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Robust Discretisation and a Posteriori Control for Strongly Oscillating Solutions of the Stationary Schrödinger Equation

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Abstract — We consider an example of a boundary value problem on an interval where the solution can show strong oscillations. In order to solve such a problem numerically, standard methods require meshes that resolve these oscillations and will thus need a prohibitively large number of unknowns. In our approach we use special problem dependent basis functions in the finite element method and provide an analysis for a priori and a posteriori bounds. In this way we can construct an efficient approximation method for the solution of such boundary value problems.

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1. Introduction

The numerical solution of linear elliptic boundary value problems that define a symmetric positive form is very elaborated in terms of a priori and a posteriori analysis. Also for the singularly perturbed operators (when the operators' coefficients become very small or large) one can describe robust methods. However, if one studies these problems with a negative potential, especially when it is large in modulus, standard methods face some difficulties. For example, the solutions of such problems may exhibit strong oscillations and an approximation by piecewise polynomials will require small computational cells that resolve these oscillations. However, this will result in a very expensive method. In this work we will, for the one-dimensional case, perform the Galerkin discretisation with local Green's functions that have been used and analysed up to now mainly in case of positive coefficients [7] [19] [1] [5]. For negative coefficients the basis functions are strongly oscillating and the question is to which extend they can approximate the solution of the problem. Since the setting is that of a general finite element method, we can follow the general guideline to prove a priori and a posteriori error estimates once we provided the necessary interpolation estimates. This work generalises the corresponding attempt in [3, Ch. 4.1] to non-constant coefficients and a posteriori error estimations and mesh-refinement.

Note, that the idea of exponential fitting of such type of equations have been considered in the context of finite difference methods, see for example [8] and the literature cited therein. However, our methods and results are quite different from this approach.

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Oscillating basis functions have also been used in a similar type of problem, the Helmholtz equation with constant coefficients and with wave-number dependent Robin boundary conditions. The methods and results are different to those used here [11] [10] [13] [20].

The techniques are essentially for the one-dimensional case. We think that the method might be of interest for the time evolution of 1D oscillating pulses. Furthermore, it has been demonstrated in [5] how to use such one-dimensional results on tensorial grids. In [3, Ch. 7.2] [15] it is demonstrated that a local choice of a number of 1D plane waves can be used to approximate multi-dimensional problems.

2. The stationary Schrödinger equation

The *stationary Schrödinger equation* is the boundary value problem

$$L(\gamma)u := -u'' + \gamma u = f \qquad \text{in } \Omega := (0,1), \tag{2.1}$$

$$u(0) = u(1) = 0 \qquad \text{in } \partial\Omega = \{0,1\} \tag{2.2}$$

with a bounded function $\gamma : \Omega \to \mathbb{R}$. Note that any bounded interval can be transformed (affine linearly) to $(0,1)$ and nonhomogeneous boundary values can be removed by substraction of a suitable linear function from u.

The weak form of this boundary value problem results from multiplying (2.1) by $v \in C_0^\infty(\Omega)$ and integration by parts

$$B(u,v) := \int_\Omega \{u'v' + \gamma uv\} = F(v) := \int_\Omega fv \qquad \text{for all } v \in C_0^\infty(\Omega). \tag{2.3}$$

To formulate this problem on a Hilbert space we let

$$\kappa := \sqrt{|\gamma|}, \quad \kappa_* := \max\{\kappa, \pi\}, \quad \gamma_* := \kappa_*^2,$$

and define for $G \subseteq \Omega$

$$|\!|\!| v |\!|\!|_{\kappa;G} := \Big(\|v'\|^2_{L^2(G)} + \|\kappa_* v\|^2_{L^2(G)} \Big)^{1/2}.$$

Assuming that γ is a bounded function, we take $V := H_0^1(\Omega)$ equipped with the norm $|\!|\!| \cdot |\!|\!|_{\kappa;\Omega}$ and observe immediately the continuity of the bilinear form B on $V \times V$

$$|B(v,w)| \leqslant \int_\Omega \big\{ |v'|\,|w'| + |\gamma|\,|v|\,|w| \big\} \leqslant |\!|\!| v |\!|\!|_{\kappa;\Omega} |\!|\!| w |\!|\!|_{\kappa;\Omega}.$$

In case γ is strictly positive we have furthermore for all $v \in V$

$$\begin{aligned} |B(v,v)| &= \int_\Omega \big\{ |v'|^2 + \gamma |v|^2 \big\} \geqslant \int_\Omega |v'|^2 + \int_{\Omega\cap\{\kappa\geqslant\pi\}} \gamma|v|^2 \\ &\geqslant \frac{1}{2}\int_\Omega |v'|^2 + \frac{1}{2}\pi^2 \int_{\Omega\cap\{\kappa<\pi\}} |v|^2 + \int_{\Omega\cap\{\kappa\geqslant\pi\}} \gamma|v|^2 \geqslant \frac{1}{2} |\!|\!| v |\!|\!|^2_{\kappa;\Omega}, \end{aligned}$$

where we have used Poincaré's inequality $\|v\|_{L^2(\Omega)} \leqslant (1/\pi)\,\|v'\|_{L^2(\Omega)}$ and $\{\kappa < \pi\}$ as an abbreviation for the set $\{x \in \Omega : \kappa(x) < \pi\}$, etc. This inequality, called coercivity, guarantees unique solvability of problem (2.3) for any continuous $F : V \to \mathbb{R}$ on V [6, Lem. 2.2]. This, however, cannot be applied for γ with (not necessarily small) negative values and it is in fact not true in general. Thus we will have to assume invertibility of $L(\gamma)$ for the given data γ, stated as the *inf-sup-condition*

$$\inf_{v\in V\setminus\{0\}} \sup_{w\in V\setminus\{0\}} \left\{\frac{B(v,w)}{\|v\|_{\kappa;\Omega}\|w\|_{\kappa;\Omega}}\right\} \geqslant c_\gamma \tag{2.4}$$

for some positive constant $c_\gamma > 0$ [6, Thm. 2.6]. We finally observe that F satisfies the bound

$$|F(v)| \leqslant \int_\Omega |f|\,|v| = \int_\Omega \frac{1}{\kappa_*}|f|\,\kappa_*|v| \leqslant \left\|\frac{1}{\kappa_*}f\right\|_{L^2(\Omega)}\|v\|_{\kappa;\Omega}.$$

In summary, we have the following existence and stability result.

Theorem 2.1. *Assume that for given $\gamma \in L^\infty(\Omega)$ assumption* (2.4) *holds true. Then there exists a unique solution $u \in V$ of* (2.3) *for any $f \in L^2(\Omega)$ and it satisfies the bound*

$$\|u\|_{\kappa;\Omega} \leqslant \frac{1}{c_\gamma}\left\|\frac{1}{\kappa_*}f\right\|_{L^2(\Omega)}.$$

In case γ is strictly positive assumption (2.4) *is satisfied with $c_\gamma = 1/2$.*

3. The finite element method

3.1. Galerkin method

For a numerical approximation we choose the Galerkin method that consists of choosing a finite dimensional space $V_h \subset V$ and to formulate the discrete boundary value problem as

$$B_h(u_h, v_h) := \int_\Omega \left\{u_h' v_h' + \gamma_h u_h v_h\right\} = F_h(v_h) := \int_\Omega f_h v_h \qquad \text{for all } v_h \in V_h, \tag{3.1}$$

where γ_h and f_h are suitable approximations to γ and f, respectively. B and B_h are connected via

$$B_h(v_h, w_h) = B(v_h, w_h) + \int_\Omega (\gamma_h - \gamma) v_h w_h.$$

Since for all $v_h, w_h \in V_h$

$$\left|\int_\Omega (\gamma - \gamma_h) v_h w_h\right| \leqslant \int_\Omega \frac{|\gamma - \gamma_h|}{\gamma_*}\,\kappa_*|v_h|\,\kappa_*|w_h| \leqslant \Gamma(\gamma,\gamma_h)\|v_h\|_{\kappa;\Omega}\|w_h\|_{\kappa;\Omega},$$

with $\Gamma(\gamma,\gamma_h) := \|(\gamma - \gamma_h)/\gamma_*\|_{L^\infty(\Omega)}$, we find that

$$|B_h(v_h, w_h)| \leqslant \big(1 + \Gamma(\gamma,\gamma_h)\big)\|v_h\|_{\kappa;\Omega}\|w_h\|_{\kappa;\Omega}.$$

However, a *discrete inf-sup-condition*

$$\inf_{v_h\in V_h\setminus\{0\}} \sup_{w_h\in V_h\setminus\{0\}} \left\{\frac{B_h(v_h,w_h)}{\|v_h\|_{\kappa;\Omega}\,\|w_h\|_{\kappa;\Omega}}\right\} \geqslant \beta c_\gamma \tag{3.2}$$

for some constant $\beta \in (0,1)$ that is independent of the dimension of V_h cannot be deduced from (2.4) for general coefficients γ and spaces V_h (although it will be true asymptotically for $h \to 0$ under reasonable conditions [16]). In the following we will assume that the condition (3.2) is satisfied. In this situation the application of the Strang lemma yields the a priori error bound [6, Lem. 2.27]

$$\begin{aligned}\|\!| u - u_h |\!\|_{\kappa;\Omega} \leqslant \Big(1 + \frac{1+\Gamma(\gamma,\gamma_h)}{\beta c_\gamma}\Big) \inf_{v_h \in V_h} \Big\{ \|\!| u - v_h |\!\|_{\kappa;\Omega} \Big\} \\ + \frac{1}{\beta c_\gamma} \Big\| \frac{\gamma - \gamma_h}{\gamma_*}\, \kappa_* u \Big\|_{L^2(\Omega)} + \frac{1}{\beta c_\gamma} \Big\| \frac{1}{\kappa_*}(f - f_h) \Big\|_{L^2(\Omega)}. \end{aligned} \tag{3.3}$$

3.2. Standard finite element spaces

Let $\mathcal{G} = \{x_i : i = 0, \dots, N+1\}$ be a set of points with $0 = x_0 < x_1 < \dots < x_N < x_{N+1} = 1$ and $\mathcal{K} = \{K_i := [x_{i-1}, x_i] : i = 0, \dots, N+1\}$ be a decomposition of $\overline{\Omega}$ into intervals. We define $h_i := |K_i|$ and h to be the piecewise constant function $h(x) := h_i$ for $x \in K_i$. In a standard finite element discretisation the space V_h would consist of continuous piecewise (with respect to the decomposition) polynomials of a, say constant in Ω, degree p. For a stable discretisation, when (3.2) holds, the error estimate follows from (3.3) with the interpolation estimate

$$\inf_{v_h \in V_h} \Big\{ \|\!| u - v_h |\!\|_{\kappa;\Omega} \Big\} \leqslant \min_{l=1,\dots,p} C_l \Big(\frac{\kappa h}{l} \Big)^l,$$

sufficient regularity assumed [17, Thm. 3.17] [14, p. 24]. Moreover, this bound requires $\kappa h < l$ for some $l \in \{1, \dots, p\}$. This is a very strong requirement on the resolution of the grid for large κ and a problem in many applications. Our aim is to create a finite element space with approximation properties that are better suited to this problem.

