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A NATURAL ADAPTIVE NONCONFORMING FEM OF QUASI-OPTIMAL COMPLEXITY

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Abstract — In recent years, the question on the convergence and optimality in the context of adaptive finite element methods has been the subject of intensive studies. However, for nonstandard FEMs such as mixed or nonconforming ones, the lack of Galerkin's orthogonality requires new mathematical arguments. The presented adaptive algorithm for the Crouzeix-Raviart finite element method and the Poisson model problem is of quasi-optimal complexity. Furthermore it is natural in the sense that collective marking rather than a separate marking is applied or the estimated error and the volume term.

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1. Introduction

This paper introduces an adaptive algorithm for the nonconforming finite element method (FEM) for the Poisson model problem, which is proven to be of quasi-optimal complexity.

For given $f \in L^2(\Omega)$, the Poisson model problem with the unknown solution u reads

$$\Delta u = -f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega.$$

Let ∇_ℓ be the piecewise action of the gradient on the triangulation $\mathcal{T}_\ell$ and $P_1(T)$ be the space of affine functions on an element $T \in \mathcal{T}_\ell$ and the conforming and nonconforming finite element spaces be given by

$$\begin{aligned} P_1(\mathcal{T}_\ell) &:= \left\{ v_\ell \in L^2(\Omega) \mid v_\ell|_T \in P_1(T) \text{ for } T \in \mathcal{T}_\ell \right\}, \\ P_1^C(\mathcal{T}_\ell) &:= P_1(\mathcal{T}_\ell) \cap C(\Omega), \\ P_1^{NC}(\mathcal{T}_\ell) &:= \{ v_\ell \in P_1(\mathcal{T}_\ell) \mid v_\ell \text{ continuos in } \operatorname{mid}(E) \text{ for } E \in \mathcal{E}_\ell \}, \\ P_{1,0}^{NC}(\mathcal{T}_\ell) &:= \left\{ v_\ell \in P_1^{NC}(\mathcal{T}_\ell) \mid v_\ell(\operatorname{mid}(E)) = 0 \text{ for } E \subseteq \partial\Omega \right\}. \end{aligned}$$

The discrete weak formulation reads: Seek $u_\ell^{NC} \in P_{1,0}^{NC}(\mathcal{T}_\ell)$, such that for all $v_\ell^{NC} \in P_{1,0}^{NC}(\mathcal{T}_\ell)$

$$\left(\nabla_\ell u_\ell^{NC}, \nabla_\ell v_\ell^{NC} \right)_{L^2(\Omega)} = \left(f, v_\ell^{NC} \right)_{L^2(\Omega)}. \tag{1.1}$$

The adaptive finite element methods consists in general of successive loops of the sequence

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}.$$

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The regions where the error is estimated to be large are refined locally, while the others may stay relatively coarse. Thus, the element size does not necessarily tends to zero all over Ω.

The convergence and optimality of adaptive algorithms has been studied in detail in recent years. For mixed finite elements there are some results on quasi-optimal convergence of adaptive algorithms in the sense of Stevenson [11] established for conforming methods. They include adaptive schemes based on separate marking strategies [7] and strategies where oscillations are reduced separately by some preprocessing algorithm [1, 9].

The contraction property for adaptive nonconforming finite element methods of [6] involves small volume contributions $\|h_\ell f\|_{L^2(\Omega)}$. As a first approach optimal convergence is obtained in [2] by two alternative separate bulk criteria, which appears unnatural in view of [8]. To overcome the disadvantage of separate marking, this work here follows the idea of [6] with one refinement indicator combining the volume term and the edge terms. Optimal convergence is achieved in terms of the weighted term

$$\xi_\ell^2 := \eta_\ell^2 + \alpha \|h_\ell f\|_{L^2(\Omega)}^2 + \beta \varepsilon_\ell^2$$

of the edge-based error estimator η_ℓ, the volume term $\|h_\ell f\|_{L^2(\Omega)}$, and the flux error ε_ℓ with proper weights α and β. Parallel to this work is [10] without the discrete Poincaré inequality (Lemma 4.1) for nonconforming finite element functions.

The main result of this paper is Theorem 3.1, which proves that the outcome of the subsequent adaptive algorithm ANCFEM is quasi-optimal in the sense of Stevenson [11] with respect to the approximation class

$$\mathcal{A}_s := \left\{(u,f) \mid \|(u,f)\|_{\mathcal{A}_s} < \infty\right\} \text{ and its norm}$$

$$\|(u,f)\|_{\mathcal{A}_s} := \sup_{N\in\mathbb{N}} \left(N^s \inf_{|\mathcal{T}|-|\mathcal{T}_0|\leqslant N} \left(\varepsilon_{\mathcal{T}}^2 + \|h_{\mathcal{T}} f\|_{L^2(\Omega)}^2 \right)^{1/2} \right).$$

Here and in the sequel, $\mathcal{T}_\ell$ is a regular triangulation of the domain Ω into triangles $T \in \mathcal{T}_\ell$ with its set of all edges $\mathcal{E}_\ell^\Omega$ and interior edges $\mathcal{E}_\ell$. The set of edges of a triangle is denoted by $\mathcal{E}(T)$. One of the edges of a triangle is designated to be its reference edge $E(T)$. The flux error is $\varepsilon_\ell^2 := \left\|p_\ell^{NC} - p\right\|_{L^2(\Omega)}^2$ with the discrete and exact fluxes $p_\ell^{NC} := \nabla_\ell u_\ell^{NC}$, and $p := \nabla u$. Furthermore, $|\cdot|$ is context-sensitive and denotes the number of elements of some finite set or the area of some domain. $A \lesssim B$ represents $A \leqslant CB$ for some mesh-independent, positive generic constant C, whereas $A \approx B$ represents $A \lesssim B \lesssim A$. Moreover, the standard notation of Lebesgue and Sobolev spaces is employed.

