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LAVRENTIEV REGULARIZATION AND BALANCING PRINCIPLE FOR SOLVING ILL-POSED PROBLEMS WITH MONOTONE OPERATORS

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Abstract — The paper considers a method for solving nonlinear ill-posed problems with monotone operators. The approach combines the Lavrentiev method, the fixed-point method, and the balancing principle for selection of the regularization parameter. The method's optimality has been proved for some set of smooth solutions. A test example proves the efficiency of the proposed method.

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1. Statement of the problem

The investigation of modern methods for solving nonlinear ill-posed problems in the regularization theory is connected with some well-developed methods for linear problems. The Tikhonov method is considered to be standard for regularization of both linear and nonlinear problems (see, for example, [6, 11]). However, in the nonlinear case this method yields a normal equation, where a Frechet derivative of the forward operator is used. Finding such an operator is not an easy task in practice, but in some cases we can simplify the corresponding computational algorithm. If the forward operator is monotone, then the Tikhonov method reduces to the Lavrentiev method and the approximate solution can be found without the Frechet derivative.

The Lavrentiev method for regularization of nonlinear ill-posed problems with a monotone operator was considered in, e.g., [5, 8, 9, 1]. The optimality of the Lavrentiev method was proved for the case of smooth solution in terms of the sourcewise condition and the general source condition in [9] and [8], respectively. In [1], only the convergence of the Lavrentiev method was established in combination with the fixed-point method. However, in [9] and [8] the obtained regularized equations remain nonlinear and one may have difficulties solving them numerically. That is why in many cases we deal with a sequence of approximations to the solution of the regularized equation, where the corresponding numerical solution is chosen with a given accuracy. In addition, the smaller is the value of the regularization parameter, the more instable is the corresponding equation. Hence the application of numerical techniques requires a further investigation.

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In the present paper, following [1] we propose to solve a nonlinear regularized equation by the fixed-point method. This is possible due to the strict monotonicity of the operator obtained by the Lavrentiev regularization. To achieve optimality in the considered method, we propose to choose an appropriate regularization parameter according to the balancing principle [8]. The main advantage of the balancing principle is its wider applicability in comparison with the discrepancy principle and a simpler realization in comparison, for example, with the monotone rule [10]. Moreover, we establish the optimality of the numerical method from [1] on the basis of the Lavrentiev regularization without strict assumptions on the smoothness of a minimal norm solution.

2. Statement of the problem

We consider the following nonlinear equation in a Hilbert space X with the inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$:

$$F(x) = f, \tag{2.1}$$

where the operator $F : D(F) \subset X \to X$ has a locally uniformly bounded Frechet derivative $F'(\cdot)$ in the domain $D(F)$. Denote by $x = x^\dagger$ the minimal norm solution of equation (2.1). It is natural to assume that (2.1) is an ill-posed problem.

Let F be a monotone operator, namely for all $x, y \in D(F)$ the inequality

$$(F(x) - F(y), x - y) \geqslant 0 \tag{2.2}$$

holds. Suppose that the Lipschitz condition is satisfied for the operator F, namely there exists a constant R such that for all $x, y \in D(F)$

$$||F(x) - F(y)|| \leqslant R||x - y||. \tag{2.3}$$

Note that condition (2.3) can be simplified. It turns out that it is enough to require that (2.3) should hold only for all x from some ball around $x^\dagger$. For simplicity, in this paper we will consider that condition (2.3) is satisfied for all $x, y \in D(F)$.

Assume that instead of an exact right-hand side f we are given only its perturbation $f_\delta \in X$, such that

$$\|f - f_\delta\| \leqslant \delta,$$

where δ is the known error level.

Now our aim is to propose a numerical solution method for (2.1) that is simple for practical implementation, optimal in order, and does not require computing the Frechet derivative. In the case of monotonicity of F, the Lavrentiev method is simpler than other methods. Let the regularized approximation x_α^δ have the following form:

$$x_\alpha^\delta = R_\alpha(f_\delta + \alpha x^0), \tag{2.4}$$

where $R_\alpha = (F + \alpha I)^{-1}$, the parameter $\alpha > 0$ is the regularization parameter, and x^0 is the known initial guess.

