n, Y(t_n;r))\right]. \end{aligned} \tag{4.6}$$

We define

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r)+\tfrac{1}{8}F\left[t_n, y(t_n;r)\right],\\ y_2(t_{n+1};r) &= y_2(t_n;r)+\tfrac{1}{8}G\left[t_n, y(t_n;r)\right]. \end{aligned} \tag{4.7}$$

The following lemmas will be applied to show the convergence of these approximations. That is

$$\lim_{h\to 0} y_1(t;r) = Y_1(t;r),$$
$$\lim_{h\to 0} y_2(t;r) = Y_2(t;r).$$

Lemma 4.1. *[10] Let the sequence of numbers $\{W_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \leqslant A|W_n| + B,\ 0 \leqslant n \leqslant N-1,$$

for some given positive constants A and B. Then

$$|W_n| \leqslant A^n|W_0| + B\frac{A^n-1}{A-1},\ 0 \leqslant n \leqslant N.$$

Lemma 4.2. *[10] Let the sequence of numbers$\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \leqslant |W_n| + A\max\{|W_n|,|V_n|\} + B,$$
$$|V_{n+1}| \leqslant |V_n| + A\max\{|W_n|,|V_n|\} + B,$$

for some given positive constants A and B, and denote

$$U_n = |W_n| + |V_n|, \quad 0 \leqslant n \leqslant N.$$

Then

$$U_n \leqslant \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A}-1},\ 0 \leqslant n \leqslant N,$$

where $\overline{A} = 1+2A$ and $\overline{B} = 2B$.

Let $F(t,u,v)$ and $G(t,u,v)$ be obtained by substituting $[y(t)]_r=[u,v]$ into (4.5),

$$F[t,y(t;r)]=2k_{1,1}(t,y(t;r)+3k_{3,1}(t,y(t;r))+3k_{3,1}(t,y(t;r)),$$
$$G[t,y(t;r)]=2k_{1,2}(t,y(t;r)+3k_{3,2}(t,y(t;r))+3k_{3,1}(t,y(t;r)).$$

The domain where F and G are defind is therefore

$$K=\{(t,u,v)|0\leqslant t\leqslant T,\quad -\infty<v<\infty\ ,\quad -\infty<u\leqslant v\}.$$

Theorem 4.1. *Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^3(k)$ and let the partial derivatives of F and G be bounded over K. Then, for arbitrary fixed $r, 0\leqslant r\leqslant 1$, the approximate solutions (4.6) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It suffices to show

$$\lim_{h\to 0} y_1(t_N;r)=Y_1(t_N;r),$$
$$\lim_{h\to 0} y_2(t_N;r)=Y_2(t_N;r),$$

where $t_N=T$. For $n=0,1,\cdots,N-1$, by using the Taylor theorem we get

$$\begin{aligned}Y_1(t_{n+1};r)&=Y_1(t_n;r)+\tfrac{1}{8}F\left[t_n,Y(t_n;r)\right]+\tfrac{25}{108}h^4ML^3+O(h^5),\\ Y_2(t_{n+1};r)&=Y_2(t_n;r)+\tfrac{1}{8}G\left[t_n,Y(t_n;r))\right]+\tfrac{25}{108}h^4ML^3+O(h^5),\end{aligned}\tag{4.8}$$

$$W_n=Y_1(t_n;r)-y_1(t_n;r),$$
$$V_n=Y_2(t_n;r)-y_2(t_n;r).$$

Hence from (4.7) and (4.8)

$$\begin{aligned}W_{n+1}=W_n+\frac{1}{8}\left\{F\left[t_n,Y_1(t_n;r),Y_2(t_n;r)\right]-F\left[t_n,y_1(t_n;r),y_2(t_n;r)\right]\right\}\\ +\frac{25}{108}h^4ML^3+O(h^5),\\ V_{n+1}=V_n+\frac{1}{8}\left\{G\left[t_n,Y_1(t_n;r),Y_2(t_n;r)\right]-G\left[t_n,y_1(t_n;r),y_2(t_n;r)\right]\right\}\\ +\frac{25}{108}h^4ML^3+O(h^5).\end{aligned}$$

Then

$$|W_{n+1}|\leqslant|W_n|+\frac{1}{4}Ph\cdot\max\{|W_n|,|V_n|\}+\frac{25}{108}h^4ML^3+O(h^5),$$
$$|V_{n+1}|\leqslant|V_n|+\frac{1}{4}Ph\cdot\max\{|W_n|,|V_n|\}+\frac{25}{108}h^4ML^3+O(h^5),$$

for $t\in[0,T]$ and $P>0$ is a bound for the partial derivatives of F and G. Thus, by Lemma 4.2

$$|W_n|\leqslant(1+\frac{1}{2}Ph)^n|U_0|+\left(\frac{25}{54}h^4ML^3+O(h^5)\right)\frac{(1+\frac{1}{2}Ph)^n-1}{\frac{1}{2}Ph},$$
$$|V_n|\leqslant(1+\frac{1}{2}Ph)^n|U_0|+\left(\frac{25}{54}h^4ML^3+O(h^5)\right)\frac{(1+\frac{1}{2}Ph)^n-1}{\frac{1}{2}Ph},$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_N| \leqslant (1+\frac{1}{2}Ph)^N|U_0| + \left(\frac{25}{28}h^3ML^3 + O(h^4)\right)\frac{(1+\frac{1}{2}Ph)^{\frac{T}{h}}-1}{P},$$
$$|V_N| \leqslant (1+\frac{1}{2}Ph)^N|U_0| + \left(\frac{25}{28}h^3ML^3 + O(h^4)\right)\frac{(1+\frac{1}{2}Ph)^{\frac{T}{h}}-1}{P}.$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \leqslant \frac{25}{28}ML^3\left(\frac{e^{\frac{1}{2}Ph}-1}{P}\right)h^3 + O(h^4),$$
$$|V_N| \leqslant \frac{25}{28}ML^3\left(\frac{e^{\frac{1}{2}Ph}-1}{P}\right)h^3 + O(h^4),$$

