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On the Recovery of Continuous Functions from Noisy Fourier Coefficients

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Abstract — We consider the classical ill-posed problem of the recovery of continuous functions from noisy Fourier coefficients. For the classes of functions given in terms of generalized smoothness, we present a priori and a posteriori regularization parameter choice realizing an order-optimal error bound.

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1. INTRODUCTION

Let $L_2 = L_2(\Omega)$ be the space of real-valued functions which are Lebesgue square summable (integrable) on a compact measurable space Ω. Denote by $C = C(\Omega)$ the space of continuous functions on Ω.

Assume that the system of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ is orthonormal in L_2 with respect to the standard scalar product $\langle \cdot, \cdot \rangle$, and

$$\sum_{k=1}^{\infty} y_k \cdot \varphi_k(t)$$

is a Fourier series of the function $y(t) \in C$.

Suppose that instead of coefficients y_k their approximate values are given, i.e., we consider a sequence of numbers

$$y_\delta := \left\{ y_{\delta,k} \right\}_{k=1}^{\infty},$$

such that

$$y_{\delta,k} := y_k + \delta \cdot \xi_k, \quad k = 1, 2, ...,$$

where $\xi := \{\xi_k\}_{k=1}^{\infty}$ is the noise. It is assumed that $\delta \in (0, 1)$ and

$$\|\xi\|_{l_2} := \left[\sum_{k=1}^{\infty} |\xi_k|^2 \right]^{1/2} \leqslant 1.$$

As noted in [8], [9], recovery of the function $y(t)$ at any point from the approximate values of its Fourier coefficients is ill-posed, i.e., by direct summation of series

$$\sum_{k=1}^{\infty} y_{\delta,k} \cdot \varphi_k(t)$$

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we cannot obtain the recovery of the function with necessary order of accuracy.

This problem was studied by many authors, for example, in [1], [2], [3], [8], [9]. In [8] a stable method of summation was proposed. Stability of the method was shown for an example using a series of eigenfuntions of the first boundary problem for an ordinary differential equations. In [9] convergence and in [8] stability of the method were shown in the case of an arbitrary orthonormal system $\{\varphi_k(t)\}_{k=1}^{\infty}$ on the class of functions for which

$$\sum_{k=1}^{\infty} \langle y, \varphi_k \rangle^2 \cdot \psi_k < \infty$$

where $\{\psi_k\}_{k=1}^{\infty}$ is the sequence of positive numbers. The order of growth of these numbers as $k \to \infty$ is not less than $k^{2+\varepsilon}$, $\varepsilon \geqslant 0$.

Continuing the approach from [9], in this paper the smoothness of the function $y(t)$ is expressed in terms of spaces W_2^{ψ} associated with the given orthonormal system $\{\varphi_k\}_{k=1}^{\infty}$, i.e.,

$$W_2^{\psi} := \left\{ y \in L_2(\Omega) : \quad \|y\|_{\psi}^2 = \sum_{k=1}^{\infty} \psi^2(k) \cdot |\langle, y, \varphi_k \rangle|^2 < \infty \right\},$$

where $\psi(k)$ is some monotonically increasing function.

In a particular case, using $\psi(k) = k^{\mu}$ we get the space

$$W_2^{\mu} := \left\{ y \in L_2(\Omega) : \quad \|y\|_{\mu}^2 = \sum_{k=1}^{\infty} k^{2\mu} \cdot |\langle y, \varphi_k \rangle|^2 < \infty \right\}.$$

These spaces were considered by P.Mathe and S.Pereverzev in [5].

Let us consider in more detail the method of investigation as in [5]. Consider the following example.

Example 1.1. For a trigonometric system and integer μ, the space W_2^{μ} consists of periodic functions which have square summable derivatives up to the order μ.