3.3. Exponentially fitted finite element spaces

Let $\overline{\gamma}_h$ be a piecewise constant function on the decomposition $\mathcal{K}$. By $\overline{\gamma}_{h,i}$ we denote the restriction of $\overline{\gamma}_h$ on K_i. We define

$$V_h^1 := \operatorname{span}\{\psi_i : i = 1, \dots, N\},$$

where the basis functions ψ_i are uniquely determined by the local boundary value problems

$$\begin{aligned} L(\overline{\gamma}_h)\psi_i &= 0 \qquad \text{on } \cup_{j=1}^{N+1} K_j, \\ \psi_i(x_j) &= \delta_{ij} \quad \text{for } j = 0, \dots, N+1. \end{aligned}$$

It is immediately seen that ψ_i is zero outside the interval $K_i \cup K_{i+1}$ for $i = 1, \dots, N$. On the interval K_i we have explicitly

$$\psi_{i-1}(x) = \begin{cases} \frac{\sin(\overline{\kappa}_{h,i}(x_i - x))}{\sin(\overline{\kappa}_{h,i} h_i)} & \text{for } \overline{\gamma}_{h,i} < 0, \\ \frac{x_i - x}{h_i} & \text{for } \overline{\gamma}_{h,i} = 0, \\ \frac{\sinh(\overline{\kappa}_{h,i}(x_i - x))}{\sinh(\overline{\kappa}_{h,i} h_i)} & \text{for } \overline{\gamma}_{h,i} > 0, \end{cases} \qquad \psi_i(x) = \begin{cases} \frac{\sin(\overline{\kappa}_{h,i}(x - x_{i-1}))}{\sin(\overline{\kappa}_{h,i} h_i)} & \text{for } \overline{\gamma}_{h,i} < 0, \\ \frac{x - x_{i-1}}{h_i} & \text{for } \overline{\gamma}_{h,i} = 0, \\ \frac{\sinh(\overline{\kappa}_{h,i}(x - x_{i-1}))}{\sinh(\overline{\kappa}_{h,i} h_i)} & \text{for } \overline{\gamma}_{h,i} > 0, \end{cases} \tag{3.4}$$

where $\overline{\kappa}_{h,i} := \sqrt{|\overline{\gamma}_{h,i}|}$ and where $\overline{\kappa}_{h,i} h_i \notin \pi\mathbb{N}$ has to be assumed in case of $\overline{\gamma}_{h,i} < 0$. We further define the enhanced space

$$V_h^2 := V_h^1 \cup \operatorname{span}\left\{\psi_{i-\frac{1}{2}} : i = 1, \ldots, N+1\right\},$$

with $\psi_{i-1/2}$ given as the unique solution of

$$\begin{aligned} L(\overline{\gamma}_h)\psi_{i-\frac{1}{2}} &= \delta_{ij} \qquad && \text{on } K_j \text{ for } j = 1, \ldots, N+1, \\ \psi_{i-\frac{1}{2}}(x_j) &= 0 && \text{for } j = 0, \ldots, N+1. \end{aligned}$$

Clearly, $\psi_{i-1/2}$ is compactly supported inside K_i and it can easily be computed using the functions $\{\psi_{i-1}, \psi_i\}$ on K_i defined above. In fact, for $x \in K_i$,

$$\psi_{i-\frac{1}{2}}(x) = \begin{cases} \frac{1}{\overline{\gamma}_{h,i}}\big(1 - \psi_{i-1}(x) - \psi_i(x)\big) & \text{for } \overline{\gamma}_{h,i} \neq 0, \\ \frac{1}{2}\big(x - x_{i-1}\big)\big(x_i - x\big) & \text{for } \overline{\gamma}_{h,i} = 0. \end{cases} \tag{3.5}$$

This type of basis functions are also called *local Green's functions* [7].

3.4. The discrete problem

In general u_h will be of the form

$$u_h(x) = \sum_{i=1}^{N} a_i \psi_i(x) + \sum_{j=1}^{N+1} b_j \psi_{j-\frac{1}{2}}(x), \tag{3.6}$$

for coefficients a_i, b_j to be determined. For $u_h \in V_h^1$ we have $b_j = 0$ for all j. To solve the problem for $u_h \in V_h^1$ we end up with a linear system with a tridiagonal matrix $A = [A_{ij}]_{i,j=1,\ldots,N}$ given by

$$A_{ij} = \int_\Omega \{\psi_j' \psi_i' + \gamma_h \psi_j \psi_i\} = -[\psi_j' \psi_i]_{x_i} + \int_\Omega \psi_i L(\gamma_h)\psi_j = -[\psi_j']_{x_i} + \int_\Omega (\gamma_h - \overline{\gamma}_h)\psi_j \psi_i$$

since $L(\overline{\gamma}_h)\psi_j = 0$ on each interval. Here, $[g]_{x_i}$ denotes the jump of a piecewise continuous function g in x_i: $[g]_{x_i} := \lim_{s \to 0+}\big(g(x_i + s) - g(x_i - s)\big)$.

For the problem in V_h^2 we find in case $j = 1, \ldots, N$

$$\begin{aligned} A_{i-1/2,j} &= \int_\Omega \{\psi_j' \psi_{i-1/2}' + \gamma_h \psi_j \psi_{i-1/2}\} = -[\psi_j' \psi_{i-1/2}]_{x_i} + \int_\Omega \psi_{i-1/2} L(\gamma_h)\psi_j \\ &= \int_\Omega (\gamma_h - \overline{\gamma}_h)\psi_j \psi_{i-1/2} \end{aligned}$$

and likewise

$$\begin{aligned} A_{i-1/2,i-1/2} &= \int_\Omega \{\psi_{i-1/2}' \psi_{i-1/2}' + \gamma_h \psi_{i-1/2} \psi_{i-1/2}\} = \int_\Omega \psi_{i-1/2} L(\gamma_h)\psi_{i-1/2} \\ &= \int_\Omega \psi_{i-1/2} + \int_\Omega (\gamma_h - \overline{\gamma}_h)\psi_{i-1/2}\psi_{i-1/2} \end{aligned}$$

since $L(\overline{\gamma}_h)\psi_{i-1/2} = 1$ on K_i. The right-hand side of the system is a vector F with components

$$F_i = \int_\Omega f_h \psi_i, \qquad F_{i-1/2} = \int_\Omega f_h \psi_{i-1/2}.$$

In the special case $\gamma_h = \overline{\gamma}_h$ and $f_h = \overline{f}_h$ we obtain $A_{ij} = -[\psi_j']_{x_i}$, $A_{i-1/2,j} = 0$ (the equations for a_i, b_i decouple), while $A_{i-1/2,i-1/2} = \int_{K_i} \psi_{i-1/2}$ and $\int_\Omega f_h \psi_{i-1/2} = \overline{f}_{h,i} \int_\Omega \psi_{i-1/2}$ yield $b_i = \overline{f}_{h,i}$ directly.

Definition 3.1. For a given piecewise constant function $\overline{\gamma}_h$ on a decomposition $\mathcal{K}$ we define the index set

$$\overline{\Lambda}_- := \big\{ i \in \{1, \dots, N+1\} : \overline{\gamma}_{h,i} < 0 \big\}$$

and the interpolation constant

$$C_{\mathrm{I}} := \begin{cases} \max\big\{ \frac{1}{|\sin(\overline{\kappa}_{h,i} h_i)|} \ : \ i \in \overline{\Lambda}_-, \ \overline{\kappa}_{h,i} h_i \geqslant \pi/2 \big\} & \text{if the set is not empty,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.7}$$

We end this section with an a priori error estimate for this kind of discretisation (in contrast to the estimate given in Section 3.2).

Theorem 3.1. *Let $u \in H^2(\Omega)$ be a solution to* (2.1)–(2.2) *and $u_h^{(1)} \in V_h^1$ and $u_h^{(2)} \in V_h^2$ be the respective solutions of the discrete problem* (3.1) *on a grid where C_{I} given by* (3.7) *is bounded. Then,*

$$\begin{aligned} \|u - u_h^{(\ell)}\|_{\kappa;\Omega} \leqslant{}& 4C_{\mathrm{I}}\Big(1 + \frac{1 + \Gamma(\gamma, \gamma_h)}{\beta c_\gamma}\Big)\big(1 + \Gamma(\overline{\kappa}_h, \kappa)\big)\Big(\|hf\|_{L^2(\Omega)} + \|h(\gamma - \overline{\gamma}_h)u\|_{L^2(\Omega)}\Big) \\ &+ \frac{1}{\beta c_\gamma}\Big\|\frac{\gamma - \gamma_h}{\gamma_*}\,\kappa_* u\Big\|_{L^2(\Omega)} + \frac{1}{\beta c_\gamma}\Big\|\frac{1}{\kappa_*}(f - f_h)\Big\|_{L^2(\Omega)} \end{aligned}$$

for $\ell \in \{1, 2\}$, and especially for $\ell = 2$ also

$$\begin{aligned} \|u - u_h^{(2)}\|_{\kappa;\Omega} \leqslant{}& 4C_{\mathrm{I}}\Big(1 + \frac{1 + \Gamma(\gamma, \gamma_h)}{\beta c_\gamma}\Big)\big(1 + \Gamma(\overline{\kappa}_h, \kappa)\big)\Big(\|h^2 f'\|_{L^2(\Omega)} + \|h^2\big((\gamma - \overline{\gamma}_h)u\big)_h'\|_{L^2(\Omega)}\Big) \\ &+ \frac{1}{\beta c_\gamma}\Big\|\frac{\gamma - \gamma_h}{\gamma_*}\,\kappa_* u\Big\|_{L^2(\Omega)} + \frac{1}{\beta c_\gamma}\Big\|\frac{1}{\kappa_*}(f - f_h)\Big\|_{L^2(\Omega)}. \end{aligned}$$

$(\,.\,)_h'$ *denotes the piecewise derivative with respect to $\mathcal{K}$.*

Proof. Note first, that a grid with the required property can be obtained by the construction presented in Section 5.4. Using the error estimate (3.3), the remaining task is to estimate the infimum after insertion of a suitable interpolant. Let us consider the case of a discrete solution in V_h^1. Then, by Theorem 4.1 and a further triangle inequality,

$$\|u - I_h^1 u\|_{\overline{\kappa}_h;\Omega} \leqslant 4C_{\mathrm{I}}\|hL(\overline{\gamma}_h)u\|_{L^2(\Omega)} \leqslant 4C_{\mathrm{I}}\big(\|hf\|_{L^2(\Omega)} + \|h(\gamma - \overline{\gamma}_h)u\|_{L^2(\Omega)}\big)$$

and this shows the first estimate for $\ell = 1$ with $\|u - I_h^1 u\|_{\kappa;\Omega} \leqslant \big(1 + \Gamma(\overline{\kappa}_h, \kappa)\big)\|u - I_h^1 u\|_{\overline{\kappa}_h;\Omega}$. The assertion for $\ell = 2$ works likewise with Theorem 4.3. ☐