Let us describe the approximation properties of the Lavrentiev method more precisely. We denote by x_α the solution of (2.4) when $\delta = 0$. From the general regularization theory it is known that the sequence of operators R_α at $\alpha \to 0$ converges pointwise to F^{-1}. Under some selection of the regularization parameter, this property is sufficient for the convergence of approximations x_α^δ to $x^\dagger$ at $\delta \to 0$, but is not sufficient for the convergence with the optimal rate. Thus, we should introduce a stronger condition on the set of minimal-norm solutions. Taking into account the above consideration, we assume that there exists an increasing function $\varphi(\alpha) := \varphi(\alpha, F, f)$ such that $0 = \varphi(0) \leqslant \varphi(\alpha) \leqslant 1$ and

$$\|x_\alpha - x^\dagger\| \leqslant \varphi(\alpha). \tag{2.5}$$

Remark 2.1. In the general case, the relation (2.5) holds if $x^\dagger \in M$, where M is some compact set. For details see Section 4.2, where a wide class of possible compact sets will be described.

The stability property of the regularization operator R_α is as follows. Since R_α is a strictly monotone operator and its inverse satisfies the Lipshitz condition (2.3) with constant $1/\alpha$ (the proof of this fact follows from [3, pp.97,100]), the norm of the element $\|x_\alpha - x_\alpha^\delta\|$ can be bounded by δ/α. However, this bound is given for the general class of operators and may be significantly improved in special cases. We assume that there exists $c_1 = c_1(F, y)$ such that

$$\|x_\alpha - x_\alpha^\delta\| \leqslant c_1 \frac{\delta}{\alpha}.$$

However, Eq. (2.4) still remains nonlinear, so it is necessary to use a known numerical method for its solving. Now we consider the fixed-point method that consists in constructing iterations according to the rule

$$x_{\alpha_i,k+1}^\delta = G_\alpha(x_{\alpha_i,k}^\delta), \tag{2.6}$$

where $G_\alpha(x) = (I - \gamma(F + \alpha I))(x) + \alpha\gamma x^0 + \gamma f_\delta$, and $\gamma > 0$ is some arbitrary parameter of the method.

Assume that G_α is contractive with the coefficient $\beta < 1$, and the accuracy of the regularized approximation x_α^δ is given by

$$\|x_\alpha^\delta - x^i\| \leqslant \frac{c_z\delta}{\alpha} \tag{2.7}$$

for some constant c_z. Here, by $x^i := x_{\alpha_i,k}^\delta$ we mean the approximate solution of equation (2.4) at $\alpha = \alpha_i$ obtained as a result of stopping the iterative process (2.6) after k steps.

Hence, the general error bound of the method $M_{\alpha,\gamma}$ has the following form:

$$\|x^i - x^\dagger\| \leqslant \|x^\dagger - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\| + \|x_\alpha^\delta - x^i\| \leqslant \varphi(\alpha) + c_z^1 \frac{\delta}{\alpha}, \tag{2.8}$$

where $c_z^1 = c_1 + c_z$.

It is evident that the minimal value of the error bound for method (2.6) is given by

$$\|x^i - x^\dagger\| \leqslant c \inf_\alpha \{\varphi(\alpha) + c_z^1 \frac{\delta}{\alpha}\},$$

and infimum is reached at the point $\alpha_{opt} = \varphi^{-1}(\delta) c_z^1 \delta$ defined as the cross point of the increasing function $\varphi(\lambda)$ and the decreasing function λ/α. Unfortunately, such a priori selection of the parameter cannot be always realized in practice because the precise form of the function $\varphi(\lambda)$ can be unknown. To solve equation (2.6) efficiently, it is necessary to select an appropriate error level δ and regularization parameter α in the computational process. The a posteriori known rules for the Lavrentiev regularization include the discrepancy principle proposed by Bakyshinski and Smirnova [1]. In their paper a similar approach to (2.6) was proposed, but only the convergence of the obtained approximation was proved. One more a posteriori rule was proposed by Tautenhahn [9]. He investigated the Lavrentiev method for solving a nonlinear equation with a monotone operator in the case of sourcewise representation of the solution $x^\dagger$, where α was taken as a solution of the following nonlinear equation:

$$\|\alpha(F'(x_\alpha^\delta) + \alpha I)^{-1}[F(x_\alpha^\delta) - y^\delta]\| = C\delta.$$

In this paper, we propose to choose the parameter α according to the balancing principle established by Pereverzev and Shock [8] for solving ill-posed problems. We denote by D_M the set of possible values of the parameter α

$$D_M = \{\alpha_i = \alpha_0 q^i, i = 0, 1, ..., M\}, \quad q > 1.$$

Then the selection of the numerical value i_+ for the parameter α according to the balancing principle is performed using the rule

$$i_+ = \max\{i : \alpha_i \in D_M^+\}, \tag{2.9}$$

where

$$D_M^+ = \{\alpha_i \in D_M : \|x^i - x^j\| \leqslant \frac{(3c_q + 1)c_z^1\delta}{\alpha_i}, j = 0, 1, ..., i-1\}$$

for some constant $c_q = q^{j-i}$.