Let $\{\varphi_k\}_{k=1}^{\infty}$ be a system of Legendre polynomials, then the respective space W_2^{μ} consists (see [7], [10]) of the functions $y(t)$ for which the derivatives $y^{(i)}$, $i = 1, 2, ..., \mu - 1$ are absolutely continuous on each $[a, b] \subset (0, 1)$ and

$$\int_0^1 |y^{(\mu)}|^2 \cdot t^{\mu} \cdot (1-t)^{\mu} dt < \infty.$$

This means that the highest derivative $y^{(\mu)}$ may have singularities at the points 0 and 1.

Il'in and Pozniak [3] have proved for the trigonometric system that for the case

$$|\langle y, \varphi_k \rangle| = O(k^{-p}), \quad p > 1 + s/2, \quad s > 1/2$$

and the Tikhonov regularization

$$T^{\alpha,s}(y_\delta) := \sum_{k=1}^{\infty} \frac{y_{\delta,k}}{1 + \alpha \cdot k^{2s}} \cdot \varphi_k \tag{1.1}$$

yields

$$\|y - T^{\alpha,s}\|_C \leqslant c \cdot \left(\sqrt{\alpha} + \frac{\delta}{\alpha}\right). \tag{1.2}$$

Here and throughout the paper c denotes constants that can vary from appearance to appearance. In [1], such an estimate was obtained for more general case of any orthonormal system with uniformly bounded functions

$$\|\varphi_k\|_C \leqslant c. \tag{1.3}$$

Note that the uniform boundary condition is not fulfilled for such an important case as the system of Legendre polynomials.

We emphasize that the optimal choice of the regularization parameter α, or the discretization level n, which depends on the noise level δ, is one of the major topics within the theory of ill-posed problems. It is easy to see that the optimal estimate (1.2) is obtained when $\alpha_0 = \delta^{2/3}$

$$\|y - T^{\alpha_0,s}\|_C \leqslant c \cdot \delta^{1/3}.$$

Note that from [1], [3] it is not clear whether this estimate will improve with increasing smoothness of the function y.

This question will be researched in the present paper.

As in [5], we consider general summation methods determined by

$$T_n^\lambda(y_\delta) := \sum_{k=1}^{n} \lambda_k^n \cdot y_{\delta,k} \cdot \varphi_k, \tag{1.4}$$

where $\lambda = \{\lambda_k^n; \quad k = 1, 2, ..., n; \quad n \in N\}$ is a certain triangular array. Such summation methods are called λ-methods (see [4]), and play an important role in the Fourier series theory. The quality of the summation methods $T_n^\lambda(y_\delta)$ depends on the truncation level n and on the properties of $\lambda = \{\lambda_k^n; \quad k = 1, 2, ..., n; \quad n \in N\}$.

To describe these properties, we will assume that there is a constant c and some θ such that

$$|1 - \lambda_k^n| \leqslant c \cdot \left(\frac{k}{n}\right)^\theta, \quad 1 \leqslant k \leqslant n, \quad n \in N. \tag{1.5}$$

In this case, we say that T_n^λ is of the degree θ. From the examples in [5] it is seen that assumption (1.5) is rather natural. In particular, (1.5) holds for the Tikhonov regularization (1.1). It is easy to see that $T^{\alpha,s}(y_\delta)$ as $\alpha = \alpha(n) \leqslant c \cdot n^{-\rho}$ has degree $2s$, if $\rho \geqslant 2s$ and 0, if $\rho < 2s$.

Moreover, we indicate the Bernstein-Rogozinsky summation method with the trigonometric orthonormal system, where $\lambda_{2l}^n := \lambda_{2l-1}^n := \cos \frac{\pi l}{2m}$, $l = 0, 1, ..., m; \quad n = 2m + 1.$ Since

$$|1 - \lambda_{2l}^n| = |1 - \lambda_{2l-1}^n| = \left|1 - \cos \frac{\pi l}{2m}\right| = 2 \cdot \sin^2 \frac{\pi l}{2m} \leqslant \frac{\pi^2}{8} \cdot \left(\frac{l}{m}\right)^2,$$

one can see that the Bernstein-Rogozinsky method has degree $\theta = 2$.