Given a subset $\mathcal{F} \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell$ of interior edges $\mathcal{E}_\ell$ and elements, the refinement indicator μ_ℓ is given via

$$\mu_\ell^2(\mathcal{F}) := \sum_{T\in\mathcal{F}\cap\mathcal{T}_\ell} \|h_\ell f\|_{L^2(T)}^2 + \sum_{E\in\mathcal{F}\cap\mathcal{E}_\ell} h_E \left\|[p_\ell^{NC}]_E \cdot \tau_E\right\|_{L^2(E)}^2 .$$

with the edge contributions η_ℓ

$$\eta_\ell := \eta_\ell(\mathcal{E}_\ell), \quad \eta_\ell^2(\mathcal{F}) := \sum_{E\in\mathcal{F}\cap\mathcal{E}_\ell} \eta_\ell^2(E),$$

$$\eta_\ell(E) := |E|^{1/2} \left\|[p_\ell^{NC}]_E \cdot \tau_E\right\|_{L^2(E)} \text{ for any } E \in \mathcal{E}_\ell$$

with $[p_\ell^{NC}]_E := p_\ell^{NC}\big|_{T_+} - p_\ell^{NC}\big|_{T_-}$ denoting the jump of the discrete flux p_ℓ^{NC} across the interior edge $E = T_+ \cap T_-$ shared by the two elements $T_\pm \in \mathcal{T}_\ell$, and $\omega_E := T_+ \cup T_-$. Note that $\eta_\ell(E)$

vanishes at all boundary edges $\mathcal{E}_\ell^\Omega \backslash \mathcal{E}_\ell$. In addition, $\nu_E = \nu_{T_+}$ is the unit normal vector exterior to T_+ along E and τ_E is the unit tangential vector along $E|_{T_+}$, respectively. The piecewise constant jump vector allows a decomposition $\left|[p_\ell^{NC}]_E\right|^2 = ([p_\ell^{NC}]_E \cdot \nu_E)^2 + ([p_\ell^{NC}]_E \cdot \tau_E)^2$ [6].

Let h_ℓ be a piecewise constant mesh-size function with $h_\ell|_T := |T|^{1/2}$ depending on the area of the elements $T \in \mathcal{T}_\ell$. The volume part for the right-hand side $f \in L^2(\Omega)$ is defined via

$$\|h_\ell f\|_{\mathcal{F}}^2 := \sum_{T \in \mathcal{F} \cap \mathcal{T}_\ell} \|h_\ell f\|_{L^2(T)}^2, \text{ thus } \|h_\ell f\|_{L^2(\Omega)} = \|h_\ell f\|_{\mathcal{T}_\ell}.$$

Finally, the total error is denoted by $\mu_\ell^2 = \eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2$.

The algorithm **ANCFEM** reads as follows.

Input: Initial coarse triangulation $\mathcal{T}_0$, $0 < \theta < \theta_0 \leqslant 1$, cf. Theorem 3.1.
Loop: For $\ell = 0, 1, \ldots$

Update ℓ

SOLVE problem (1.1) on $\mathcal{T}_\ell$.
ESTIMATE the refinement indicator μ_ℓ^2.
MARK Compute a quasi-minimal subset $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell$ of elements and interior edges satisfying the bulk criterion, i.e.,

$$\theta \mu_\ell^2 \leqslant \mu_\ell^2(\mathcal{M}_\ell) \quad \text{and} \tag{1.2}$$

$$|\mathcal{M}_\ell| \approx \min\left\{|\mathcal{F}| \,\middle|\, \theta\mu_\ell^2 \leqslant \mu_\ell^2(\mathcal{F}),\, \mathcal{F} \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell\right\}. \tag{1.3}$$

REFINE all edges in the closure $\mathcal{C}\ell(\mathcal{M}_\ell^E)$ with $\mathcal{M}_\ell^E := (\mathcal{M}_\ell \cap \mathcal{E}_\ell) \cup E(\mathcal{M}_\ell \cap \mathcal{T}_\ell)$,

$$\text{and } \left\{E(T) \in \mathcal{E}_\ell \,\middle|\, T \in \mathcal{T}_\ell \text{ and } \mathcal{E}(T) \cap \mathcal{C}\ell\left(\mathcal{M}_\ell^E\right) \neq \emptyset\right\} \subseteq \mathcal{C}\ell\left(\mathcal{M}_\ell^E\right)$$

using Newest-Vertex-Bisection (NVB) to generate a new regular triangulation $\mathcal{T}_{\ell+1}$. The possible refinements of a triangle $T \in \mathcal{T}_\ell$ are depicted in Fig. 1.1 and depend on the set of edges in $\mathcal{E}(T) \cap \mathcal{C}\ell(\mathcal{M}_\ell^E)$.
Output: Sequence of triangulations $(\mathcal{T}_k)$, and discrete solutions (u_k^{NC}).

The remaining part of the paper is organized as follows. Section 2 summarizes the necessary preliminaries in order to prove the contraction property and optimal convergence of ANCFEM in Section 3. The technical proof of discrete reliability of Theorem 2.1 is postponed to the last section.

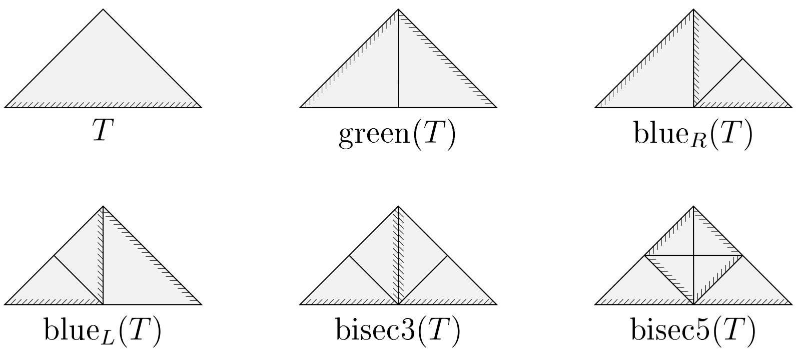

Fig. 1.1. Possible refinements of a triangle T on one level using NVB. The reference edges of T and each subtriangle are accented. In the case that $\mathcal{E}(T) \subseteq \mathcal{C}\ell(\mathcal{M}_\ell^E)$, either bisec3$(T)$ or bisec5(T) can be applied

2. Preliminaries

To prove the convergence and optimal rates, this section discusses such important characteristics as efficiency, reliability, quasi-orthogonality and discrete reliability. The proofs can be found in the given references, and in the case of discrete reliability in Section 4.

Throughout this section, let $\mathcal{T}_\ell$ be a triangulation of Ω and $\mathcal{T}_{\ell+k}$ be some NVB-refinement of $\mathcal{T}_\ell$ on $k \geqslant 1$ levels of refinement, and $\mathcal{F} := \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+k} \cup \mathcal{E}_\ell \setminus \mathcal{T}_{\ell+k}$ be the elements and edges that have been refined from level ℓ to $\ell+k$.