In this paper, we apply the balancing principle to choose the regularization parameter in approximately solving (2.1) by the iterative process (2.6). Our task is to prove the optimality of the method under consideration and to find the number of iterations (2.6) required for a given accuracy.

3. Optimality theorem

The following theorem shows that if α and δ are correlated according to the balancing principle, then the method (2.6) is optimal in order.

Theorem 3.1. *Assume that the operator F of Eq. (2.1) is monotone and condition (2.3) takes place. If the solution $x^\dagger$ satisfies condition (2.5), then choosing the index i_+ according to (2.9) ensures that method (2.6) is optimal in order, i.e., the following bound holds:*

$$\|x^\dagger - x^{i_+}\| \leqslant \frac{c\delta}{\alpha_{opt}} = c\varphi(\alpha_{opt}), \tag{3.1}$$

where the constant c is independent of δ.

Proof. The proof of this theorem is based on the technique used in Theorem 3.1 of [6]. Introduce the index

$$i_\star = \max\{i : \alpha_i \in D_M^\star\}, \tag{3.2}$$

where

$$D_M^\star = \{\alpha_i \in D_M : \varphi(\alpha_i) \leqslant \frac{c_z^1\delta}{\alpha_i}\}.$$

Then for all numbers $j \leqslant i_\star$, taking into account (2.8), (3.2), the monotonicity property of the function $\varphi(\lambda)$, and the relation $\alpha_\star = \alpha_j q^{i_\star - j}$ we have

$$\begin{aligned}
\|x^j - x^{i_\star}\| &\leqslant \|x^\dagger - x^j\| + \|x^\dagger - x^{i_\star}\| \\
&\leqslant \varphi(\alpha_j) + \frac{c_z^1\delta}{\alpha_j} + \varphi(\alpha_{i_\star}) + \frac{c_z^1\delta}{\alpha_{i_\star}} \\
&\leqslant \varphi(\alpha_{i_\star}) + \frac{c_z^1\delta}{\alpha_j} + \varphi(\alpha_{i_\star}) + \frac{c_q c_z^1\delta}{\alpha_j} \\
&\leqslant 2\varphi(\alpha_{i_\star}) + \frac{(1+c_q)c_z^1\delta}{\alpha_j} \leqslant \frac{2c_q c_z^1\delta}{\alpha_j} + \frac{(1+c_q)c_z^1\delta}{\alpha_j} \leqslant \frac{(3c_q+1)c_z^1\delta}{\alpha_j},
\end{aligned}$$

where $c_q = q^{j-i_\star} \leqslant 1$.

Therefore, using the definition of the set D_M^+ in (2.9) and the above relation, we obtain that $\alpha_{i_\star} \in D_M^+$ and

$$i_\star \leqslant i_+. \tag{3.3}$$

Then from (2.5) and (3.3) it follows that

$$\|x^\dagger - x^{i_+}\| \leqslant \|x^\dagger - x^{i_\star}\| + \|x^{i_\star} - x^{i_+}\| \leqslant \frac{2c_z^1\delta}{\alpha_{i_\star}} + \frac{4c_z^1\delta}{\alpha_{i_\star}} = \frac{6c_z^1\delta}{\alpha_{i_\star}}. \tag{3.4}$$

Since (3.2) holds only for $i < i_\star$ and $\delta = c_z^1\varphi(\alpha_{opt})\alpha_{opt}$ by the definition of α_{opt} we obtain that for the next element $q\alpha_{i_\star} > \alpha_{i_\star}$ the following relation holds:

$$\varphi(q\alpha_{i_\star})q\alpha_{i_\star} > c_z^1\delta = c_z^1\varphi(\alpha_{opt})\alpha_{opt} > \varphi(\alpha_{opt})\alpha_{opt}. \tag{3.5}$$

From (3.5) in view of the monotonicity of the function φ it follows that

$$\alpha_{opt} < q\alpha_{i_\star}. \tag{3.6}$$

Substituting (3.6) into (3.4) we obtain that

$$\|x^\dagger - x^{i_+}\| \leqslant \frac{6\delta c_z^1}{\alpha_{i_\star}} = \frac{6q\delta c_z^1}{q\alpha_{i_\star}} < \frac{6qc_z^1\delta}{\alpha_{opt}},$$

which proves the statement.