We say that an orthonormal system $\{\varphi_k(t)\}_{k=1}^\infty$ belongs to the class (K^β), if

$$\|\varphi_k\|_C \asymp k^\beta, \quad k = 1, 2, ...$$

for some $\beta \geqslant 0$.

The trigonometric orthonormal system has the property (K^β) for $\beta = 0$, and thus satisfies (1.3) considered in [1], [3]. Where as the system of Legendre does not satisfy (1.3), and belongs to class (K^β) with $\beta = 1$.

In this paper, we consider orthonormal systems $\{\varphi_k(t)\}_{k=1}^\infty$ belonging to the class (K^β) . Note that we thus significantly expand the analysis of the considered problem as compared to [1], [3], where the case of a uniform bounded system, i.e., $\beta = 0$ was considered. However, a much more important fact is that we consider a generalized smoothness expressed in terms of spaces W_2^ψ, where, as in the previous investigations [1], [2], [3], [5], only smoothness measured on the power scale, i.e., $\psi(k) = k^\mu$ was considered. Another novelty of this paper is that in contrast to [1], [2], [3], [5] we investigate the choice of the regularization parameter practically of the most important case where the smoothness of the recoverable function is unknown.

2. The case of the known smoothness

In this section, we investigate the case where the function ψ describing the smoothness of the recoverable function y is known a priori.

First, we will prove the following lemma.

Lemma 2.1. *Let $\{\varphi_k(t)\}_{k=1}^\infty$ belong to the class (K^β) and $y \in W_2^\psi$, where the function $\psi(k)$ increases faster than k^p for $p > \beta + 1/2$. Then the inequality*

$$\left\| \sum_{k=n+1}^{\infty} \langle y, \varphi_k \rangle \varphi_k \right\|_C \leqslant c_1 \cdot \frac{(n+1)^{\beta+1/2}}{\psi(n+1)} \cdot \|y\|_\psi.$$

holds.

Proof. Let us recall $y_k = \langle y, \varphi_k \rangle$. Applying the Cauchy-Schwarz inequality, we have the following estimate:

$$\begin{aligned} \left\| \sum_{k=n+1}^{\infty} y_k \varphi_k \right\|_C &\leqslant \sum_{k=n+1}^{\infty} k^p \cdot \frac{1}{|\psi(k)|} \cdot |\psi(k)| \cdot |y_k| \cdot \frac{\|\varphi_k\|_C}{k^p} \\ &\leqslant c \cdot \frac{(n+1)^p}{\psi(n+1)} \cdot \left[\sum_{k=n+1}^{\infty} |\psi(k)|^2 \cdot |y_k|^2 \right]^{1/2} \cdot \left[\sum_{k=n+1}^{\infty} \frac{1}{k^{2p-2\beta}} \right]^{1/2} \\ &\leqslant c_1 \cdot \frac{(n+1)^{\beta+1/2}}{\psi(n+1)} \cdot \|y\|_\psi. \end{aligned}$$

Here we have used the fact that the sequence $\frac{k^p}{\psi(k)}$, $k = n+1, \ldots.,$ is unincreasing. The proof is completed. □

Theorem 2.1. *Let $\{\varphi_k(t)\}_{k=1}^\infty$ belong to the class (K^β) and $T_n^\lambda(y_\delta)$ be any summation method (1.4) of degree θ. If $\psi(k)$ increases not faster than k^θ and increases faster than k^p for $p > \beta + 1/2$, then for $n \asymp \psi^{-1}(1/\delta)$ we have the estimate*

$$\sup_{\|y\|_\psi \leqslant 1} \sup_{\|\xi\|_{l_2} \leqslant 1} \left\| y - T_n^\lambda(y_\delta) \right\|_C \leqslant c \cdot \delta \cdot \left[\psi^{-1}(1/\delta) \right]^{\beta+1/2}, \tag{2.1}$$

where c does not depends on δ.