Lemma 2.1 (Reliability & Efficiency). *For positive, generic constants c_{eff}, C_{rel} there is efficiency and reliability of η_ℓ [2, 5, 6]*

$$c_{\text{eff}}\eta_\ell^2 \leqslant \varepsilon_\ell^2 \leqslant C_{\text{rel}}\left(\eta_\ell^2 + \|h_\ell f\|^2_{L^2(\Omega)}\right).$$

Thus, efficiency and reliability hold for the refinement indicator μ_ℓ as follows:

$$c_{\text{eff}}\left(\mu_\ell^2 - \|h_\ell f\|^2_{L^2(\Omega)}\right) \leqslant \varepsilon_\ell^2 \leqslant C_{\text{rel}}\mu_\ell^2. \tag{2.1}$$

Lemma 2.2 (Quasi-orthogonality). *Let $\mathcal{K} := \mathcal{F} \cap \mathcal{T}_\ell$. The CR-solutions $p_\ell^{NC} \in P_{1,0}^{NC}(\mathcal{T}_\ell)$ and $p_{\ell+k}^{NC} \in P_{1,0}^{NC}(\mathcal{T}_{\ell+k})$ of* (1.1) *satisfy the quasi-optimality [2]*

$$\left|\left(p - p_{\ell+k}^{NC}, p_\ell^{NC} - p_{\ell+k}^{NC}\right)_{L^2(\Omega)}\right| \leqslant C_{\text{qo}}^{1/2}\varepsilon_{\ell+k}\|h_\ell f\|_{\mathcal{K}}.$$

This leads to

$$\begin{aligned}\left\|p_{\ell+k}^{NC} - p_\ell^{NC}\right\|^2_{L^2(\Omega)} &\leqslant \varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 + 2C_{\text{qo}}^{1/2}\|h_\ell f\|_{\mathcal{F}}\,\varepsilon_{\ell+k},\\ \varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 &\leqslant \left\|p_{\ell+k}^{NC} - p_\ell^{NC}\right\|^2_{L^2(\Omega)} + 2C_{\text{qo}}^{1/2}\|h_\ell f\|_{\mathcal{F}}\,\varepsilon_{\ell+k}.\end{aligned}$$

Lemma 2.3 (Bounding the Overhead of Closure). *Let $\mathcal{T}_\ell$ be a regular triangulation refined from $\mathcal{T}_0$ by an adaptive algorithm using NVB and Closure and $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell$ be the set of marked elements and edges in step* Mark. *Then, the overhead of closure is bounded as follows:*

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{j=0}^{\ell-1}|\mathcal{M}_j|. \tag{2.2}$$

Proof. Let $\mathcal{M}_\ell^\star \subseteq \mathcal{E}_\ell \cup \mathcal{T}_\ell$ be the minimal set satisfying the bulk criterion (1.2). With respect to the result of Refine $\mathcal{M}_\ell^\star$ is equivalent to the set of marked edges

$$\begin{aligned}\mathcal{M}_\ell^{E\star} &:= \{E \in \mathcal{E}_\ell \,|\, E \in \mathcal{M}_\ell^\star \text{ or } E = E(T) \text{ with } T \in \mathcal{M}_\ell^\star\}\\ &= (\mathcal{M}_\ell^\star \cap \mathcal{E}_\ell) \cup E(\mathcal{M}_\ell^\star \cap \mathcal{T}_\ell).\end{aligned}$$

Hence, (2.2) is a direct consequence of [3, 7] and (1.3), i.e.,

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{j=0}^{\ell-1}\left|\mathcal{M}_j^{E\star}\right| \approx \sum_{j=0}^{\ell-1}\left|\mathcal{M}_j^\star\right| \approx \sum_{j=0}^{\ell-1}|\mathcal{M}_j|. \qquad \square$$

The following theorem states the discrete reliability.

Theorem 2.1 Discrete Reliability. *For $\mathcal{F} := \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k} \cup \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+k}$ the set of refined elements and edges form $\mathcal{T}_\ell$ to $\mathcal{T}_{\ell+k}$. Then, $|\mathcal{F}| \lesssim |\mathcal{T}_{\ell+k}| - |\mathcal{T}_\ell|$ and*

$$\left\|p_{\ell+k}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2 \leqslant C_{\mathrm{drel}} \left(\eta_\ell^2(\mathcal{F}) + \|h_\ell f\|_{\mathcal{F}}^2\right). \tag{2.3}$$

Proof. The proof is lengthy and technical, and therefore is deferred to Section 4. □

3. Optimal convergence rates

In this section, we consider the convergence and optimality of the sequence of CR-solutions $p_\ell^{NC} \in P_{1,0}^{NC}(\mathcal{T}_\ell)$ of (1.1) generated by the algorithm ANCFEM starting with the initial coarse and regular triangulation $\mathcal{T}_0$.

Lemma 3.1 (Estimator Reduction). *There exist $\Lambda > 0$ and a contraction factor $0 < \rho < 1$ such that on each level $\ell \geqslant 0$ of* ANCFEM *there is reduction of the refinement indicator in the following sense*

$$\eta_{\ell+1}^2 + \|h_{\ell+1} f\|_{L^2(\Omega)}^2 \leqslant \rho\left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2\right) + \Lambda \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2. \tag{3.1}$$

Proof. Let $\mathcal{F} = \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+1} \cup \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ be the set of all elements and edges in $\mathcal{T}_\ell$ that have been refined in $\mathcal{T}_{\ell+1}$. For any edge $E \in \mathcal{E}_\ell$ and any $\delta > 0$

$$\left|[p_\ell^{NC}] \cdot \tau_E\right|^2 \leqslant (1 + 1/\delta)\left|[p_\ell^{NC} - p_{\ell+1}^{NC}] \cdot \tau_E\right|^2 + (1+\delta)\left|[p_{\ell+1}] \cdot \tau_E\right|^2$$

holds. Since $p_{\ell+1}^{NC} - p_\ell^{NC}$ is a piecewise constant function on $\mathcal{T}_{\ell+1}$, it follows that

$$\sum_{E \in \mathcal{E}_{\ell+1}} h_E^2 \left|[p_{\ell+1}^{NC} - p_\ell^{NC}] \cdot \tau_E\right|^2 \lesssim \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2.$$

For some positive generic constant C, the incorporation of the bulk criterion leads to

$$\begin{aligned}
&\eta_{\ell+1}^2 + \|h_{\ell+1} f\|_{L^2(\Omega)}^2 \\
&\quad\leqslant (1+\delta)\eta_\ell^2(\mathcal{E}_\ell \cap \mathcal{E}_{\ell+1}) + (1+\delta)/2\,\eta_\ell^2(\mathcal{F}) + \left(1/2\,\|h_\ell f\|_{\mathcal{F}}^2 + \|h_\ell f\|_{\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}}^2\right) \\
&\quad+ (1+1/\delta) C \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2 \\
&\quad\leqslant (1+\delta)\eta_\ell^2 - (1+\delta)/2\,\eta_\ell^2(\mathcal{F}) + (1+\delta)\|h_\ell f\|_{L^2(\Omega)}^2 - (1+\delta)/2\,\|h_\ell f\|_{\mathcal{F}}^2 \\
&\quad- \delta\left(\|h_\ell f\|_{\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}}^2 + 1/2\,\|h_\ell f\|_{\mathcal{F}}^2\right) + (1+1/\delta) C \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2 \\
&\quad\leqslant (1+\delta)(1-\theta/2)\left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2\right) - \delta\left(\|h_\ell f\|_{\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}}^2 + 1/2\,\|h_\ell f\|_{\mathcal{F}}^2\right) \\
&\quad+ (1+1/\delta) C \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2 \\
&\quad\leqslant \rho\left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2\right) + C(1+1/\delta)\left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