4. Interpolation estimates

We study suitable linear continuous operators $I_h^k : V \to V_h^k$ (for $k \in \{1, 2\}$) and give estimates for $\|v - I_h^k v\|_{L^2(\Omega)}$ in terms of v in $H^1(\Omega)$ or $H^2(\Omega)$. For $v \in H^1(\Omega)$ let us define

$$I_h^1 v(x) := v(x_{i-1})\psi_{i-1}(x) + v(x_i)\psi_i(x) \qquad \text{for } x \in K_i, \ i \in \{1, \dots, N+1\} \tag{4.1}$$

with the basis functions from (3.4). Note that for constant $v = v_0$ on K_i we get from (3.5) $I_h^1 v(x) = v_0\big(\psi_{i-1}(x) + \psi_i(x)\big) = v_0\big(1 - \overline{\gamma}_{h,i}\psi_{i-1/2}(x)\big)$ on K_i which *not* equals v_0. On the enhanced space V_h^2 we can define for $v \in H^1(\Omega)$

$$I_h^2 v(x) := I_h^1 v(x) + \frac{\overline{\gamma}_{h,i}}{2}\big(v(x_{i-1}) + v(x_i)\big)\psi_{i-1/2}(x) \qquad \text{for } x \in K_i,\ i \in \{1, \dots, N+1\}. \tag{4.2}$$

For a constant function $v = v_0$ on K_i we get $I_h^2 v(x) = v_0\big(1-\overline{\gamma}_{h,i}\psi_{i-1/2}(x)\big)+v_0\overline{\gamma}_{h,i}\psi_{i-1/2}(x) = v_0$ on K_i. Alternatively, we define for $v \in H^2(\Omega)$

$$\widetilde{I}_h^2 v(x) := I_h^1 v(x) + \Big(\frac{1}{h_i}\int_{K_i} L(\overline{\gamma}_{h,i})v\Big)\psi_{i-1/2}(x) \qquad \text{for } x \in K_i,\ i \in \{1, \dots, N+1\}, \tag{4.3}$$

that also has the property $\widetilde{I}_h^2 v_0 = v_0$ for constant v_0 on K_i. We also need interpolation operators with less regularity requirements than those presented so far. Let for $i \in \{1, \dots, N+1\}$

$$F_i(v) := \frac{1}{h_{m(i)}}\int_{K_{m(i)}} v, \qquad \text{where} \quad m(i) := \begin{cases} i & \text{if } \overline{\kappa}_{h,i} \geqslant \overline{\kappa}_{h,i+1} \\ i+1 & \text{if } \overline{\kappa}_{h,i+1} > \overline{\kappa}_{h,i} \end{cases}, \tag{4.4}$$

and define for $v \in L^2(\Omega)$

$$Q_h^1 v(x) := F_{i-1}(v)\psi_{i-1}(x) + F_i(v)\psi_i(x) \quad \text{for } x \in K_i. \tag{4.5}$$

4.1. Estimates for the interpolation onto V_h^1

Theorem 4.1. *Let $v \in H^1(\Omega)$ and $I_h^1 v \in V_h^1$ be as in (4.1) with $h \leqslant 1/2$. Then, $I_h^1 v$ is well defined if we require that $C_{\mathrm{I}} < \infty$, C_{I} from (3.7), and we have the estimate*

$$\|v - I_h^1 v\|_{L^2(\Omega)} \leqslant 2C_{\mathrm{I}}\Big(\sum_{i=1}^{N+1} h_i^2 \|v\|^2_{\overline{\kappa}_{h,i};K_i}\Big)^{1/2}, \tag{4.6}$$

while for $v \in H^2(\Omega)$

$$2\|h^{-1}(v - I_h^1 v)\|_{L^2(\Omega)} + \|v - I_h^1 v\|_{\overline{\kappa}_h;\Omega} \leqslant 4C_{\mathrm{I}}\|hL(\overline{\gamma}_h)v\|_{L^2(\Omega)}. \tag{4.7}$$

Proof. It is convenient to consider the interval $(0, h)$ in place of K_i and some $v \in H^1(0, h)$. For this let $\overline{\gamma} := \overline{\gamma}_{h,i}$, $\overline{\kappa} := \overline{\kappa}_{h,i}$ (i. e., $\overline{\gamma} = \pm\overline{\kappa}^2$), $\overline{L} := L(\overline{\gamma})$ and $\overline{G}(\,.\,,\,.\,) \equiv \overline{G}(\,.\,,\,.\,;\overline{\gamma})$ be Green's function for the homogeneous Dirichlet boundary value problem for $\overline{L}$ on $(0, h)$ (see Section 4.3). Let $v_h := I_h^1 v$. Since $(v - v_h)(0) = (v - v_h)(h) = 0$ and $\overline{L}v_h = 0$, we get by Green's representation formula and integration by parts

$$\begin{aligned}(v - v_h)(x) &= \int_0^h \overline{G}(x,y)\overline{L}(v - v_h)(y)\,dy = \int_0^h \overline{G}(x,y)(-v'' + \overline{\gamma}v)(y)\,dy \\ &= \int_0^h \big\{\partial_2\overline{G}(x,y)v'(y) + \overline{\gamma}\overline{G}(x,y)v(y)\big\}\,dy\end{aligned}$$

for all $x \in (0, h)$. This leads to the pointwise inequality

$$|(v - v_h)(x)| \leqslant \|\partial_2\overline{G}(x,\,.\,)\|_{L^2(0,h)}\|v'\|_{L^2(0,h)} + \overline{\kappa}^2\|\overline{G}(x,\,.\,)\|_{L^2(0,h)}\|v\|_{L^2(0,h)}$$

and taking the L^2-norm over $(0,h)$ yields

$$\|v - v_h\|_{L^2(0,h)} \leqslant h^{1/2} \sup_{x\in(0,h)} \Big\{ \|\partial_2 \overline{G}(x,\,.\,)\|_{L^2(0,h)} \|v'\|_{L^2(0,h)} + \overline{\kappa}^2 \|\overline{G}(x,\,.\,)\|_{L^2(0,h)} \|v\|_{L^2(0,h)} \Big\}.$$

Now we use the estimates for Green's function from Theorem 4.4 in the three different cases. Note that $h \leqslant 1/2$ implies $\overline{\kappa} \geqslant \pi$ in Cases 2 and 3.

Case 1 $\overline{\gamma} h^2 \leqslant -(\pi/2)^2$ (i. e., $\overline{\kappa} h \geqslant \pi/2$). Then

$$\begin{aligned}\|v - v_h\|_{L^2(0,h)} &\leqslant \frac{h}{|\sin(\overline{\kappa}h)|} \|v'\|_{L^2(0,h)} + \frac{\overline{\kappa}h}{|\sin(\overline{\kappa}h)|} \|v\|_{L^2(0,h)} \\ &\leqslant \frac{h}{|\sin(\overline{\kappa}h)|} \big(\|v'\|_{L^2(0,h)} + \overline{\kappa} \|v\|_{L^2(0,h)} \big) \leqslant \frac{2}{|\sin(\overline{\kappa}h)|} h \|v\|_{\overline{\kappa};(0,h)}.\end{aligned}$$

Case 2 $|\overline{\gamma}| h^2 < (\pi/2)^2$ (i. e., $\overline{\kappa} h < \pi/2$). Then

$$\|v - v_h\|_{L^2(0,h)} \leqslant h \big(\|v'\|_{L^2(0,h)} + \overline{\kappa}(\overline{\kappa}h) \|v\|_{L^2(0,h)} \big) \leqslant 2h \|v\|_{\overline{\kappa};(0,h)}.$$

Case 3 $\overline{\gamma} h^2 \geqslant (\pi/2)^2$ (i. e., $\overline{\kappa} h \geqslant \pi/2$). Then

$$\begin{aligned}\|v - v_h\|_{L^2(0,h)} &\leqslant \frac{h^{1/2}}{\overline{\kappa}^{1/2}} \|v'\|_{L^2(0,h)} + (\overline{\kappa}h)^{1/2} \|v\|_{L^2(0,h)} \\ &\leqslant \frac{h^{1/2}}{\overline{\kappa}^{1/2}} \big(\|v'\|_{L^2(0,h)} + \overline{\kappa} \|v\|_{L^2(0,h)} \big) \leqslant 2h \|v\|_{\overline{\kappa};(0,h)}.\end{aligned}$$

So we can bound $\|v - v_h\|_{L^2(0,h)}$ by $2h/|\sin(\overline{\kappa}h)|\, \|v\|_{\overline{\kappa};(0,h)}$ in Case 1 and by $2h\|v\|_{\overline{\kappa};(0,h)}$ in Cases 2 and 3.

If $\overline{L}v$ exists, we start with the representation

$$(v - v_h)(x) = \int_0^h \overline{G}(x,y) \overline{L}v(y)\,dy$$

to get

$$\|v - v_h\|_{L^2(0,h)} \leqslant h^{1/2} \sup_{x\in(0,h)} \Big\{ \|\overline{G}(x,\,.\,)\|_{L^2(0,h)} \Big\} \|\overline{L}v\|_{L^2(0,h)}.$$

For the factor in front of $\|\overline{L}v\|_{L^2(0,h)}$ we get the bounds $h^{1/2}\,h^{1/2}/(\overline{\kappa}|\sin(\overline{\kappa}h)|) \leqslant 2h^2/|\sin(\overline{\kappa}h)|$ (Case 1), $h^{1/2}\,h^{3/2} \leqslant 2h^2$ (Case 2), and $h^{1/2}/\overline{\kappa}^{3/2} \leqslant 2h^2$ (Case 3). To derive the result for the energy norm we start from

$$\overline{\kappa}_* \|v - v_h\|_{L^2(0,h)} \leqslant \overline{\kappa}_* h^{1/2} \sup_{x\in(0,h)} \Big\{ \|\overline{G}(x,\,.\,)\|_{L^2(0,h)} \Big\} \|\overline{L}v\|_{L^2(0,h)}.$$

For the factor in front of $\|\overline{L}v\|_{L^2(0,h)}$ we get now the bounds $\overline{\kappa}h^{1/2}\,h^{1/2}/(\overline{\kappa}|\sin(\overline{\kappa}h)|) \leqslant 2h/|\sin(\overline{\kappa}h)|$ (Case 1), $\overline{\kappa}_* h^{1/2}\,h^{3/2} \leqslant 2h$ (Case 2), and $\overline{\kappa}h^{1/2}/\overline{\kappa}^{3/2} \leqslant 2h$ (Case 3).

Finally, to bound derivatives we start from

$$(v - v_h)'(x) = \int_0^h \partial_1 \overline{G}(x,y) \overline{L}v(y)\,dy$$

to get

$$\begin{aligned}\|(v-v_h)'\|^2_{L^2(0,h)} &= \int_0^h \Big(\int_0^h \partial_1\overline{G}(x,y)\overline{L}v(y)\,dy\Big)^2 dx\\ &= \int_0^h\int_0^h \Big(\int_0^h \partial_1\overline{G}(x,y)\partial_1\overline{G}(x,z)\,dx\Big)\overline{L}v(y)\overline{L}v(z)\,dy\,dz\\ &\leqslant \int_0^h\int_0^h \|\partial_1\overline{G}(\,.\,,y)\|_{L^2(0,h)}\|\partial_1\overline{G}(\,.\,,z)\|_{L^2(0,h)}|\overline{L}v(y)|\,|\overline{L}v(z)|\,dy\,dz\\ &\leqslant h\Big(\sup_{y\in(0,h)}\{\|\partial_1\overline{G}(\,.\,,y)\|_{L^2(0,h)}\}\|\overline{L}v\|_{L^2(0,h)}\Big)^2.\end{aligned}$$

By symmetry of Green's function we have $\|\partial_1\overline{G}(\,.\,,y)\|_{L^2(0,h)} = \|\partial_2\overline{G}(y,\,.\,)\|_{L^2(0,h)}$ and we can thus refer to the same estimates as above to bound $\|(v-v_h)'\|_{L^2(0,h)}$ by $2h/|\sin(\overline{\kappa}h)|\,\|\overline{L}v\|_{L^2(0,h)}$ in Case 1 and $2h\|\overline{L}v\|_{L^2(0,h)}$ in Cases 2 and 3.