□

4. Discussion

4.1. Fixed-point iteration

Next we find the value of γ guaranteeing that the mapping G_α given in the form (2.6) is contractive, and find the convergence rate of the fixed-point iterations.

Note that the operator $F + \alpha I$ acting in the Hilbert space X is strictly monotone, i.e., the following relation takes place for all $\alpha > 0$:

$$((F+\alpha I)(x) - (F+\alpha I)(y), x - y) \geqslant \alpha\|x-y\|^2, \quad x, y \in X. \tag{4.1}$$

Theorem 4.1. *Let the operator $F + \alpha I$ be strictly monotone and Lipschitz-continuous. Moreover, assume that $\gamma\alpha < 1$. Then the operator G_α is contractive with the constant $\beta = \sqrt{(1-\alpha\gamma)^2 + \gamma^2 R^2}$ if $\gamma < \frac{2\alpha}{\alpha^2+R^2}$,and the following bound takes place:*

$$\|x^\delta_{\alpha_i} - x^i\| \leqslant \frac{\beta^k ||x_1 - x_0||}{1-\beta}. \tag{4.2}$$

Bound (4.2) attains the best value at $\gamma = \frac{\alpha}{\alpha^2+R^2}$ and $\beta = \frac{R}{\sqrt{\alpha^2+R^2}}$.

Proof. To prove this statement for any $x, y \in D(G_\alpha)$, we write the expression $||G_\alpha(x) - G_\alpha(y)||^2$ in the following form:

$$\begin{aligned} &\|G_\alpha(x) - G_\alpha(y)\|^2 = \|(1-\alpha\gamma)(x-y) - \gamma(F(x) - F(y))\|^2 \\ &= ((1-\alpha\gamma)(x-y) - \gamma(F(x) - F(y)), (1-\alpha\gamma)(x-y) - \gamma(F(x) - F(y))) \\ &= (1-\alpha\gamma)^2\|x-y\|^2 - 2(1-\alpha\gamma)\gamma(F(x) - F(y), x-y) + \gamma^2\|F(x) - F(y)\|^2. \end{aligned} \tag{4.3}$$

If $\gamma < 1/\alpha$, due to the monotonicity of the operator F, the second term in (4.3) is nonpositive. Therefore, from (4.3) it follows that

$$\|G_\alpha(x) - G_\alpha(y)\|^2 \leqslant (1-\alpha\gamma)^2\|x-y\|^2 + \gamma^2\|F(x) - F(y)\|^2. \tag{4.4}$$

Next, using (2.3) we obtain

$$\|G_\alpha(x) - G_\alpha(y)\|^2 \leqslant [(1-\alpha\gamma)^2 + \gamma^2 R^2]\|x-y\|^2. \tag{4.5}$$

Thus, the Lipschitz constant β for mapping G_α has the form

$$\beta = \sqrt{(1-\alpha\gamma)^2 + \gamma^2 R^2}. \tag{4.6}$$

Let us find the value of γ guaranteeing that the expression on the right-hand side of (4.6) is less than one. Recall that the condition $\gamma < 1/\alpha$ has already been enforced. It is easy to see that $\beta < 1$ if the following condition holds:

$$\gamma < \frac{2\alpha}{\alpha^2 + R^2}. \tag{4.7}$$

Thus, by the fixed point theorem [4, section 4.3.4], the sufficient condition for the convergence of the fixed point method to the solution of Eq. (2.4) is satisfied by selecting γ in the form

$$\gamma < \min\{\frac{1}{\alpha}, \frac{2\alpha}{\alpha^2 + R^2}\}. \tag{4.8}$$

Besides, the solution of (2.6) is unique for each positive α, and (4.2) holds.