For $\delta < \theta/(2-\theta)$ reduction of μ_ℓ with $\rho := (1+\delta)(1-\theta/2) < 1$ and $\Lambda := C(1+1/\delta)$ is ensured, which proves the assertion. □

Lemma 3.2 (Convergence). *There exist positive α, β and $0 < \varrho < 1$ depending on the positive generic constants C_{rel}, C_{qo}, and Λ from Lemmas 2.1, 2.2, and 3.1 such that on each level $\ell \geqslant 0$ of* ANCFEM *there is a contraction*

$$\xi_{\ell+1}^2 \leqslant \varrho \xi_\ell^2$$

of the weighted term $\xi_\ell^2 := \eta_\ell^2 + \alpha \|h_\ell f\|_{L^2(\Omega)}^2 + \beta\varepsilon_\ell^2$.

Proof. Again, let $\mathcal{F} = \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+1} \cup \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ be the set of all elements and edges refined from level ℓ to $\ell+1$. Lemma 3.1 shows that for any $0 < \theta < 1$ there exist $0 < \rho < 1$ and $0 < \Lambda$ such that

$$\mu_{\ell+1}^2 \leqslant \rho\mu_\ell^2 + \Lambda \left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2, \text{ and } \|h_{\ell+1} f\|_{L^2(\Omega)}^2 \leqslant \|h_\ell f\|_{L^2(\Omega)}^2 - \frac{1}{2}\|h_\ell f\|_{\mathcal{F}}^2.$$

The quasi-orthogonality of Lemma 2.2 proves

$$\left\|p_{\ell+1}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}^2 \leqslant \varepsilon_\ell^2 - \varepsilon_{\ell+1}^2 + 2C_{\mathrm{qo}}^{1/2}\|h_\ell f\|_{\mathcal{F}}\,\varepsilon_{\ell+1}. \tag{3.2}$$

Let $\beta := \Lambda(1 - 1/\gamma_1)$. Given $0 < \delta < \theta(2-\theta)$ and $\rho = (1+\delta)(1-\theta/2) < 1$ from the previous lemma, there exist γ_1, γ_2, α satisfying the following conditions:

$$0 < \gamma_2 < \min\left\{\Lambda, \frac{1-\rho}{C_{\mathrm{rel}}}\right\}, \qquad 1 < \frac{\Lambda}{\gamma_2} < \gamma_1, \qquad 0 < 2C_{\mathrm{qo}}\Lambda\gamma_1 - 1 < \alpha.$$

Thus, the Young inequality and (3.2) imply

$$\mu_{\ell+1}^2 \leqslant \rho\mu_\ell^2 + \Lambda\left(\varepsilon_\ell^2 - \varepsilon_{\ell+1}^2 + C_{\mathrm{qo}}\gamma_1\|h_\ell f\|_{\mathcal{F}}^2 + \varepsilon_{\ell+1}^2/\gamma_1\right),$$

which results in

$$\begin{aligned}
&\eta_{\ell+1}^2 + \alpha\|h_{\ell+1} f\|_{L^2(\Omega)}^2 + \Lambda(1 - 1/\gamma_1)\varepsilon_{\ell+1}^2 \\
&\quad\leqslant \rho\eta_\ell^2 + \left(C_{\mathrm{qo}}\Lambda\gamma_1 - \frac{\alpha-1}{2}\right)\|h_\ell f\|_{\mathcal{F}}^2 + (\rho + \alpha - 1)\|h_\ell f\|_{L^2(\Omega)}^2 + \Lambda\varepsilon_\ell^2 \\
&\quad\leqslant (\rho + C_{\mathrm{rel}}\gamma_2)\eta_\ell^2 + \left(C_{\mathrm{qo}}\Lambda\gamma_1 - \frac{\alpha-1}{2}\right)\|h_\ell f\|_{\mathcal{F}}^2 \\
&\quad\quad + (\rho + \alpha - 1 + C_{\mathrm{rel}}\gamma_2)\|h_\ell f\|_{L^2(\Omega)}^2 + (\Lambda - \gamma_2)\varepsilon_\ell^2.
\end{aligned}$$

Hence, $\xi_\ell^2 := \eta_\ell^2 + \alpha\|h_\ell f\|_{L^2(\Omega)}^2 + \beta\varepsilon_{\ell+1}^2$ satisfies the contraction property $\xi_{\ell+1}^2 < \varrho\xi_\ell^2$ with α, β as above, and

$$0 < \varrho := \max\left\{\rho + C_{\mathrm{rel}}\gamma_2, 1 - \frac{1-\rho-C_{\mathrm{rel}}\gamma_2}{\alpha}, \frac{\Lambda - \gamma_2}{\Lambda(1-1/\gamma_1)}\right\} < 1. \qquad \square$$

The remaining part of this section proves the optimal convergence of ANCFEM in the spirit of Stevenson [11]. Given $s > 0$, the approximation class $\mathcal{A}_s$ is defined via

$$\begin{aligned}
\mathcal{A}_s &:= \left\{(u,f) \mid \|(u,f)\|_{\mathcal{A}_s} < \infty\right\} \text{ and} \\
\|(u,f)\|_{\mathcal{A}_s}^2 &:= \sup_{N\in\mathbb{N}}\left(N^s \inf_{|\mathcal{T}|-|\mathcal{T}_0|\leqslant N}\left(\varepsilon_{\mathcal{T}}^2 + \|h_{\mathcal{T}} f\|_{L^2(\Omega)}^2\right)^{1/2}\right).
\end{aligned}\tag{3.3}$$

Thus, $(u,f) \in \mathcal{A}_s$ if and only if for all $\epsilon > 0$ there exists an admissible triangulation $\mathcal{T}_\epsilon$ refined from $\mathcal{T}_0$ such that the NC-solution satisifies

$$\varepsilon^2(\mathcal{T}_\epsilon) + \|h_\epsilon f\|^2_{L^2(\Omega)} \leqslant \epsilon^2 \text{ and } |\mathcal{T}_\epsilon| - |\mathcal{T}_0| \lesssim \epsilon^{-1/s} \|(u,f)\|^{1/s}_{\mathcal{A}_s}.$$

There is equivalence of ξ_ℓ with $\varepsilon_\ell^2 + \|h_\ell f\|^2_{L^2(\Omega)}$, i.e.,

$$\varepsilon_\ell^2 + \|h_\ell f\|^2_{L^2(\Omega)} \approx \eta_\ell^2 + \|h_\ell f\|^2_{L^2(\Omega)} + \varepsilon_\ell^2 \approx \xi_\ell^2 \tag{3.4}$$

with constants depending on α, and β, as well as on the efficiency and reliability.