These local estimates can directly be rewritten as estimates on K_i and the estimates (4.6)–(4.7) for $v-v_h$ on Ω are obtained from this after squaring and summation over all K_i. □

Theorem 4.2. *Let $v\in H^1(\Omega)$ and $Q_h^1 v\in V_h^1$ be as in (4.5) with $h\leqslant 1/2$. Then, $Q_h^1 v$ is well defined if we require that $C_{\mathrm{I}}<\infty$, C_{I} from (3.7), and we have the estimates*

$$\|\overline{\kappa}_h(v-Q_h^1 v)\|_{L^2(\Omega)} \leqslant \sqrt{3}(1+\sqrt{2\sigma})C_{\mathrm{I}}\|\overline{\kappa}_h v\|_{L^2(\Omega)}, \tag{4.8}$$

$$\|v-Q_h^1 v\|_{L^2(\Omega)} \leqslant \sqrt{3}(2+\sqrt{2\sigma})C_{\mathrm{I}}\Big(\sum_{i=1}^{N+1} h_i^2 \|v\|^2_{\overline{\kappa}_h;K_i}\Big). \tag{4.9}$$

Here, $\sigma := \max_{i=1,\dots,N}\{h_i/h_{i+1}, h_{i+1}/h_i\}$.

Proof. From its definition (4.4) we see $|F_i(v)|\leqslant 1/h_{m(i)}^{1/2}\,\|v\|_{L^2(K_{m(i)})}$, and thus

$$\begin{aligned}\|Q_h^1 v\|_{L^2(K_i)} &\leqslant |F_{i-1}(v)|\,\|\psi_{i-1}\|_{L^2(K_i)} + |F_i(v)|\,\|\psi_i\|_{L^2(K_i)}\\ &\leqslant \Big(h_{m(i-1)}|F_{i-1}(v)|^2 + h_{m(i)}|F_i(v)|^2\Big)^{1/2}\Big(\frac{1}{h_{m(i-1)}}\|\psi_{i-1}\|^2_{L^2(K_i)} + \frac{1}{h_{m(i)}}\|\psi_i\|^2_{L^2(K_i)}\Big)^{1/2}\\ &\leqslant C_{\mathrm{I}}\Big(\|v\|^2_{L^2(K_{m(i-1)})} + \|v\|^2_{L^2(K_{m(i)})}\Big)^{1/2}\Big(\frac{h_i}{h_{m(i-1)}} + \frac{h_i}{h_{m(i)}}\Big)^{1/2}\\ &\leqslant \sqrt{2\sigma}C_{\mathrm{I}}\|v\|_{L^2(K_{m(i-1)}\cup K_{m(i)})}.\end{aligned}$$

By definition of $m(i-1), m(i)$ it holds

$$\overline{\kappa}_{h,i}\|v\|_{L^2(K_{m(i-1)}\cup K_{m(i)})} \leqslant \|\overline{\kappa}_h v\|_{L^2(K_{m(i-1)}\cup K_{m(i)})} \leqslant \|\overline{\kappa}_h v\|_{L^2(K_{i-1}\cup K_i\cup K_{i+1})}$$

and therefore

$$\overline{\kappa}_{h,i}\|v-v_h\|_{L^2(K_i)} \leqslant \|\overline{\kappa}_h v\|_{L^2(K_i)} + \overline{\kappa}_{h,i}\|v_h\|_{L^2(K_i)} \leqslant (1+\sqrt{2\sigma})C_{\mathrm{I}}\|\overline{\kappa}_h v\|_{L^2(K_{i-1}\cup K_i\cup K_{i+1})}.$$

To prove the next assertion we start from the decomposition

$$v - Q_h^1 v = v - I_h^1 v + \big(v(x_{i-1}) - F_{i-1}(v)\big)\psi_{i-1} + \big(v(x_i) - F_i(v)\big)\psi_i,$$

which can be estimated using previous results by

$$
\begin{aligned}
&\|v - Q_h^1 v\|_{L^2(K_i)} \\
&\quad = \|v - I_h^1 v\|_{L^2(K_i)} + h_{m(i-1)}^{1/2}\|v'\|_{L^2(K_{m(i-1)})}\|\psi_{i-1}\|_{L^2(K_i)} + h_{m(i)}^{1/2}\|v'\|_{L^2(K_{m(i)})}\|\psi_i\|_{L^2(K_i)} \\
&\quad \leqslant 2C_{\mathrm{I}} h_i \|v\|_{\overline{\kappa}_{h,i};K_i} + \sqrt{2\sigma} C_{\mathrm{I}} \|hv'\|_{L^2(K_{i-1}\cup K_i \cup K_{i+1})}.
\end{aligned}
$$

The final result for the L^2-norms follows by summation over i. □

4.2. Estimates for the interpolation onto V_h^2

Theorem 4.3. *Let $v \in H^1(\Omega)$ and $I_h^2 v, \widetilde{I}_h^2 v \in V_h^2$ be as in (4.2) and (4.3), respectively, with $h \leqslant 1/2$. Then, both interpolants are well defined if we require that $C_{\mathrm{I}} < \infty$, C_{I} from (3.7), and we have the estimate*

$$
\|v - I_h^2 v\|_{L^2(\Omega)} \leqslant 2C_{\mathrm{I}} \Big(\sum_{i=1}^{N+1} h_i^2 \|v'\|_{L^2(K_i)}^2 \Big)^{1/2}. \tag{4.10}
$$

The estimate (4.7) holds correspondingly with $\widetilde{I}_h^2$ for $v \in H^2(\Omega)$ and, furthermore, for $v \in H^3(\Omega)$

$$
2\|h^{-1}(v - \widetilde{I}_h^2 v)\|_{L^2(\Omega)} + \|v - \widetilde{I}_h^2 v\|_{\overline{\kappa}_h;\Omega} \leqslant 4C_{\mathrm{I}} \|h^2 (L(\overline{\gamma}_h)v)_h'\|_{L^2(\Omega)}. \tag{4.11}
$$

$(\,.\,)_h'$ denotes the piecewise derivative with respect to $\mathcal{K}$.

Proof. We use the notation introduced in the proof of Theorem 4.1. From definition (4.2) we obtain

$$
\begin{aligned}
(v - I_h^2 v)(x) &= v(x) - \Big(v(0)\psi_0(x) + v(h)\psi_1(x) + \frac{\overline{\gamma}}{2}(v(0) + v(h))\psi_{1/2}(x) \Big) \\
&= v(x) - \Big(v(0)\psi_0(x) + v(h)\psi_1(x) + \frac{1}{2}(v(0) + v(h)) \big(1 - \psi_0(x) - \psi_1(x)\big) \Big) \\
&= v(x) - \frac{1}{2}(v(0) + v(h)) - \frac{1}{2}\big(v(h) - v(0)\big) \big(\psi_1(x) - \psi_0(x)\big)
\end{aligned}
$$

and this yields the pointwise bound

$$
\big|(v - I_h^2 v)(x)\big| \leqslant \Big(1 + \frac{1}{2}\|\psi_0\|_{L^\infty(0,h)} + \frac{1}{2}\|\psi_1\|_{L^\infty(0,h)}\Big) \int_0^h |v'(y)|\, dy.
$$

For $\overline{\gamma} > 0$, or $\overline{\gamma} < 0$ and $\overline{\kappa}h < \pi/2$, we have $\|\psi_0\|_{L^\infty(0,h)} = \|\psi_1\|_{L^\infty(0,h)} = 1$ and therefore

$$
\|v - I_h^2 v\|_{L^2(0,h)} \leqslant 2h\|v'\|_{L^2(0,h)},
$$

while for $\overline{\gamma} < 0$ and $\overline{\kappa}h \geqslant \pi/2$ we obtain

$$
\|v - I_h^2 v\|_{L^2(0,h)} \leqslant \frac{2h}{|\sin(\overline{\kappa}h)|}\|v'\|_{L^2(0,h)}.
$$

This can directly be rewritten as an estimate for $v - v_h$ on K_i and the estimate (4.10) for $v - v_h$ on Ω is obtained from this after squaring and summation over all K_i.

We let $v_h := \widetilde{I}_h^2 v$ and get by the representation formula (with $\overline{G}(x,y) = G(x,y;\overline{\gamma})$)

$$(v - v_h)(x) = \int_0^h \overline{G}(x,y)\overline{L}(v - v_h)(y)\,dy = \int_0^h \overline{G}(x,y)\Big(\overline{L}v(y) - \frac{1}{h}\int_0^h \overline{L}v\Big)\,dy.$$

If we only exploit the L^2-stability of the mean value, we can repeat the arguments from Theorem 4.1 to get the same results as for I_h^1. Now let $v \in H^3(\Omega)$. Using $\|\overline{L}v - \int_0^h \overline{L}v/h\|_{L^2(0,h)} \leqslant h\|(\overline{L}v)'\|_{L^2(0,h)}$, we can achieve the pointwise bound

$$|(v - v_h)(x)| \leqslant h^{3/2} \sup_{x\in(0,h)} \big\{\|\overline{G}(x,\,.\,)\|_{L^2(0,h)}\big\}\|(\overline{L}v)'\|_{L^2(0,h)}.$$

Similarly, we get the pointwise estimate for $(v - v_h)'(x)$ with $\partial_1\overline{G}$ on the left-hand side and the remainder is as in the proof of Theorem 4.1. $\square$

4.3. Estimates for Green's functions

The boundary value problem (2.1)–(2.2) has a Green's function $G : \Omega \times \Omega \to \mathbb{R}$, which allows the solution to be represented as

$$u(x) = \int_0^1 G(x,y)Lu(y)\,dy = \int_0^1 G(x,y)f(y)\,dy \tag{4.12}$$

for every $f \in L^1(\Omega)$. To construct G one can determine the fundamental system $\{\Psi_0, \Psi_1\}$ that fulfills the boundary value problems $L(\gamma)\Psi_0 = 0$ in Ω, $\Psi_0(0) = 1$, $\Psi_0(1) = 0$, and $L(\gamma)\Psi_1 = 0$ in Ω, $\Psi_1(0) = 0$, $\Psi_1(1) = 1$. Then G is given by

$$G(x,y) = \frac{1}{\Psi_0'(0)} \begin{cases} \Psi_0(x)\Psi_1(y) & \text{for } y \leqslant x, \\ \Psi_1(x)\Psi_0(y) & \text{for } x \leqslant y. \end{cases}$$

G is explicitly known only in very special cases, for example, when γ is constant.

The results of the previous section required estimates on Green's function $G(\,.\,,\,.\,;\overline{\gamma})$ for the boundary value problem (2.1)–(2.2) with constant coefficient $\overline{\gamma}$ on $(0,h)$. The following theorem provides these estimates (some details were elaborated in the diploma thesis of A. Shutovich that has been prepared under the supervision of the author).