The next question we address is, which γ value results in the fixed point method with the fastest convergence, i.e., we need to find $\gamma = \gamma_{min}$ minimizing the expression in (4.6). We obtain

$$\gamma_{min} = \frac{\alpha}{\alpha^2 + R^2}. \tag{4.9}$$

It is evident that the given value of γ satisfies bound (4.8). The corresponding value for β_{min} is

$$\beta_{min} = \frac{R}{\sqrt{\alpha^2 + R^2}}. \tag{4.10}$$

Finally, the rate of convergence of the fixed-point method is determined by (4.2). □

Corollary 4.1. *As follows from Theorem 4.1, the number of iterations required to achieve some accuracy ϵ can be determined as follows:*

$$N_{iter} \leqslant \log_{\beta} \frac{\epsilon(1-\beta)}{\|x_1 - x_0\|}. \tag{4.11}$$

This number depends on the contractive coefficient β and the value of $\|x_1 - x_0\|$. As observed from the numerical experiments, bound (4.11) is a very rough estimate, especially for small α.

We propose to estimate the number of the remaining iterations after each step by the following formula:

$$N_{iter}^{n} = \log_{\beta} \frac{\epsilon(1-\beta)}{\|x_n - x_{n-1}\|}. \tag{4.12}$$

As soon as the value of N_{iter}^{n} is smaller than one, the computation by the fixed-point method is stopped. It appears that the number of necessary iterations may be smaller compared to bound (4.11).

Remark 4.1. If the conditions of Theorem 4.1 are satisfied not for the entire domain $D(F)$, but only for all $x, y \in B_r(x^{\dagger})$, where $r = 2(\delta/\alpha + \|x^{\dagger} - x^0\|)$, then it is easy to show that Theorem 4.1 also holds true.

4.2. General source condition

In this section, we establish additional assumptions to guarantee condition (2.5). Let us introduce the smoothness properties of the minimal-norm solution given in terms of the general source condition.

Assumption 1. (A1) Let there exist such v that the relation

$$x^0 - x^{\dagger} = \phi(F'(x^{\dagger}))v \tag{4.13}$$

holds, where the index function ϕ, $\phi(0) = 0$ is a continuous nondecreasing function defined on some interval $[0, \sigma]$ containing the spectrum of $F'(x^{\dagger})$.

It is known that for the optimal accuracy of the regularization method the index function ϕ should be covered by the qualification of the regularization method. For the Lavrentiev method under consideration, denote the qualification by $\rho(\alpha)$.

The theory states that the considered method has a low qualification in the case of $\phi(\lambda) = \lambda^p$, namely $\rho(\alpha) := \alpha^p$, $\quad 0 < p \leqslant 1$. Thus, if $0 < p \leqslant 1$, then the accuracy of the method is bounded by the value $O(\delta^{\frac{p}{p+1}})$, but in the case of a sufficiently smooth solution, i.e., $p > 1$, one obtains only an order of accuracy $\delta^{1/2}$. At the same time, in the case of $\phi(\lambda) = \ln^{-p}(\lambda)^{-1}$, the qualification is given by $\rho(\alpha) := \ln^{-p}(\alpha)^{-1}$, where $p \geqslant 1$. Hence we obtain a convergence with the rate $O(\ln^{-p}(\delta^{-1}))$ for any $p > 1$.

Now, based on definition 2 in [7], we make the following assumption:

Assumption 2. (A2) Let there exist æ > 0, such that the relation

$$æ\frac{\rho(\alpha)}{\phi(\alpha)} \leqslant \inf_{\alpha \leqslant \lambda \leqslant \sigma} \frac{\rho(\lambda)}{\phi(\lambda)}, \quad 0 < \alpha \leqslant \sigma. \tag{4.14}$$

takes place.

In addition, we should make one more assumption concerning the Frechet derivative $F'(x)$ introduced in [9].

Assumption 3. (A3) Let there exist a constant $k_0 \geqslant 0$ such that for all $x \in D(F)$ and $\omega \in X$ there is some element $k(x, x^\dagger, \omega) \in X$ with the property

$$[F'(x) - F'(x^\dagger)]\omega = F'(x)k(x, x^\dagger, \omega) \quad \text{and} \quad \|k(x, x^\dagger, \omega)\| \leqslant k_0\|\omega\|.$$

Proposition 4.1. *Let assumptions A1, A2, A3 be satisfied. Then for all $\alpha > 0$*

$$\|x_\alpha - x^\dagger\| \leqslant (1 + k_0)æ^{-1}\|v\|\phi(\alpha). \tag{4.15}$$