Without violating the quasi-orthogonality in Lemma 2.2 we may enlarge C_{qo} in order to satisfy $1/4 \leqslant C_{\mathrm{qo}}$ as assumed in Lemma 3.3 or even $\max\{2, c_{\mathrm{eff}}\} < 8C_{\mathrm{qo}}$ in Theorem 3.1.

Lemma 3.3. *Let $1/4 \leqslant C_{\mathrm{qo}}$ with C_{qo} from Lemma 2.2 and $\mathcal{T}_{\ell+\epsilon} := \mathcal{T}_\epsilon \oplus \mathcal{T}_\ell$ be the overlay, which is the coarsest common refinement, of two regular triangulations $\mathcal{T}_\epsilon$ and $\mathcal{T}_\ell$ refined by NVB from $\mathcal{T}_0$. Then the flux-error on $\mathcal{T}_{\ell+\epsilon}$ is bounded as follows:*

$$\varepsilon^2_{\ell+\epsilon} \leqslant 8C_{\mathrm{qo}}(\varepsilon_\epsilon^2 + \|h_\epsilon f\|^2_{L^2(\Omega)} - \|h_{\ell+\epsilon} f\|^2_{L^2(\Omega)}).$$

Proof. Let $\mathcal{F}_\epsilon := \mathcal{E}_\epsilon \setminus \mathcal{E}_{\ell+\epsilon} \cup \mathcal{T}_\epsilon \setminus \mathcal{T}_{\ell+\epsilon}$ be the set of edges and elements in $\mathcal{T}_\epsilon$ that have been refined in $\mathcal{T}_{\ell+\epsilon}$. Hence, Lemma 2.2, the Young inequality, and $\|h_{\ell+\epsilon} f\|^2_{L^2(\Omega)} \leqslant \|h_\epsilon f\|^2_{L^2(\Omega)} - \frac{1}{2}\|h_\epsilon f\|^2_{\mathcal{F}_\epsilon}$ lead to the assertion

$$\varepsilon^2_{\ell+\epsilon} \leqslant 2\varepsilon_\epsilon^2 + 4C_{\mathrm{qo}}\|h_\epsilon f\|^2_{\mathcal{F}_\epsilon} \leqslant 2\varepsilon_\epsilon^2 + 8C_{\mathrm{qo}}\|h_\epsilon f\|^2_{L^2(\Omega)} - 8C_{\mathrm{qo}}\|h_{\ell+\epsilon} f\|^2_{L^2(\Omega)}. \qquad \square$$

Theorem 3.1 (Optimal Convergence Rates). *Let $\max\{2, c_{\mathrm{eff}}\} < 8C_{\mathrm{qo}}$ for positive generic constants c_{eff}, and C_{qo} from Lemmas 2.1 and 2.2. Furthermore, let $C_{\mathrm{drel}} > 0$ be given from Theorem 2.1, and positive constants α, β from Lemma 3.2, and u be the exact solution and f the right-hand side of the Poisson model problem* (1.1)*. Then, for $(u,f) \in \mathcal{A}_s$ and $0 < \theta < 1$ sufficiently small, i.e.,*

$$0 < \theta < \theta_0 := \min\left\{1, \frac{c_{\mathrm{eff}}}{C_{\mathrm{drel}}}, \frac{c_{\mathrm{eff}}}{C_{\mathrm{drel}} + C_{\mathrm{qo}} + c_{\mathrm{eff}}}\right\},$$

the algorithm ANCFEM *starting on a coarse, regular triangulation $\mathcal{T}_0$ generates a sequence of triangulations with an optimal number of elements*

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \xi_\ell^{-1/s} \quad \text{with } \xi_\ell^2 := \eta_\ell^2 + \alpha\|h_\ell f\|^2_{L^2(\Omega)} + \beta\varepsilon_\ell^2.$$

Proof. Given c_{eff}, C_{qo}, C_{drel}, α, β, and $\theta < \theta_0$ as assumed, then τ can be chosen such that

$$0 < \tau^2 < \min\left\{\frac{c_{\mathrm{eff}} - \theta C_{\mathrm{drel}}}{c_{\mathrm{eff}} + 1}, \frac{c_{\mathrm{eff}} - \theta(C_{\mathrm{drel}} + C_{\mathrm{qo}} + c_{\mathrm{eff}})}{c_{\mathrm{eff}}(1-\theta) + 1}, 1\right\} < 1 \tag{3.5}$$

and set

$$\epsilon^2 := \frac{\tau^2}{8C_{\mathrm{qo}}}\left(\eta_\ell^2 + \|h_\ell f\|^2_{L^2(\Omega)} + \varepsilon_\ell^2\right).$$

Thus, due to (3.3) on any level $\ell \geqslant 0$ there exists a regular triangulation $\mathcal{T}_\epsilon$ refined from $\mathcal{T}_0$ using NVB satisfying

$$\varepsilon_\epsilon^2 + \|h_\epsilon f\|_{L^2(\Omega)}^2 \leqslant \epsilon^2, \text{ and } |\mathcal{T}_\epsilon| - |\mathcal{T}_0| \lesssim \epsilon^{-1/s}.$$

The number of elements of the overlay $\mathcal{T}_{\ell+\epsilon} := \mathcal{T}_\epsilon \oplus \mathcal{T}_\ell$ is bounded as follows [7, 8]:

$$|\mathcal{T}_{\ell+\epsilon}| - |\mathcal{T}_\ell| = |\mathcal{T}_\epsilon \oplus \mathcal{T}_\ell| - |\mathcal{T}_\ell| \leqslant |\mathcal{T}_\epsilon| - |\mathcal{T}_0|.$$