Theorem 4.4. *Let G be Green's function of the boundary value problem* (2.1)–(2.2) *with constant coefficient $\overline{\gamma} = \pm\overline{\kappa}^2$ on $(0,h)$, then*

$$\sup_{x\in(0,h)} \big\{\|G(x,\,.\,;\overline{\gamma})\|_{L^2(0,h)}\big\} \leqslant h^{3/2} \begin{cases} \frac{1}{\overline{\kappa}h|\sin(\overline{\kappa}h)|} & \text{if } \overline{\gamma} < 0,\ \overline{\kappa}h > \pi/2, \\ \frac{1}{12} & \text{if } \overline{\kappa}h < \pi/2, \\ \frac{1}{(\overline{\kappa}h)^{3/2}} & \text{if } \overline{\gamma} > 0,\ \overline{\kappa}h > \pi/2, \end{cases}$$

and

$$\sup_{x\in(0,h)} \big\{\|\partial_2 G(x,\,.\,;\overline{\gamma})\|_{L^2(0,h)}\big\} \leqslant h^{1/2} \begin{cases} \frac{1}{|\sin(\overline{\kappa}h)|} & \text{if } \overline{\gamma} < 0,\ \overline{\kappa}h > \pi/2, \\ 1 & \text{if } \overline{\kappa}h < \pi/2, \\ \frac{1}{(\overline{\kappa}h)^{1/2}} & \text{if } \overline{\gamma} > 0,\ \overline{\kappa}h > \pi/2. \end{cases}$$

Proof. If G is Green's function of the boundary value problem (2.1)–(2.2) with constant coefficient on $(0,h)$, then $\widehat{G}(\xi,\eta;z) = 1/h\, G(h\xi, h\eta; z/h^2)$ is the corresponding Green's function on $(0,1)$. Therefore, we obtain after scaling

$$\begin{aligned}\|G(x,\,.\,;\overline{\gamma})\|_{L^2(0,h)} &= h^{3/2}\|\widehat{G}(x/h,\,.\,;\overline{\gamma}h^2)\|_{L^2(0,1)},\\ \|\partial_2 G(x,\,.\,;\overline{\gamma})\|_{L^2(0,h)} &= h^{1/2}\|\partial_2\widehat{G}(x/h,\,.\,;\overline{\gamma}h^2)\|_{L^2(0,1)}.\end{aligned}$$

In the following, we will estimate

$$g_0(\xi;z) := \|\widehat{G}(\xi,\,.\,;z)\|^2_{L^2(0,1)} \quad \text{and} \quad g_1(\xi;z) := \|\partial_2\widehat{G}(\xi,\,.\,;z)\|^2_{L^2(0,1)}$$

for different values of z.

Case 1: $z<0$, $k := \sqrt{|z|} \geqslant \pi/2$, $\sin(k) \neq 0$ We start calculating $g_0(\xi;-k^2)$ to

$$g_0(\xi;-k^2) = -\frac{\sin(k\xi)\sin(k(1-\xi))}{2k^3|\sin(k)|} + \frac{(1-\xi)\sin(k\xi)^2 + \xi\sin(k(1-\xi))^2}{2k^2\sin(k)^2}.$$

Since $\xi \in (0,1)$, $(1-\xi)\sin(k\xi)^2 + \xi\sin(k(1-\xi))^2 \leqslant (1-\xi)+\xi = 1$, and $\sin(k)/k \leqslant 1$, one gets that

$$g_0(\xi;-k^2) \leqslant \frac{1}{2k^3|\sin(k)|} + \frac{1}{2k^2\sin(k)^2} \leqslant \frac{1}{k^2\sin(k)^2},$$

thus

$$\sup_{\xi\in(0,1)}\left\{\|\widehat{G}(\xi,\,.\,;-k^2)\|_{L^2(0,1)}\right\} \leqslant \frac{1}{k|\sin(k)|}.$$

Similarly, we get for $g_1(\xi;-k^2)$

$$\begin{aligned}g_1(\xi;-k^2) &= \frac{\sin(k\xi)\sin(k(1-\xi))}{2k|\sin(k)|} + \frac{(1-\xi)\sin(k\xi)^2 + \xi\sin(k(1-\xi))^2}{2\sin(k)^2}\\ &\leqslant \frac{1}{2k|\sin(k)|} + \frac{1}{2\sin(k)^2} \leqslant \frac{1}{\sin(k)^2},\end{aligned}$$

resulting in

$$\sup_{\xi\in(0,1)}\left\{\|\partial_2\widehat{G}(\xi,\,.\,;-k^2)\|_{L^2(0,1)}\right\} \leqslant \frac{1}{|\sin(k)|}.$$

Case 2: $z>0$, $k := \sqrt{|z|} \geqslant \pi/2$ We now get

$$\begin{aligned}g_0(\xi;k^2) &= \frac{\sinh(k\xi)\sinh(k(1-\xi))}{2k^3\sinh(k)} - \frac{(1-\xi)\sinh(k\xi)^2 + \xi\sinh(k(1-\xi))^2}{2k^2\sinh(k)^2}\\ &=: T_1 - T_2.\end{aligned}$$

Since all terms g_0, T_1, T_2 are positive, we conclude $T_2 \leqslant T_1$ and $g_0(\xi;k^2) \leqslant T_1$. By elementary calculus we see that $e^k/4 \leqslant (e^k - e^{-k})/2$ for $k \geqslant \log(2)/2$ and thus

$$T_1 \leqslant \frac{1}{4}\,\frac{e^k}{2k^3\sinh(k)} \leqslant \frac{1}{2}\,\frac{\sinh(k)}{2k^3\sinh(k)} \leqslant \frac{1}{4k^3}$$

showing

$$\sup_{\xi\in(0,1)}\big\{\|\widehat{G}(\xi,\,.\,;k^2)\|_{L^2(0,1)}\big\}\leqslant\frac{1}{2k^{3/2}}.$$

For $g_1(\xi;k^2)$ we get

$$g_1(\xi;k^2)=k^2(T_1+T_2)\leqslant 2k^2T_1\leqslant\frac{1}{2k}$$

and so

$$\sup_{\xi\in(0,1)}\big\{\|\partial_2\widehat{G}(\xi,\,.\,;k^2)\|_{L^2(0,1)}\big\}\leqslant\frac{1}{k^{1/2}}.$$

Case 3: $z<0$, $k:=\sqrt{|z|}<\pi/2$ Taking the derivative of $g_0(\,.\,;-k^2)$ (given above) we arrive at

$$g_0'(\xi;-k^2)=\frac{\xi(1-\xi)}{\sin(k)^2}\Big(\frac{\sin(2k\xi)}{2k\xi}-\frac{\sin(2k(1-\xi))}{2k(1-\xi)}\Big).$$

By monotonicity of $t\mapsto\sin(t)/t$ in $(0,\pi)$ we see that $g_0'(\,.\,;-k^2)$ is positive for $\xi\in(0,1/2)$ and negative for $\xi\in(1/2,1)$. Thus the maximum of $g_0(\,.\,;-k^2)$ is attained for $\xi=1/2$. Using $\sin(t)\geqslant t-t^3/6$ we obtain

$$\sup_{\xi\in(0,1)}\big\{\|\widehat{G}(\xi,\,.\,;-k^2)\|_{L^2(0,1)}\big\}\leqslant g_0\big(\frac{1}{2};k^2\big)=\frac{k-\sin(k)}{2k^3}\frac{\sin(k/2)^2}{\sin(k)^2}\leqslant\frac{k^3/6}{2k^3}=\frac{1}{12}.$$

For $g_1(\,.\,;-k^2)$ we get

$$g_1'(\xi;-k^2)=\frac{\cos(2k\xi)-\cos(2k(1-\xi))+k\big((1-\xi)\sin(2k\xi)-\xi\sin(2k(1-\xi))\big)}{1-\cos(2k)}.$$

As before (using monotonicity of the mappings $\xi\mapsto\cos(2k\xi)-\cos(2k(1-\xi))$ and $\xi\mapsto(1-\xi)k\sin(2k\xi)-\xi k\sin(2k(1-\xi)))$ we conclude that $g_1(\,.\,;-k^2)$ attains its maximum at $\xi=1/2$ and this gives

$$\sup_{\xi\in(0,1)}\big\{\|\partial_2\widehat{G}(\xi,\,.\,;-k^2)\|_{L^2(0,1)}\big\}\leqslant g_1\big(\frac{1}{2}\big)=\frac{k+\sin(k)}{2k}\frac{\sin(k/2)^2}{\sin(k)^2}\leqslant\frac{2k}{2k}=1.$$

Case 4: $z>0$, $k:=\sqrt{|z|}<\pi/2$ As above we see from

$$g_0'(\xi;k^2)=\frac{\xi(1-\xi)}{\sinh(k)^2}\Big(\frac{\sinh(2k(1-\xi))}{2k(1-\xi)}-\frac{\sinh(2k\xi)}{2k\xi}\Big)$$

and monotonicity of $t\mapsto\sinh(t)/t$ that the maximum of $g_0(\,.\,;k^2)$ is attained for $\xi=1/2$. Using trigonometric relations and $\sinh(t)\leqslant t+t^3/6\,\cosh(t)$ we obtain

$$\sup_{\xi\in(0,1)}\big\{\|\widehat{G}(\xi,\,.\,;k^2)\|_{L^2(0,1)}\big\}\leqslant g_0\big(\frac{1}{2};k^2\big)=\frac{\sinh(k)-k}{2k^3}\frac{\sinh(k/2)^2}{\sinh(k)^2}=\frac{\sinh(k)-k}{4k^3(1+\cosh(k))}$$
$$\leqslant\frac{k^3/6\,\cosh(k)}{4k^3(1+\cosh(k))}\leqslant\frac{1}{24}.$$

For $g_1(\,.\,;k^2)$ we get

$$g_1'(\xi;k^2) = \frac{\cosh(2k\xi) - \cosh(2k(1-\xi)) - k\big((1-\xi)\sinh(2k\xi) - \xi\sinh(2k(1-\xi))\big)}{1-\cosh(2k)}$$

and the reasoning is similar to the one used before to conclude that $g_1(\,.\,;k^2)$ is maximal at $\xi = 1/2$. Using $\sinh(t) \leqslant t\cosh(t)$ we obtain

$$\begin{aligned}\sup_{\xi\in(0,1)} \big\{\|\partial_2\widehat{G}(\xi,\,.\,;k^2)\|_{L^2(0,1)}\big\} \leqslant g_1\big(\tfrac{1}{2},k^2\big) &= \frac{k+\sinh(k)}{2k}\frac{\sinh(k/2)^2}{\sinh(k)^2} = \frac{k+\sinh(k)}{2k(1+\cosh(k))} \\ &\leqslant \frac{k}{2k} = \frac{1}{2}.\end{aligned}$$

□

5. A posteriori error estimates

5.1. Error representation

Using the solution properties of $u \in V$ and $u_h \in V_h \subset V$, respectively, we get, for arbitrary $v \in V$ and $v_h \in V_h$, the representation

$$\begin{aligned}B(u-u_h,v) &= \int_0^1 \big\{(u-u_h)'v' + \gamma(u-u_h)v\big\} \\ &= \sum_{i=1}^{N}[u_h']_{x_i}(v-v_h)(x_i) - \int_0^1 r_h(v-v_h) + \int_0^1 \big\{(f-f_h)v - (\gamma-\gamma_h)u_h v\big\}\end{aligned} \tag{5.1}$$

for the error $u-u_h$ with the *derivative jump* in the grid point x_i

$$[u_h']_{x_i} := \lim_{s\to 0}\big(u_h'(x_i+s) - u_h'(x_i-s)\big)$$

and the *residual* r_h piecewise defined by

$$r_h := -u_h'' + \gamma_h u_h - f_h \qquad \text{on } K$$

for all $K \subset \mathcal{K}$. For $u_h \in V_h^1$ the residual is $r_h = L(\gamma_h)u_h - f_h = L(\overline{\gamma}_h)u_h + (\gamma_h - \overline{\gamma}_h)u_h - f_h = (\gamma_h - \overline{\gamma}_h)u_h - f_h$. The choice $\gamma_h = \overline{\gamma}_h$ then gives $r_h = -f_h$. In case $u_h \in V_h^2$ the residual is $r_h = L(\overline{\gamma}_h)u_h + (\gamma_h - \overline{\gamma}_h)u_h - f_h$. Note that $L(\overline{\gamma}_h)u_h = b_h$, where $b_h = b_j$ on $K = K_j$ (see (3.6)) is directly accessible from the discrete solution. In general we thus have $r_h = b_h - f_h + (\gamma_h - \overline{\gamma}_h)u_h$. If we take $\gamma_h = \overline{\gamma}_h$, then we find (see Sect. 3.4) $b_j = \int_{K_j} f_h\psi_{j-1/2}/\int_{K_j}\psi_{j-1/2}$ and therefore we get $r_h = 0$ for piecewise constant f_h.