Proof. For any summation method $T_n^\lambda(y_\delta)$ we have

$$\left\|y - T_n^\lambda(y_\delta)\right\|_C \leqslant \left\|\sum_{k=1}^{n}(1-\lambda_k^n)\cdot y_k\cdot\varphi_k\right\|_C + \left\|\sum_{k=n+1}^{\infty} y_k\cdot\varphi_k\right\|_C + \delta\cdot\left\|\sum_{k=1}^{n}\lambda_k^n\cdot\xi_k\cdot\varphi_k\right\|_C. \quad (2.2)$$

The first and last summands in (2.2) can be estimated by using the Nikolski type inequality for any polynomials with respect to the system $\{\varphi_k\}_{k=1}^\infty$

$$\left\|\sum_{k=1}^{n} a_k\cdot\varphi_k\right\|_C \leqslant c\cdot\sum_{k=1}^{n}|a_k|\cdot k^\beta \leqslant c\cdot\left[\sum_{k=1}^{n}|a_k|^2\right]^{1/2}\cdot\left[\sum_{k=1}^{n}k^{2\beta}\right]^{1/2} \leqslant c\cdot n^{\beta+1/2}\cdot\left[\sum_{k=1}^{n}|a_k|^2\right]^{1/2}.$$

Then for the last summand in (2.2) we have

$$\left\|\sum_{k=1}^{n}\lambda_k^n\cdot\xi_k\cdot\varphi_k\right\|_C \leqslant c_\lambda\cdot\|\xi\|_{l_2}\cdot n^{\beta+1/2}. \quad (2.3)$$

Note that the constant in (2.3) depends only on the method T_n^λ , and it can be effectively computed.

The first summand in (2.2) can be estimated as

$$\begin{aligned}\left\|\sum_{k=1}^{n}(1-\lambda_k^n)\cdot y_k\cdot\varphi_k\right\|_C &\leqslant c\cdot n^{\beta+1/2}\cdot\left[\sum_{k=1}^{n}\Big(\frac{k}{n}\Big)^{2\theta}\cdot\frac{1}{\psi^2(k)}\cdot\psi^2(k)\cdot|y_k|^2\right]^{1/2}\\ &\leqslant c_2\cdot n^{\beta+1/2}\cdot\left[\sum_{k=1}^{n}\max_{1\leqslant k\leqslant n}\left(\frac{k^\theta}{\psi(k)}\right)^2\cdot\frac{1}{n^{2\theta}}\cdot\psi^2(k)\cdot|y_k|^2\right]^{1/2}\\ &= c_2\cdot n^{\beta+1/2}\cdot\left[\sum_{k=1}^{n}\frac{n^{2\theta}}{\psi^2(n)}\cdot\frac{1}{n^{2\theta}}\cdot\psi^2(k)\cdot|y_k|^2\right]^{1/2}\\ &= c_2\cdot\frac{n^{\beta+1/2}}{\psi(n)}\cdot\|y\|_\psi.\end{aligned} \quad (2.4)$$

The middle summand in (2.2) was estimated in Lemma 2.1, that finally leads to the inequality

$$\left\|y - \sum_{k=1}^{n}\lambda_k^n\cdot y_{\delta,k}\cdot\varphi_k\right\|_C \leqslant n^{\beta+1/2}\cdot\left[\frac{c_1+c_2}{\psi(n)} + c_\lambda\cdot\delta\right]\cdot\|y\|_\psi. \quad (2.5)$$

We choose n such that $\frac{c_1+c_2}{\psi(n)} \asymp c_\lambda\cdot\delta$, consequently $n \asymp \psi^{-1}(\frac{1}{\delta})$. Then we have the following estimate

$$\sup_{\|y\|_\psi\leqslant 1}\ \sup_{\|\xi\|_{l_2}\leqslant 1}\left\|y - T_n^\lambda(y_\delta)\right\|_C \leqslant c\cdot\delta\cdot\left[\psi^{-1}\Big(\frac{1}{\delta}\Big)\right]^{\beta+1/2} \quad (2.6)$$

where the constant c depends on c_1, c_2 and c_λ.