and if $h \to 0$, we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

5. Numerical Examples

Example 5.1. Consider the fuzzy differential equation

$$\begin{cases} y'(t) = -y(t), \quad t \geqslant 0, \\ y(0) = [0.96+0.04r,\ 1.01-0.01r]. \end{cases} \tag{5.1}$$

The exact solution is given by

$$Y(t;r) = \left[(0.96+0.04r)e^{-t}, (1.01-0.01r)e^{-t}\right].$$

At $t = 0.1$ we get

$$Y(0.1;r) = \left[(0.96+0.04r)e^{-0.1}, (1.01-0.01r)e^{-0.1}\right].$$

With the use of the third-order Runge-kutta Nystrom method the approximate solution is

$$y_1(t_{n+1};r) = y_1(t_n;r)\left(1-h+\frac{h^2}{2!}-\frac{h^3}{3!}\right),$$
$$y_2(t_{n+1};r) = y_2(t_n;r)\left(1-h+\frac{h^2}{2!}-\frac{h^3}{3!}\right).$$

The exact and approximate solutions obtained by the Euler method and by the third-orde Runge-Kutta Nystrom method are compared and plotted in Fig. 5.1.

Example 5.2. Consider the fuzzy differential equation

$$\begin{cases} y'(t) = ty(t), \quad t \in [0,1] \\ y(0) = \left[\sqrt{e}-0.5(1-r), \sqrt{e}+0.5(1-r)\right]. \end{cases} \tag{5.2}$$

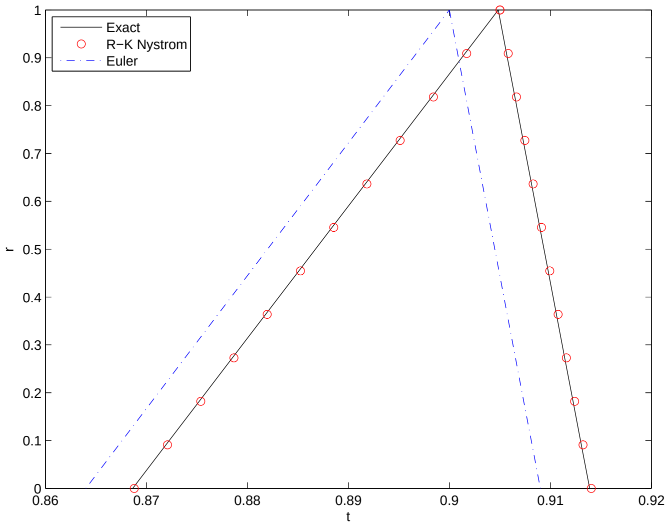

Fig. 5.1. (h=0.2)

The exact solution is given by $Y(t;r) = \left[(\sqrt{e} - 0.5(1-r))e^{\frac{t^2}{2}},\ (\sqrt{e} + 0.5(1-r))e^{\frac{t^2}{2}}\right]$. At $t = 0.1$ we get $Y(0.1;r) = [(\sqrt{e} - 0.5(1-r))e^{0.005},\ (\sqrt{e} + 0.5(1-r))e^{0.005}]$. The exact and approximate solutions obtained by the third-order Runge-Kutta Nystrom method (Eqs. (4.3) - (4.7)), are compared and plotted in Fig. 5.2.

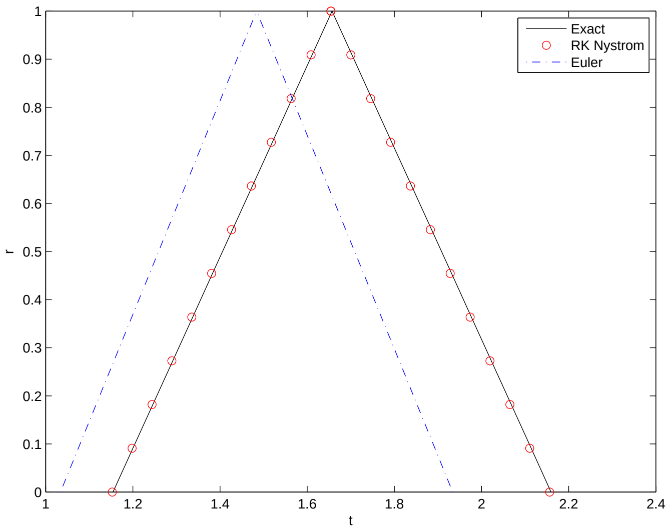

Fig. 5.2. (h=0.5)

6. Conclusions

In this work, we have used the third-order Runge-Kutta Nystrom method to find a numerical solution of fuzzy differential equations. Taking into account the convergence order of the Euler method is $O(h)$ (as given in [10]), a higher order of convergence $O(h^3)$ is obtained by the proposed method. Comparison of the solutions of examples 5.1 and 5.2 shows that the proposed method gives a better solution than the Euler method does.

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