Let $\mathcal{F}_\ell := \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+\epsilon} \cup \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+\epsilon}$ be the set of edges and elements in $\mathcal{T}_\ell$ being refined in $\mathcal{T}_{\ell+\epsilon}$. By Theorem 2.1, and the aforementioned estimates and (3.4) the set $\mathcal{F}_\ell$ satisfies

$$|\mathcal{F}_\ell| \lesssim |\mathcal{T}_{\ell+\epsilon}| - |\mathcal{T}_\ell| \leqslant |\mathcal{T}_\epsilon| - |\mathcal{T}_0| \lesssim \epsilon^{-1/s} \approx \left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2 + \varepsilon_\ell^2\right)^{-1/s} \approx \xi_\ell^{-1/s}.$$

Next, we prove that $\mathcal{F}_\ell$ satisfies the bulk criterion $\theta\mu_\ell^2 \leqslant \mu_\ell^2(\mathcal{F}_\ell)$. A direct consequence of Lemma 3.3 reads

$$\varepsilon_{\ell+\epsilon}^2 \leqslant \tau^2 \left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2 + \varepsilon_\ell^2\right) - 8C_{\text{qo}} \|h_{\ell+\epsilon} f\|_{L^2(\Omega)}^2.$$

Together with the quasi-orthogonality (cf. Lemma 2.2) and the discrete reliability (2.3) this proves

$$\begin{aligned}\varepsilon_\ell^2 &\leqslant C_{\text{drel}}\eta_\ell^2(\mathcal{F}_\ell) + (C_{\text{drel}} + C_{\text{qo}}) \|h_\ell f\|_{\mathcal{F}_\ell}^2 + 2\varepsilon_{\ell+\epsilon}^2 \\ &\leqslant C_{\text{drel}}\eta_\ell^2(\mathcal{F}_\ell) + (C_{\text{drel}} + C_{\text{qo}}) \|h_\ell f\|_{\mathcal{F}_\ell}^2 + \tau^2\left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2 + \varepsilon_\ell^2\right) - 8C_{\text{qo}} \|h_{\ell+\epsilon} f\|_{L^2(\Omega)}^2.\end{aligned}$$

Furthermore, efficiency (2.1) shows

$$\begin{aligned}c_{\text{eff}}\eta_\ell^2 \leqslant \varepsilon_\ell^2 \leqslant{}& \frac{1}{1-\tau^2}\left(C_{\text{drel}}\eta_\ell^2(\mathcal{F}_\ell) + (C_{\text{drel}} + C_{\text{qo}}) \|h_\ell f\|_{\mathcal{F}_\ell}^2\right) \\ &+ \frac{1}{1-\tau^2}\left(\tau^2\left(\eta_\ell^2 + \|h_\ell f\|_{L^2(\Omega)}^2\right) - 8C_{\text{qo}} \|h_{\ell+\epsilon} f\|_{L^2(\Omega)}^2\right).\end{aligned}$$

Finally some reordering leads to

$$\begin{aligned}\left(c_{\text{eff}} - \frac{\tau^2}{1-\tau^2}\right)\mu_\ell^2 \leqslant \frac{C_{\text{drel}}}{1-\tau^2}\eta_\ell^2(\mathcal{F}_\ell) &+ \left(\frac{C_{\text{drel}} + C_{\text{qo}}}{1-\tau^2} + c_{\text{eff}}\right)\|h_\ell f\|_{\mathcal{F}_\ell}^2 \\ &+ \left(c_{\text{eff}} - \frac{8C_{\text{qo}}}{1-\tau^2}\right)\|h_\ell f\|_{\mathcal{T}_\ell \cap \mathcal{T}_{\ell+\epsilon}}^2.\end{aligned}$$

Due to $\max\{2, c_{\text{eff}}\} < 8C_{\text{qo}}$ and the choice of τ in (3.5) the last term is negative, i.e., $c_{\text{eff}} - 8C_{\text{qo}}/(1-\tau^2) < 0$, and $\mathcal{F}_\ell$ satisfies the bulk criterion $\theta\mu_\ell^2 \leqslant \mu_\ell^2(\mathcal{F}_\ell)$ on each level ℓ for $\theta < \theta_0$.

Since α, $\beta > 0$ are chosen according to the contraction property of Lemma 3.2, the assertion follows by $|\mathcal{F}_\ell| \lesssim \xi_\ell^{-1/s}$, i.e.,

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leqslant \sum_{j=0}^{\ell-1} |\mathcal{T}_{j+1}| - |\mathcal{T}_j| \lesssim \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \lesssim \sum_{j=0}^{\ell-1} |\mathcal{F}_j| \leqslant \sum_{j=0}^{\ell-1} \xi_j^{-1/s}$$

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \xi_\ell^{-1/s} \sum_{j=1}^{\ell-1} \varrho^{-j/(2s)} = \frac{1-\varrho^{-(\ell+1)/(2s)}}{1-\varrho^{-1/(2s)}}\xi_\ell^{-1/s} \lesssim \xi_\ell^{-1/s}. \qquad \square$$

4. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. One main argument therein is the following discrete Poincaré inequality, which is a result of [4] and will be verified first.

Lemma 4.1 (Discrete Poincaré Inequality). *Let $\alpha_{\ell+k}^{NC} \in P_{1,0}^{NC}(\mathcal{T}_{\ell+k})$, and $\alpha_\ell^{NC} \in P_{1,0}^{NC}(\mathcal{T}_\ell)$ with integral means satisfying $\fint_E \alpha_{\ell+k}^{NC}\,\mathrm{d}s = \fint_E \alpha_\ell^{NC}\,\mathrm{d}s$ for any $E \in \mathcal{E}_\ell$. Then, for any $T \in \mathcal{T}_\ell$ the following discrete Poincaré inequality holds:*

$$\left\|\alpha_{\ell+k}^{NC} - \alpha_\ell^{NC}\right\|_{L^2(T)} \lesssim |T|^{1/2} \left\|\nabla_{\ell+k}\alpha_{\ell+k}^{NC}\right\|_{L^2(T)}.$$

Proof. Let $T \in \mathcal{T}_\ell$, and $\alpha_T := \fint_T u^{NC}\,\mathrm{d}x$ be a piecewise constant function on $\mathcal{T}_\ell$ with $u^{NC} := \alpha_{\ell+k}^{NC} - \alpha_\ell^{NC} \in P_1^{NC}(\mathcal{T}_{\ell+k})$. Then, the affine transformation of $T \in \mathcal{T}_\ell$ on T_{ref} yields functions defined on T_{ref}, each marked by $\hat{\cdot}$ (e.g., $\hat\alpha_{\ell+k}^{NC}$, $\hat\alpha_\ell^{NC}$, $\hat u^{NC}$, $\hat\alpha_T$).