The proof is completed. □

Note that estimate (2.1) in terms of the functions ψ is new. In the case of the power function $\psi(k) = k^\mu$, estimate (2.1) coincides with the assertion of Theorem 3.1 from [5].

3. The case of the unknown smoothness

In the previous section, we have proved the error estimation for the constructed function $T_n^\lambda(y_\delta)$ in terms of ψ. In practice, we have chosen $n \asymp \psi^{-1}(\frac{1}{\delta})$. Recall that ψ describes the smoothness of the recoverable function y. In practice, this information is often unavailable. The problem is to choose n without the knowledge of ψ but with the same order of accuracy $\delta \cdot [\psi^{-1}(\frac{1}{\delta})]^{\beta+1/2}$. This section is devoted to the solution of this problem where we use the idea from [6].

Let us introduce the following set:

$$M^\star := \Big\{n : \quad \frac{c_1 + c_2}{\psi(n)} \leqslant c_\lambda \cdot \delta\Big\}$$

and the number

$$n_\star := \min\big\{n : \quad n \in M^\star\big\},$$

where c_1, c_2, c_λ are constants that appear in the estimates (2.2)-(2.5).

Lemma 3.1. *For a sufficiently small $\delta \in (0;1)$, we have the inequality*

$$n_\star^{\beta+1/2} \leqslant c_\beta \cdot \left[\psi^{-1}\Big(\frac{1}{\delta}\Big)\right]^{\beta+1/2},$$

where the constant c_β depends only on β.

Proof. Denote

$$n_{opt} = \psi^{-1}\Big(\frac{1}{\delta}\Big).$$

It is obvious that

$$n_\star - 1 \leqslant n_{opt} \leqslant n_\star.$$

Consequently,

$$n_\star^{\beta+1/2} \leqslant \big(n_{opt} + 1\big)^{\beta+1/2} = \left[\psi^{-1}\Big(\frac{1}{\delta}\Big) + 1\right]^{\beta+1/2}$$

by the assumption $\psi^{-1}\big(\frac{1}{\delta}\big) \geqslant 1$, from this it follows that

$$n_\star^{\beta+1/2} \leqslant 2^{\beta+1/2} \cdot \left[\psi^{-1}\Big(\frac{1}{\delta}\Big)\right]^{\beta+1/2} = c_\beta \cdot \left[\psi^{-1}\Big(\frac{1}{\delta}\Big)\right]^{\beta+1/2}.$$

The lemma is proved. □

It is obvious that from $n \in M^\star$ and $m > n$ follows $m \in M^\star$. Then by virtue of (15) we have

$$\begin{aligned}
\Big\|T_n^\lambda(y_\delta) - T_m^\lambda(y_\delta)\Big\|_C &\leqslant \Big\|y - T_n^\lambda(y_\delta)\Big\|_C + \Big\|y - T_m^\lambda(y_\delta)\Big\|_C \\
&\leqslant \left[\frac{c_1 + c_2}{\psi(n)} + c_\lambda \cdot \delta\right] \cdot n^{\beta+1/2} \cdot \|y\|_\psi + \left[\frac{c_1 + c_2}{\psi(m)} + c_\lambda \cdot \delta\right] \cdot m^{\beta+1/2} \cdot \|y\|_\psi \\
&\leqslant \big[2 \cdot c_\lambda \cdot \delta \cdot n^{\beta+1/2} + 2 \cdot c_\lambda \cdot \delta \cdot m^{\beta+1/2}\big] \cdot \|y\|_\psi \leqslant 4 \cdot c_\lambda \cdot \delta \cdot m^{\beta+1/2} \cdot \|y\|_\psi.
\end{aligned}$$

Now let us introduce the set

$$M^+ := \Big\{ n : \quad \forall m \geqslant n \quad \Big\| T_n^\lambda(y_\delta) - T_m^\lambda(y_\delta) \Big\|_C \leqslant 4 \cdot c_\lambda \cdot \delta \cdot m^{\beta+1/2} \cdot \|y\|_\psi \Big\}$$

and the number

$$n_+ := \min\{ n : \quad n \in M^+ \}.$$

Note that the number n_+ can be found without the knowledge of the function ψ describing the smoothness of the recoverable function $y \in W_2^\psi$. However, there is the following theorem.