For the following section recall the definitions of c_γ (cf. 2.4), $\varGamma$ (see Section 3.1), $m(i)$ (from (4.4)), and σ (Theorem 4.2).

5.2. Upper bound

Theorem 5.1. *Let $u \in V$ be a solution to* (2.1)–(2.2) *and u_h, either in V_h^1 or V_h^2, be the respective solution of the discrete problem* (3.1) *on a grid with $C_{\mathrm{I}} < \infty$, C_{I} from* (3.7)*. Then* *the following a posteriori error estimate is valid*

$$c_\gamma \|\!|u-u_h|\!\|_{\kappa;\Omega} \leqslant 4(1+\sqrt{\sigma})C_{\mathrm{I}}\Big(1+\Gamma(\kappa,\overline{\kappa}_h)\Big)\Big(\sum_{i=1}^{N+1}\eta_i^2\Big)^{1/2} + \Big\|\frac{1}{\kappa_*}(f-f_h)\Big\|_{L^2(\Omega)} + \Big\|\frac{\gamma-\gamma_h}{\gamma_*}\kappa_* u_h\Big\|_{L^2(\Omega)}$$

with the local error indicator

$$\eta_i := \Big(h_{m(i)}\min\Big\{1,\frac{1}{\overline{\kappa}_{h,m(i)}h_{m(i)}}\Big\}[u_h']_{x_i}^2 + h_i^2\min\Big\{1,\frac{1}{\overline{\kappa}_{h,i}h_i}\Big\}^2\|r_h\|^2_{L^2(K_i)}\Big)^{1/2}.$$

Proof. From (5.1) we get for arbitrary $v\in V$, $v_h\in V_h$,

$$|B(u-u_h,v)| \leqslant \sum_{i=1}^{N}\big|[u_h']_{x_i}(v-v_h)(x_i)\big| + \Big|\int_\Omega r_h(v-v_h)\Big| + \Big(\Big\|\frac{1}{\kappa_*}(f-f_h)\Big\|_{L^2(\Omega)} + \Big\|\frac{1}{\kappa_*}(\gamma-\gamma_h)u_h\Big\|_{L^2(\Omega)}\Big)\|\kappa_* v\|_{L^2(\Omega)}.$$

Taking $I_h^1 v$ for v_h we would have $(v-I_h^1 v)(x_i)=0$ and the first term on the right-hand side would disappear. However, if $\overline{\kappa}_{h,i}h_i$ is large, we need a better estimate. This can only be achieved taking $v_h = Q_h^1 v$, since Q_h^1 is continuous on L^2 (unlike I_h^1). By Theorem 4.2 (proof) we observe

$$\overline{\kappa}_{h,i}\|v-v_h\|_{L^2(K_i)} \leqslant (1+\sqrt{2\sigma})C_{\mathrm{I}}\|\overline{\kappa}_h v\|_{L^2(K_{i-1}\cup K_i\cup K_{i+1})},$$

while for more regular v one can derive

$$\|v-v_h\|_{L^2(K_i)} \leqslant (2+\sqrt{2\sigma})C_{\mathrm{I}}h_i\|\!|v|\!\|_{\overline{\kappa}_h;K_{i-1}\cup K_i\cup K_{i+1}}.$$

We are now in the position to choose the best of both choices to get

$$\|v-v_h\|_{L^2(K_i)} \leqslant (2+\sqrt{2\sigma})C_{\mathrm{I}}h_i\min\Big\{1,\frac{1}{\overline{\kappa}_{h,i}h_i}\Big\}\|\!|v|\!\|_{\overline{\kappa}_h;K_{i-1}\cup K_i\cup K_{i+1}}.$$

Thus the residual term is bounded as follows

$$\begin{aligned}\Big|\int_\Omega r_h(v-v_h)\Big| &\leqslant \sum_{i=1}^{N+1}\|r_h\|_{L^2(K_i)}\|v-v_h\|_{L^2(K_i)}\\ &\leqslant (2+\sqrt{2\sigma})C_{\mathrm{I}}\sum_{i=1}^{N+1}h_i\min\Big\{1,\frac{1}{\overline{\kappa}_{h,i}h_i}\Big\}\|r_h\|_{L^2(K_i)}\|\!|v|\!\|_{\overline{\kappa}_h;K_{i-1}\cup K_i\cup K_{i+1}}\\ &\leqslant \sqrt{3}(2+\sqrt{2\sigma})C_{\mathrm{I}}\Big(\sum_{i=1}^{N+1}h_i^2\min\Big\{1,\frac{1}{\overline{\kappa}_{h,i}h_i}\Big\}^2\|r_h\|^2_{L^2(K_i)}\Big)^{1/2}\|\!|v|\!\|_{\overline{\kappa}_h}.\end{aligned}$$

In order to estimate the jump residual term, we first observe

$$|(v-v_h)(x_i)| = |v(x_i)-F_i(v)| = \Big|\frac{1}{h_{m(i)}}\int_{K_{m(i)}}\{v(x_i)-v\}\Big| \leqslant \|\sqrt{h}v'\|_{L^2(K_{m(i)})}.$$

A different estimate is provided as follows: first note that

$$|(v - v_h)(x_i)| \leqslant |v(x_i)| + |F_i(v)| \leqslant 2\|v\|_{L^\infty(K_{m(i)})}.$$

For arbitrary $\varepsilon \in (0,1)$ we have from Lemma 5.1 by rescaling and with $j := m(i)$

$$\begin{aligned}\|v\|_{L^\infty(K_j)} &\leqslant \varepsilon h_j^{1/2}\|v'\|_{L^2(K_j)} + \frac{2}{\varepsilon h_j^{1/2}}\|v\|_{L^2(K_j)} \\ &= \varepsilon h_j^{1/2}\|v'\|_{L^2(K_j)} + \frac{2}{\varepsilon h_j^{1/2}\overline{\kappa}_{h,j}}\|\overline{\kappa}_{h,j} v\|_{L^2(K_j)} \\ &\leqslant \Big(\varepsilon^2 h_j + \frac{4}{\varepsilon^2 h_j \overline{\kappa}_{h,j}^2}\Big)^{1/2} \|v\|_{\overline{\kappa}_{h,j};K_j}.\end{aligned}$$

The choice $\varepsilon^2 h_j = 2/\overline{\kappa}_{h,j}$ yields

$$\|v\|_{L^\infty(K_j)} \leqslant \frac{2}{\overline{\kappa}_{h,j}^{1/2}} \|v\|_{\overline{\kappa}_{h,j};K_j}$$

so that we get

$$|(v - v_h)(x_i)| \leqslant \frac{4}{\overline{\kappa}_{h,j}^{1/2}} \|v\|_{\overline{\kappa}_h;K_j}.$$

We can take the best of the two choices to conclude

$$\begin{aligned}|(v - v_h)(x_i)| &\leqslant \min\Big\{h_j^{1/2}, \frac{4}{\overline{\kappa}_{h,j}^{1/2}}\Big\} \|v\|_{\overline{\kappa}_h;K_j} \\ &\leqslant 4h_j^{1/2} \min\Big\{1, \frac{1}{(\overline{\kappa}_{h,j} h_j)^{1/2}}\Big\} \|v\|_{\overline{\kappa}_h;K_j}.\end{aligned}$$

This allows us to derive

$$\begin{aligned}\sum_{i=1}^N \big|[u_h']_{x_i}\big|\,\big|(v - v_h)(x_i)\big| &\leqslant 4 \sum_{i=1}^N h_{m(i)}^{1/2} \min\Big\{1, \frac{1}{\overline{\kappa}_{h,m(i)} h_{m(i)}}\Big\}^{1/2} \big|[u_h']_{x_i}\big| \|v\|_{\overline{\kappa}_{h,m(i)};K_{m(i)}} \\ &\leqslant 8\Big(\sum_{i=1}^N h_{m(i)} \min\Big\{1, \frac{1}{\overline{\kappa}_{h,m(i)} h_{m(i)}}\Big\} [u_h']_{x_i}^2\Big)^{1/2} \|v\|_{\overline{\kappa}_h}.\end{aligned}$$

The assertion follows with

$$\|\overline{\kappa}_h v\|_{L^2(\Omega)} \leqslant \|\kappa v\|_{L^2(\Omega)} + \|(\overline{\kappa}_h - \kappa)v\|_{L^2(\Omega)} \leqslant \big(1 + \Gamma(\kappa, \overline{\kappa}_h)\big)\|\kappa_* v\|_{L^2(\Omega)}$$

and using the inf-sup condition (2.4). Note, that it suffices to consider the interpolation Q_h^1 since $V_h^1 \subset V_h^2$. □

Lemma 5.1. *Let $I := (0,1)$. Then for all $\epsilon > 0$ and all $v \in H^1(I)$ it holds that*

$$\|v\|_{L^\infty(I)} \leqslant \epsilon\|v'\|_{L^2(I)} + \frac{3/2}{\epsilon}\|v\|_{L^2(I)}.$$

Proof. Let $v \in C^1(I)$, J a subinterval in I, and $x, y \in J$. Then $v(x) = v(y) + \int_y^x v'$ and this yields

$$|v(x)|^2 \leqslant \frac{3}{2}\Big(|v(y)|^2 + 2|x-y| \int_J |v'|^2\Big).$$

Now we integrate over $y \in J$ to get that for all $x \in J$

$$|v(x)|^2 \leqslant \frac{3}{2}\Big(\frac{1}{|J|}\int_J |v|^2 + |J| \int_J |v'|^2\Big).$$

Now choose x_0 where the maximum of v is attained, choose $\epsilon > 0$ and take J as an interval of length $|J| = 2\epsilon^2/3$ around x_0. Taking the root and a density argument proves the assertion. □

5.3. Lower bound

Theorem 5.2. *Let $u \in V$ be a solution to* (2.1)–(2.2) *and $u_h \in V_h^1$ be the solution of the discrete problem* (3.1) *with $\gamma_h = \overline{\gamma}_h$ on a suitable grid. Then the lower a posteriori error estimate*

$$\begin{aligned} &h_{m(i)}^{1/2} \min\Big\{1, \frac{1}{\overline{\kappa}_{h,m(i)} h_{m(i)}}\Big\}^{1/2} \big|[u_h']_{x_i}\big| + h_i \min\Big\{1, \frac{1}{\overline{\kappa}_{h,i} h_i}\Big\} \|r_h\|_{L^2(K_i)} \\ &\quad \leqslant 10 \sum_{j=i}^{i+1} \Big\{\big(1 + \Gamma(\overline{\kappa}_{h,j}, \kappa; K_j)\big)|\!|\!| u - u_h |\!|\!|_{\kappa, K_j} + \Big\|\frac{f - f_h}{\overline{\kappa}_{h,j}}\Big\|_{L^2(K_j)} + \Big\|\frac{\gamma - \overline{\gamma}_{h,j}}{\overline{\gamma}_{h,j}} \overline{\kappa}_{h,j} u_h\Big\|_{L^2(K_j)}\Big\} \end{aligned}$$

holds for $i = 1, \ldots, N$. Here, $\Gamma(\overline{\kappa}_{h,j}, \kappa; K_j) := \|\frac{\overline{\kappa}_{h,j} - \kappa}{\overline{\kappa}_{h,j}}\|_{L^\infty(K_j)}$.