Since $\fint_T \left(u^{NC} - \alpha_T\right)\mathrm{d}x$ vanishes, the result of S. Brenner [4]

$$\left\|\left(\hat\alpha_{\ell+k}^{NC} - \hat\alpha_\ell^{NC}\right) - \hat\alpha_T\right\|_{L^2(T_{\mathrm{ref}})} \lesssim \left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})} \tag{4.1}$$

applies. Let $\mathcal{T}$ be the refinement of T_{ref} corresponding to the refinement of $T \in \mathcal{T}_\ell$ in $\mathcal{T}_{\ell+k}$. Thus, $\|\hat\alpha_T\|_{L^2(T_{\mathrm{ref}})} \lesssim \left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})}$ is proven by means of the Hölder and Friedrich inequalities, namely

$$\begin{aligned}
\|\hat\alpha_T\|_{L^2(T_{\mathrm{ref}})} &= \left|\int_{T_{\mathrm{ref}}} \hat u^{NC}\,\mathrm{div}\,x\,\mathrm{d}x\right| \lesssim \left|\sum_{K\in\mathcal{T}}\left(-\int_K \nabla_{\ell+k}\hat u^{NC}\,x\,\mathrm{d}x + \int_{\partial K}\hat u^{NC}\,x\,\mathrm{d}s\right)\right| \\
&\lesssim \left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})} + \sum_{E\in\mathcal{E}(\mathcal{T})}\int_E \left[\hat u^{NC}\right]_E x\cdot\nu_E\,\mathrm{d}s \\
&\lesssim \left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})} + \sum_{E\in\mathcal{E}(\mathcal{T})}\int_E \left[\hat u^{NC}\right]_E (x - x_E^\star)\cdot\nu_E\,\mathrm{d}s
\end{aligned}$$

with $x^\star$ chosen fixed for each edge E, such that $(x - x_E^\star)\perp\nu_E$ for all $x \in E$. Next the equivalence $\int_{T_{\mathrm{ref}}}\nabla_\ell\hat\alpha_\ell^{NC}\,\mathrm{d}x = \int_{T_{\mathrm{ref}}}\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x$ is proven,

$$\begin{aligned}
\int_{T_{\mathrm{ref}}}\nabla_\ell\hat\alpha_\ell^{NC}\,\mathrm{d}x &= \int_{\partial T_{\mathrm{ref}}}\hat\alpha_\ell^{NC}\cdot\nu_{T_{\mathrm{ref}}}\,\mathrm{d}s = \sum_{E\in\mathcal{E}(T_{\mathrm{ref}})}\int_E \hat\alpha_\ell^{NC}\cdot\nu_E\,\mathrm{d}s = \sum_{E\in\mathcal{E}(T_{\mathrm{ref}})}\int_E \hat\alpha_{\ell+k}^{NC}\cdot\nu_E\,\mathrm{d}s \\
&= \int_{T_{\mathrm{ref}}}\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x - \int_{\bigcup\mathcal{E}(\mathcal{T})\cap\mathrm{int}(T_{\mathrm{ref}})}\left[\hat\alpha_{\ell+k}^{NC}\cdot\nu_E\right]\mathrm{d}s = \int_{T_{\mathrm{ref}}}\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x.
\end{aligned}$$

The application of this equality proves $\left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})}^2 \lesssim \|\nabla_{\ell+k}\hat\alpha_{\ell+k}\|_{L^2(T_{\mathrm{ref}})}^2$, i.e.,

$$\begin{aligned}
\left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})}^2 &= \int_{T_{\mathrm{ref}}}\left(\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC} - \nabla_\ell\hat\alpha_\ell^{NC}\right)\left(\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC} - \nabla_\ell\hat\alpha_\ell^{NC}\right)\mathrm{d}x \\
&= \int_{T_{\mathrm{ref}}}\left(\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC} - \nabla_{\ell+k}\hat\alpha_\ell^{NC}\right)\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x \\
&= \|\nabla_{\ell+k}\hat\alpha_{\ell+k}\|_{L^2(T_{\mathrm{ref}})}^2 - \nabla_\ell\hat\alpha_\ell^{NC}\int_{T_{\mathrm{ref}}}\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x \\
&\lesssim \|\nabla_{\ell+k}\hat\alpha_{\ell+k}\|_{L^2(T_{\mathrm{ref}})}^2 - \int_{T_{\mathrm{ref}}}\nabla_\ell\hat\alpha_\ell^{NC}\,\mathrm{d}x\int_{T_{\mathrm{ref}}}\nabla_{\ell+k}\hat\alpha_{\ell+k}^{NC}\,\mathrm{d}x \\
&\lesssim \|\nabla_{\ell+k}\hat\alpha_{\ell+k}\|_{L^2(T_{\mathrm{ref}})}^2.
\end{aligned}$$

Finally, taking into account the previous estimates one verifies

$$\begin{aligned}\left\|\hat\alpha^{NC}_{\ell+k}-\hat\alpha^{NC}_{\ell}\right\|_{L^2(T_{\mathrm{ref}})} &\leqslant \left\|\hat u^{NC}-\hat\alpha_T\right\|_{L^2(T_{\mathrm{ref}})}+\left\|\hat\alpha_T\right\|_{L^2(T_{\mathrm{ref}})} \lesssim \left\|\nabla_{\ell+k}\hat u^{NC}\right\|_{L^2(T_{\mathrm{ref}})}\\ &\lesssim \left\|\nabla_{\ell+k}\hat\alpha^{NC}_{\ell+k}\right\|_{L^2(T_{\mathrm{ref}})}.\end{aligned}$$

A careful transformation back from T_{ref} to $T\in\mathcal{T}_\ell$ leads to the factor $|T|^{1/2}$ in the assertion

$$\left\|\alpha^{NC}_{\ell+k}-\alpha^{NC}_{\ell}\right\|_{L^2(T)} \lesssim |T|^{1/2}\left\|\nabla_{\ell+k}\alpha^{NC}_{\ell+k}\right\|_{L^2(T)}. \qquad \square$$

Proof of Theorem 2.1. A verification of $|\mathcal{F}_\ell\cap\mathcal{E}_\ell| \lesssim |\mathcal{T}_\ell|-|\mathcal{T}_{\ell+k}|$ is given, e.g., in [7]. Since $|\mathcal{F}\cap\mathcal{T}_\ell| \approx |\mathcal{F}\cap\mathcal{E}_\ell| \lesssim |\mathcal{T}_\ell|-|\mathcal{T}_{\ell+k}|$ the assertion $|\mathcal{F}| \lesssim |\mathcal{T}_{\ell+k}|-|\mathcal{T}_\ell|$ follows directly.