Proof. We will use the scheme from the proof of Theorem 3.3 in [9]. We introduce the auxiliary operator

$$L_\alpha = \int_0^1 F'(x^\dagger + t(x_\alpha - x^\dagger))dt$$

and the following notations:

$$A = F'(x^\dagger), \quad K_\alpha = \alpha(A + \alpha I)^{-1}.$$

From the Lagrange theorem in integral form we have $F(x_\alpha) - F(x^\dagger) = L_\alpha(x_\alpha - x^\dagger)$, hence, Eq. (2.4) can be written in the form

$$L_\alpha(x_\alpha - x^\dagger) + \alpha(x_\alpha - x^\dagger) = \alpha(x^0 - x^\dagger).$$

From the monotonicity of (2.2) it follows that the operators $(L_\alpha + \alpha I)$ and $(A + \alpha I)^{-1}$ exist. Then using A1 and the relations $AK_\alpha = K_\alpha A$ and $AL_\alpha = L_\alpha A$, we obtain

$$\begin{aligned} x_\alpha - x^\dagger &= (L_\alpha + \alpha I)^{-1}\alpha(x^\dagger - x^0) = \alpha(A + \alpha I)^{-1}\phi(A)v + \alpha[(L_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1}]\phi(A)v \\ &= K_\alpha\phi(A)v + (L_\alpha + \alpha I)^{-1}(A - L_\alpha)K_\alpha\phi(A)v. \end{aligned}$$

Next we use Proposition 3 from [7]. According to this proposition, one can obtain that for any decreasing function $f(\lambda)$ satisfying A2 the following relation holds:

$$\sup_{0 \leqslant \lambda \leqslant \sigma} \left| \left(1 - \frac{\lambda}{\alpha + \lambda}\right) f(\lambda) \right| = \alpha \sup_{0 \leqslant \lambda \leqslant \sigma} |(\alpha + \lambda)^{-1} f(\lambda)| \leqslant æ^{-1} f(\alpha).$$

Then, taking into account the inequality above and A3, we obtain

$$\begin{aligned} \|x_\alpha - x^\dagger\| &\leqslant \|K_\alpha\phi(A)v\| + \|(L_\alpha + \alpha I)^{-1}L_\alpha k(x, x^\dagger, \omega)K_\alpha\phi(A)v\| \\ &\leqslant \|K_\alpha\phi(A)v\| + k_0\|(L_\alpha + \alpha I)^{-1}L_\alpha\|\|K_\alpha\phi(A)v\| \\ &\leqslant æ^{-1}\phi(\alpha)\|v\| + k_0æ^{-1}\phi(\alpha)\|v\| = æ^{-1}(1 + k_0)\|v\|\phi(\alpha), \end{aligned}$$

which proves the theorem. □

Remark 4.2. From proposition 4.1 it follows that $\varphi(\alpha) = æ^{-1}(1 + k_0)\|v\|\phi(\alpha)$.

4.3. Numerical example

To show the efficiency of the proposed combination of (2.6) and (2.9), we apply our method to the nonlinear equation from [4, §8.1]. Consider the operator F acting in the Hilbert space $L_2(0, 1)$ in the following way:

$$F(x) := \int_0^1 R(t, s)x^3(s)ds = f(t), \tag{4.16}$$

where

$$R(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

First we check that the operator F is monotone. Indeed, since the function $x^3(t)$ is increasing on all real axes and $R(t,s) \geqslant 0$ for all $0 \leqslant t, s \leqslant 1$, the following relation holds for all $x(t), y(t) : x(t) > y(t)$:

$$(F(x) - F(y), x - y) = \int_0^1 \left[\int_0^1 R(t,s)(x^3 - y^3)(s)ds\right](x-y)(t)dt \geqslant 0.$$

The Frechet derivative of F at a point is given by $x^\dagger$

$$F'(x^\dagger)h(s) = 3\int_0^1 R(t,s)(x^\dagger(s))^2 h(s)ds.$$

In our computations, we use the following function for the exact solution:

$$x^\dagger(t) = t^3$$

It is easy to check that if $x^0(t) = t^3 - 3/56t^8 + 3/56t$, then for $\nu = 1$ the function $x^0 - x^\dagger$ satisfies the sourcewise condition

$$x^0 - x^\dagger = (F'(x^\dagger))^\nu \omega.$$

So, we expect to obtain the rate of convergence $O(\delta^{\frac{1}{2}})$.

There are two possible strategies for applying the balancing principle.