Proof. We rewrite the error representation (5.1) in the form

$$\sum_{i=1}^{N} [u_h']_{x_i} (v - v_h)(x_i) - \int_\Omega r_h (v - v_h) = B(u - u_h, v) - \int_\Omega \big\{(f - f_h) v - (\gamma - \gamma_h) u_h v\big\}.$$

Choose $v = \chi_i r_h$ and $v_h = 0$, where $\chi_i(x) = (x_i - x)(x - x_{i-1})$ for $x \in K_i$ and $\chi_i(x) = 0$ otherwise. By explicit calculation we obtain

$$\|\chi_i\|_{L^1(K_i)} = \frac{1}{6} h_i^3, \quad \|\chi_i\|_{L^2(K_i)} = \frac{1}{\sqrt{30}} h_i^{5/2}, \quad \|\chi_i'\|_{L^2(K_i)} = \frac{1}{\sqrt{3}} h_i^{3/2}.$$

If $\gamma_h = \overline{\gamma}_h$, then r_h is constant ($r_h = -f_h$ or $r_h = b_h - f_h$) and therefore

$$\begin{aligned} h_i \|r_h\|_{L^2(K_i)}^2 &= \frac{6}{h_i} \int_{K_i} |r_h|^2 \chi_i = \frac{6}{h_i} \int_{K_i} r_h v \\ &\leqslant \frac{6}{h_i} \Bigg(\|(u - u_h)'\|_{L^2(K_i)} \frac{h_i}{\sqrt{3}} + \|\kappa(u - u_h)\|_{L^2(K_i)} \big(1 + \Gamma(\overline{\kappa}_{h,j}, \kappa; K_j)\big) \frac{\overline{\kappa}_{h,i} h_i^2}{\sqrt{30}} \\ &\qquad + \Big(\Big\|\frac{1}{\overline{\kappa}_{h,i}} (f - f_h)\Big\|_{L^2(K_i)} + \Big\|\frac{\gamma - \gamma_h}{\overline{\gamma}_{h,i}} \overline{\kappa}_{h,i} u_h\Big\|_{L^2(K_i)} \Big) \frac{\overline{\kappa}_{h,i} h_i^2}{\sqrt{30}} \Bigg) \|r_h\|_{L^2(K_i)} \end{aligned}$$

if $\overline{\kappa}_{h,i}h_i \geqslant 1$. This gives

$$h_i\|r_h\|_{L^2(K_i)} \leqslant 4\Big(\big(1+\Gamma(\overline{\kappa}_{h,i},\kappa;K_i)\big)\|u-u_h\|_{\kappa;K_i} + \Big\|\frac{f-f_h}{\overline{\kappa}_{h,i}}\Big\|_{L^2(K_i)} + \Big\|\frac{\gamma-\gamma_h}{\overline{\gamma}_{h,i}}\overline{\kappa}_{h,i}u_h\Big\|_{L^2(K_i)}\Big) \cdot \max\{1,\overline{\kappa}_{h,i}h_i\}$$

and the estimate for the r_h-part is obtained by dividing through $\max\{1,\overline{\kappa}_{h,i}h_i\} = 1/\min\big\{1, 1/(\overline{\kappa}_{h,i}h_i)\big\}$.

Now fix $i \in \{1,\dots,N\}$ and take $v = [u_h']_{x_i}\phi_i$ and $v_h = 0$, where ϕ_i is the continuous piecewise linear function with $\phi_i(x_i) = 1$ and $\phi_i(x) > 0$ if and only if $x \in (x_i - \delta_i h_i, x_i + \delta_{i+1}h_{i+1})$ where $\delta_j := \min\{1, 1/(\overline{\kappa}_{h,j}h_j)\}$. We find, if $\overline{\kappa}_{h,j}h_j \geqslant 1$, that $\delta_j h_j = 1/\overline{\kappa}_{h,j}$ and thus

$$[u_h']^2_{x_i} \leqslant \sum_{j=i}^{i+1}\Bigg(\|(u-u_h)'\|_{L^2(K_j)} + \big(1+\Gamma(\overline{\kappa}_{h,j},\kappa;K_j)\big)\|\kappa(u-u_h)\|_{L^2(K_j)} + \Big\|\frac{f-f_h}{\overline{\kappa}_{h,j}}\Big\|_{L^2(K_j)} + \Big\|\frac{\gamma-\gamma_h}{\overline{\gamma}_{h,j}}\overline{\kappa}_{h,j}u_h\Big\|_{L^2(K_j)} + \frac{1}{\overline{\kappa}_{h,j}}\|r_h\|_{L^2(K_j)}\Bigg)\overline{\kappa}_{h,j}^{1/2}\ \big|[u_h']_{x_i}\big|.$$

We now use $\overline{\kappa}_{h,j} \leqslant \overline{\kappa}_{h,m(i)} = (1/h_{m(i)})\,\overline{\kappa}_{h,m(i)}h_{m(i)}$, the previous estimate for $\|r_h\|_{L^2(K_j)}$ and this establishes the required bound. Note that the technique to use ϕ_i has been used in [21] and [2]. $\square$

Remark 5.1. Theorem 5.2 gives a lower bound for the error in terms of the residual r_h in case $\gamma_h = \overline{\gamma}_h$. This can be generalised to the case $\gamma_h \neq \overline{\gamma}_h$ (with the same technique) if we split the residual $r_h = b_h - f_h - (\gamma_h - \overline{\gamma}_h)u_h$ and if we treat $(\gamma_h - \overline{\gamma}_h)u_h$ as a data error term. However, this estimate is trivial on V_h^2, since the data error terms, especially the one for γ, dominate the residual terms.

5.4. Grid modification

In view of the previous results, we have to guarantee that we can construct a sequence of grids such that C_{I} satisfies a uniform bound. For practical purposes, we formulate this as a modification of an arbitrary grid.

Theorem 5.3. *Let $\mathcal{K}$ be a decomposition of $[0,1]$ and let $\overline{\kappa}_h$ be piecewise constant function on $\mathcal{K}$. Then we can modify $\mathcal{K}$ to a new grid $\mathcal{K}^{\mathrm{mod}}$ with the same number of intervals and the same set of constant values for $\overline{\kappa}_h^{\mathrm{mod}}$ and such that $C_{\mathrm{I}} \leqslant 2$.*

Proof. We start checking $|\sin(\overline{\kappa}_{h,i}h_i)| \geqslant 1/2$ where $i \in \overline{\Lambda}_-$, $\overline{\kappa}_{h,i}h_i \geqslant \pi/2$ for $i = 1,2,\dots,N+1$ (see (3.7)). If $|\sin(\overline{\kappa}_{h,i}h_i)| < 1/2$ is detected, we consider $J_i := [x_{i-1}, x_{i+1}]$ for $i < n$ and $J_n := [x_{n-2}, x_n]$ for $i = n$, and our aim is to bisect J_i at an intermediate point (close to x_i) such that both new intervals have the required property. To formalise this problem it is notationally convenient to assume that $J = [0,1]$, to split J at a barycentric coordinate ξ, and use indices '1' and '2' to refer to quantities on the intervals $[0,\xi]$ and $[\xi,1]$, respectively. In this setting we have to find $\xi \in [0,1]$ such that $g(\xi) := \min\{g_1(\xi), g_2(\xi)\} \geqslant 1/2$, where $g_1(\xi) := |\sin(\max\{\kappa_1\xi,\pi/2\})|$ and $g_2(\xi) := |\sin(\max\{\kappa_2\xi,\pi/2\})|$ with $\kappa_1 := \overline{\kappa}_{h,i}(h_i + h_{i+1})$ and $\kappa_2 := \overline{\kappa}_{h,i+1}(h_i + h_{i+1})$. The barycentric location of x_i in J_i is denoted by $\xi_0 := h_i/(h_i + h_{i+1})$. Let us assume that $i+1 \in \Lambda^-$ (otherwise g_2 is not needed) and that $\kappa_1 \geqslant \kappa_2$ (otherwise we start the following argument with κ_2).

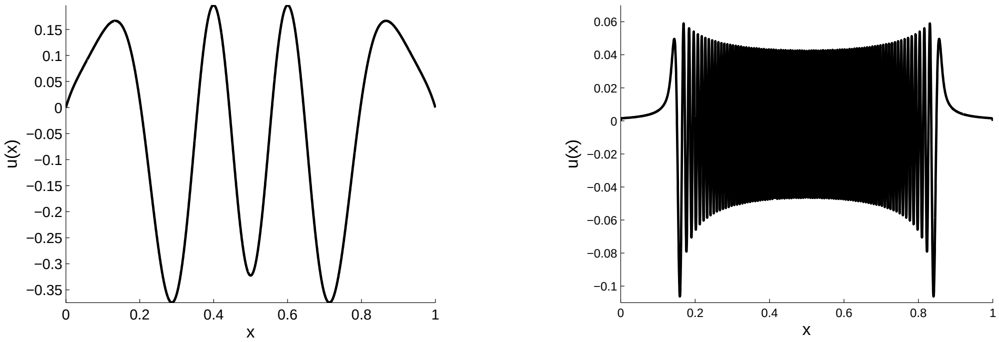

Figure 6.1. Reference solutions for the cases $[c_a, f_a] = [1.0\mathrm{e}3, 6.4\mathrm{e}1]$ (left) and $[c_a, f_a] = [1.0\mathrm{e}6, 2.0\mathrm{e}3]$ (right)

We introduce the sets $I_k^< := \{\eta \in J : g_k(\eta) < 1/2\}$ and $I_k^> := \{\eta \in J : g_k(\eta) \geqslant 1/2\}$. By assumption we have $g_1(\xi_0) < 1/2$ and $\kappa_1\xi_0 > \pi/2$, hence $\xi_0 \in I_k^<$. If $\kappa_2 \leqslant 5/(6\pi)$, then $g_2(\xi) \geqslant 1/2$ for all $\xi \in J$, hence $I_2^> = J$. Otherwise there exists at least one connected component of $I_2^>$ of length $\pi/(2\kappa_2)$ in J. We consider the component that is closest to ξ_0. Since it is of length $\pi/(2\kappa_2) \geqslant \pi/(2\kappa_1)$, it must contain a connected component of $I_1^>$ and there we find a required ξ, for example the one with minimal distance to ξ_0. □

6. Numerical results

We consider a sequence of examples with

$$\gamma(x) = c_a\big(2(2x-1)^2 - 1\big)e^{(x-1/2)^2}, \qquad f(x) = f_a,$$

where we take values

$$[c_a, f_a] \in \{[1.0\mathrm{e}3, 6.4\mathrm{e}1], [1.0\mathrm{e}4, 2.0\mathrm{e}2], [1.0\mathrm{e}5, 6.4\mathrm{e}2], [1.0\mathrm{e}6, 2.0\mathrm{e}3]\}.$$

The potentials are negative on $\Omega_- := (1/2 - 1/2\sqrt{2}, 1/2 + 1/2\sqrt{2})$ and nonnegative on $\Omega \setminus \Omega_-$, therefore the solutions of the problems are oscillating with variable frequency in Ω_- and exponentially decreasing in $\Omega \setminus \Omega_-$. Note that $f_a/\sqrt{c_a}$ is of similar size in these examples.