To prove (2.3), we recall the discrete Helmholtz decomposition of $\delta := p^{NC}_{\ell+k}-p^{NC}_{\ell}$. There exists $\alpha^{NC}_{\ell+k}\in P^{NC}_{1,0}(\mathcal{T}_{\ell+k})$, and $\beta^{C}_{\ell+k}\in\hat P_1(\mathcal{T}_{\ell+k})$ with $\hat P_1(\mathcal{T}_{\ell+k}) := \left\{v\in P^C_1(\mathcal{T}_{\ell+k}) \,\middle|\, ⨍_\Omega v=0\right\}$ such that

$$\delta := p^{NC}_{\ell+k}-p^{NC}_{\ell} = \nabla_{\ell+k}\alpha^{NC}_{\ell+k}+\operatorname{Curl}\beta^{C}_{\ell+k}.$$

To estimate $\left\|p^{NC}_{\ell+k}-p^{NC}_{\ell}\right\|^2_{L^2(\Omega)}$, each summand of the right-hand side of

$$\left\|p^{NC}_{\ell+k}-p^{NC}_{\ell}\right\|^2_{L^2(\Omega)} = \int_\Omega \delta\cdot\nabla_{\ell+k}\alpha^{NC}_{\ell+k}\,\mathrm{d}x+\int_\Omega\delta\cdot\operatorname{Curl}\beta^{C}_{\ell+k},$$

is bounded separately in the sequel. Let $\alpha^{NC}_{\ell}\in P^{NC}_1(\mathcal{T}_\ell)$ such that for all $E\in\mathcal{E}_\ell$ $⨍_E\alpha^{NC}_\ell = ⨍_E\alpha^{NC}_{\ell+k}$ holds. Thus,

$$\begin{aligned}\int_\Omega\delta\cdot\nabla_{\ell+k}\alpha^{NC}_{\ell+k}\,\mathrm{d}x &= \left(f,\alpha^{NC}_{\ell+k}\right)_{L^2(\Omega)}-\sum_{E\in\mathcal{E}_\ell}\int_E[p^{NC}_\ell\alpha^{NC}_{\ell+k}]_E\nu_E\,\mathrm{d}s\\ &= \left(f,\alpha^{NC}_{\ell+k}-\alpha^{NC}_\ell\right)_{L^2(\Omega)}-\sum_{E\in\mathcal{E}_\ell}\int_E[p^{NC}_\ell(\alpha^{NC}_{\ell+k}-\alpha^{NC}_\ell)]_E\nu_E\,\mathrm{d}s\\ &= \sum_{T\in\mathcal{T}_\ell}\left(f,\alpha^{NC}_{\ell+k}-\alpha^{NC}_\ell\right)_{L^2(T)} \leqslant \sum_{T\in\mathcal{F}\cap\mathcal{T}_\ell}\|f\|_{L^2(T)}\left\|\alpha^{NC}_{\ell+k}-\alpha^{NC}_\ell\right\|_{L^2(T)}.\end{aligned}$$

The discrete Poincaré inequality of Lemma 4.1 states

$$\left\|\alpha^{NC}_{\ell+k}-\alpha^{NC}_\ell\right\|_{L^2(T)} \lesssim |T|^{1/2}\left\|\nabla_{\ell+k}\alpha^{NC}_{\ell+k}\right\|_{L^2(T)} \leqslant |T|^{1/2}\left\|p^{NC}_{\ell+k}-p^{NC}_\ell\right\|_{L^2(\Omega)},$$

which implies

$$\begin{aligned}\int_\Omega\delta\cdot\nabla_{\ell+k}\alpha^{NC}_{\ell+k}\,\mathrm{d}x &\lesssim \left\|p^{NC}_{\ell+k}-p^{NC}_\ell\right\|_{L^2(\Omega)}\sum_{T\in\mathcal{F}}\|f\|_{L^2(T)}|T|^{1/2}\\ &\lesssim \left\|p^{NC}_{\ell+k}-p^{NC}_\ell\right\|_{L^2(\Omega)}\|h_\ell f\|_{\mathcal{F}}.\end{aligned}$$

To bound the second part of the Helmholtz decomposition, let $\mathcal{I}_\ell$ be the Scott-Zhang interpolation operator on $\mathcal{T}_\ell$ and $\beta^C_\ell := \mathcal{I}_\ell\beta^C_{\ell+k}$ with $\left\|\beta^C_{\ell+k}-\beta^C_\ell\right\|_{L^2(E)}=0$ for $E\in\mathcal{E}_{\ell+k}\cap\mathcal{E}_\ell$. In 2D, it holds

$$\left\|\beta^C_{\ell+k}-\beta^C_\ell\right\|_{L^2(E)} \leqslant C\,|E|^{1/2}\left|\beta^C_{\ell+k}\right|_{H^1(\omega_E)}.$$

Thus, the L^2 orthogonalities

$$\left(p_{\ell+k}^{NC}, \operatorname{Curl} \beta_{\ell+k}^{C}\right)_{L^2(\Omega)} = 0 = \left(p_{\ell}^{NC}, \operatorname{Curl} \beta_{\ell}^{C}\right)_{L^2(\Omega)}$$

lead to

$$\begin{aligned}
\int_\Omega (p_{\ell+k}^{NC} - p_\ell^{NC}) \cdot \operatorname{Curl} \beta_{\ell+k}^C \,\mathrm{d}x &= \int_\Omega -p_\ell^{NC} \operatorname{Curl} \beta_{\ell+k}^C = -\int_\Omega p_\ell^{NC} \operatorname{Curl}\left(\beta_{\ell+k}^C - \beta_\ell^C\right) \\
&= -\sum_{T\in\mathcal{T}_\ell} \int_T \operatorname{curl} p_\ell^{NC} (\beta_{\ell+k}^C - \beta_\ell^C)\,\mathrm{d}x + \sum_{E\in\mathcal{E}_\ell} \int_E [p_\ell^{NC}]_E \cdot \tau_E (\beta_{\ell+k}^C - \beta_\ell^C)\,\mathrm{d}s \\
&\leqslant \sum_{E\in\mathcal{F}} \left\|[p_\ell^{NC}]_E \cdot \tau_E\right\|_{L^2(E)} \left\|\beta_{\ell+k}^C - \beta_\ell^C\right\|_{L^2(E)} \\
&\lesssim \sum_{E\in\mathcal{F}\cap\mathcal{E}_\ell} \left\|[p_\ell^{NC}]_E \cdot \tau_E\right\|_{L^2(E)} |E|^{1/2} \left|\beta_{\ell+k}\right|_{H^1(\omega_E)} \\
&\lesssim \eta_\ell(\mathcal{F}) \left\|p_{\ell+k}^{NC} - p_\ell^{NC}\right\|_{L^2(\Omega)}.
\end{aligned}$$

Taking into account the estimates for both parts of the Helmholtz decomposition, the assertion follows directly. □

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