- Test the condition

$$\|x^i - x^j\| \leqslant \frac{(3c_q + 1)c_z^1 \delta}{\alpha_i}, j = 0, 1, ..., i-1, \tag{4.17}$$

  starting from a small α_0 and going forward to $\alpha_k = \alpha_0 q^k$;

- Test condition (4.17) starting from a large value α_M and continuing to a smaller regularization parameter $\alpha_{M-k} = \alpha_M q^{-k}$.

In [6], it was noted that the second strategy is more practical. At the same time there appears to be a problem with the size of the grid M, i.e., how small should the parameter α_0 be? Since we expect to obtain an error bound of order $\sqrt{\delta}$, we choose the parameter α_0 in the form

$$\alpha_0 = C\sqrt{\delta}, \tag{4.18}$$

where C is some constant. Accordingly, the size of the grid M equals

$$M = \log_q \frac{\alpha_M}{C\sqrt{\delta}}. \tag{4.19}$$

The following algorithm is a realization of our method:

1. Choose a sufficiently big $\alpha_M = 1$ and $q = 1.25$, $C = 0.8$. This allows to define the grid

$$D_M = \{\alpha_i = \alpha_0 q^i, i = 0, 1, ..., M\}.$$

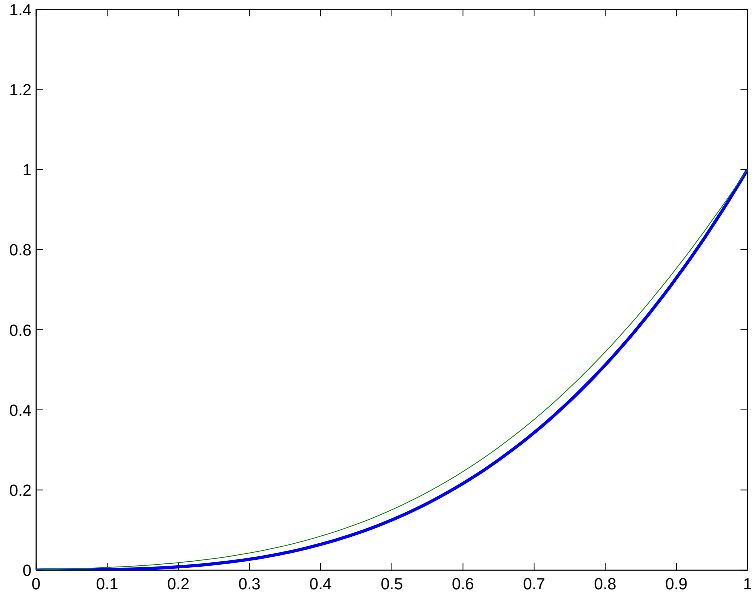

Fig. 4.1. Curve of the exact and approximate solutions

2. Compute the approximation x^i of the equation
$$(A + \alpha_i I)x = f_\delta + \alpha_i x', \tag{4.20}$$
with the accuracy $c_z\delta/\alpha_i$ for $i = M - 1, ..., 0$.
3. For $i = M, M-1, ..., 1$ check condition (4.17). As soon as for some $i = i_+$ this condition is satisfied, the test is stopped. Output the element x_{i_+} as an approximate solution of Eq. (4.20) .

The test computations were performed to find the solution of Eq. (4.16) with error levels corresponding to the data noise of 2% to 7%. The Lipschitz constant R equals approximately 0.21 for all cases. The results of the test computations are presented in Table 1, where the column named "error" contains the accuracy of approximation in $L_2(0, 1)$ metrics. From the $\log - \log$ scale we estimated the convergence rate for our method to be $O(\delta^{0.21})$. The plots of the exact solution (bold line) and the approximate solution obtained for $\delta = 5\%$ are shown in Fig. 1. Table 2 reports the numerical results for $\delta = 5\%$. In this table, the column labeled as "iter2" gives the number of iterations carried out for achieving accuracy (2.7) by (4.12), and the column "iter1" shows the number of iterations according to (4.11).

Table 4.1. **Dependence of the error of approximation on the noise level on the right-hand side**

δ	$error$	α
2%(0.000085)	0.018035	0.0863
3%(0.00013)	0.0204	0.16514
4%(0.0017)	0.02175	0.298
5%(0.00215)	0.022066	0.4164

Table 4.2. **Number of iterations computed using (4.11) and (4.12) for some values of α**

α	$iter1$	$iter2$
0.213	6	4
0.17055	9	5
0.13644	12	8
0.087	27	15
0.045	89	29
0.0229	293	84
0.01465	619	197

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