Theorem 3.1. *Under the conditions of Theorem 2.1 for any functions* $y \in W_2^\psi$

$$\Big\| y - T_n^\lambda(y_\delta) \Big\|_C \leqslant \hat{c} \cdot \delta \cdot \Big[\psi^{-1}\Big(\frac{1}{\delta}\Big) \Big]^{\beta+1/2} \cdot \|y\|_\psi,$$

where $\hat{c}$ does not depend on δ.

Proof. Let us show that for any $p \in M^\star$ it follows that $p \in M^+$. Let $s > p$. Then

$$\begin{aligned} \Big\| T_s^\lambda(y_\delta) - T_p^\lambda(y_\delta) \Big\|_C &\leqslant \Big\| y - T_s^\lambda(y_\delta) \Big\|_C + \Big\| y - T_p^\lambda(y_\delta) \Big\|_C \\ &\leqslant \Big[\frac{(c_1 + c_2) \cdot s^{\beta+1/2}}{\psi(s)} + c_\lambda \cdot s^{\beta+1/2} \cdot \delta + \frac{(c_1 + c_2) \cdot p^{\beta+1/2}}{\psi(p)} + c_\lambda \cdot p^{\beta+1/2} \cdot \delta \Big] \cdot \|y\|_\psi \\ &\leqslant 4 \cdot c_\lambda \cdot s^{\beta+1/2} \cdot \delta \cdot \|y\|_\psi. \end{aligned}$$

Consequently, $p \in M^\star$. This means that

$$M^\star \subseteq M^+.$$

Then it is obvious that

$$n_+ \leqslant n_\star.$$

Further we have

$$\begin{aligned} \Big\| y - T_{n_+}^\lambda(y_\delta) \Big\|_C &\leqslant \Big\| y - T_{n_\star}^\lambda(y_\delta) \Big\|_C + \Big\| T_{n_\star}^\lambda(y_\delta) - T_{n_+}^\lambda(y_\delta) \Big\|_C \\ &\leqslant \Big[\frac{(c_1 + c_2) \cdot n_\star^{\beta+1/2}}{\psi(n_\star)} + c_\lambda \cdot \delta \cdot n_\star^{\beta+1/2} + 4 \cdot c_\lambda \cdot \delta \cdot n_\star^{\beta+1/2} \Big] \cdot \|y\|_\psi \\ &\leqslant 6 \cdot c_\lambda \cdot \delta \cdot n_\star^{\beta+1/2} \cdot \|y\|_\psi. \end{aligned}$$

Taking into account Lemma 2, from this we have the estimate

$$\Big\| y - T_{n_+}^\lambda(y_\delta) \Big\|_C \leqslant 6 \cdot c_\lambda \cdot c_\beta \cdot \delta \cdot \Big[\psi^{-1}\Big(\frac{1}{\delta}\Big) \Big]^{\beta+1/2} \cdot \|y\|_\psi,$$

which corresponds to the assertion of the theorem at $\hat{c} = 6 \cdot c_\lambda \cdot c_\beta$.

Note that the choice of the parameter $n = n_+$ yields estimate (2.6) with the optimal order of accuracy $\delta \cdot \big[\psi^{-1}(1/\delta)\big]^{\beta+1/2}$. In contract to Theorem 2.1, the choice of $n = n_+$ can be realized without the knowledge of the smoothness of the recoverable function. We also note that this way of adaptively choosing the regularization parameter was not considered in the earlier papers [1], [2], [3], [5], [8], [9] that examined the recovery of continuous functions from noisy Fourier coefficients. □

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