Since we have no exact solution, we provide as a reference the numerical solution on a uniform grid with 4164 intervals in V_h^2 (default) or $\mathbb{P}_2(\mathcal{K})$ (if indicated). The errors are integrated on the intervals of the fine grid by a trapezoidal rule with 50 points. The reference solutions for $[c_a, f_a] = [1.0\mathrm{e}3, 6.4\mathrm{e}1]$ and $[c_a, f_a] = [1.0\mathrm{e}6, 2.0\mathrm{e}3]$, obtained as described above, are shown in Fig. 6.1.

As seen from the a priori estimate (Theorem 3.1) and the a posteriori estimate (Theorem 5.1) one might have problems if $\overline{\kappa}_i h_i$ is close to $\pi\mathbb{N}\setminus\{0\}$ for at least one index i. In Section 5.4 we proposed a method to guarantee $|\sin(\overline{\kappa}_i h_i)| \geqslant 1/2$. We show calculations that compare the unmodified with the modified grid on a sequence of uniformly refined grids for the example $[c_a, f_a] = [1.0\mathrm{e}6, 2.0\mathrm{e}3]$ in Fig. 6.2. In the following we will always use this grid correction.

Let us now compare the solutions in V_h^1 and V_h^2 to solutions from standard finite element spaces $\mathbb{P}_1(\mathcal{K})$ and $\mathbb{P}_2(\mathcal{K})$. In Figs. 6.3 and 6.4 we see on the left that both methods give comparable results for $[c_a, f_a] = [1.0\mathrm{e}3, 6.4\mathrm{e}1]$, while we observe on the right that for $[c_a, f_a] = [1.0\mathrm{e}6, 2.0\mathrm{e}3]$ only the approach in V_h^k gives acceptable results (solid and dashed lines).

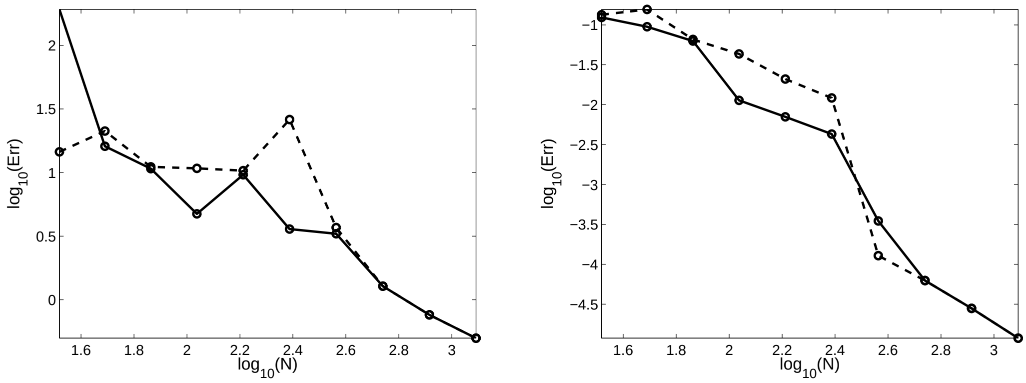

Figure 6.2. Errors on a sequence of solutions in V_h^p on uniform grids (dashed) vs. those on a corrected grid (solid) for $[c_a, f_a] = [1.0\text{e}6, 2.0\text{e}3]$ and $p = 1$ (left), $p = 2$ (right)

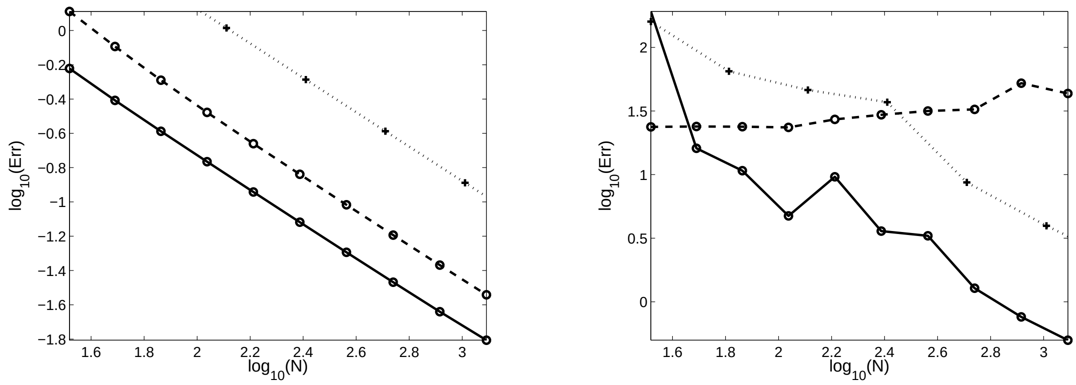

Figure 6.3. Compare errors in V_h^1 (solid) to errors in $\mathbb{P}_1(\mathcal{K})$ (dashed) for $[c_a, f_a] = [1.0\text{e}3, 6.4\text{e}1]$ (left) and $[c_a, f_a] = [1.0\text{e}6, 2.0\text{e}3]$ (right) on a sequence of uniform grids. The dotted line shows the a posteriori error on V_h^1

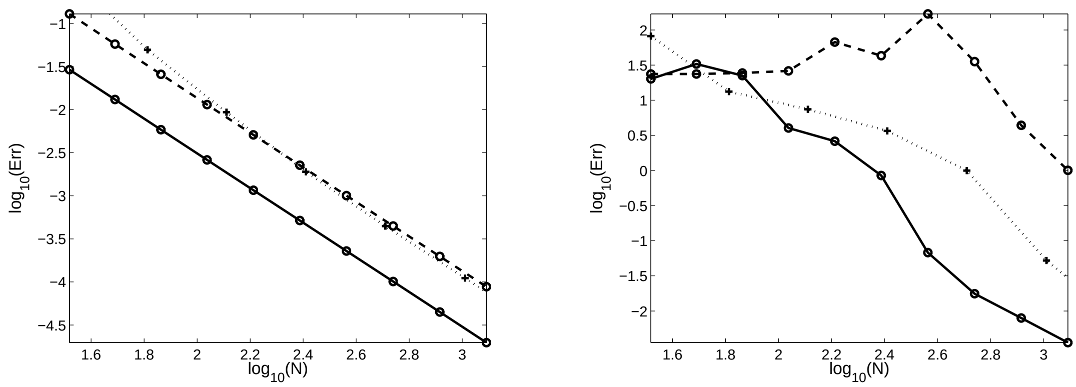

Figure 6.4. Compare errors in V_h^2 (solid) to errors in $\mathbb{P}_2(\mathcal{K})$ (dashed) for $[c_a, f_a] = [1.0\text{e}3, 6.4\text{e}1]$ (left) and $[c_a, f_a] = [1.0\text{e}6, 2.0\text{e}3]$ (right) on a sequence of uniform grids. The dotted line shows the a posteriori error on V_h^2

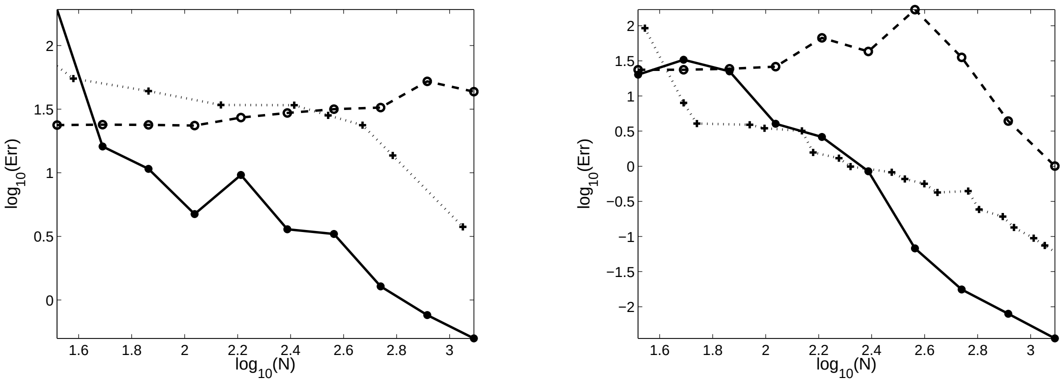

Figure 6.5. The adaptive finite element method in V_h^1 (left) and V_h^2 (right) for $[c_a, f_a] = [1.0e6, 2.0e3]$ and its error estimates (dotted). The others curves are those from Fig. 6.3 and 6.4.

Furthermore, we show the results of the a posteriori error estimates in Theorem 5.1 on the sequence of uniform grids in Figs. 6.3 and 6.4. Although there is an overestimation by a factor of about 10, the dependence on the number of unknowns is correct and this factor holds for both small and large c_a, f_a.

Finally, we want to use our a posteriori error estimates to establish an adaptive finite element algorithm. One starts with some coarse grid and then performs a sequence of steps *solve–estimate–refine* [4] until the error estimate is below a prescribed tolerance. Here we assume that 'solve' returns the exact discrete solution and that the the step 'refine' returns the exact evaluations of the right-hand side in the a posteriori error estimates. The procedure 'refine' first performs a marking of the elements as in [4] (a fixed energy fraction strategy with $\theta = 0.7$) and by bisecting all marked elements into two equal parts. Recall that we include after this refinement step our grid correction step from Section 5.4. We stop the procedure when the estimated error is below 0.01.

The results in Fig. 6.5 show that the adaptive algorithms follow a curve in the $N \mapsto \log_{10}(\mathrm{err}(N))$-diagram that is in accordance to the previous observations on uniform grids. The algorithm behaves as expected, taking the overestimation of the error into account. Observe that our example is especially bad for adaptivity since the right-hand side is constant, so that the residuals do not vary strongly. Furthermore we see that the number of adaptive steps is quite large, or, the gain per step is quite low. This could be a consequence of the large constants that appear in the estimates between the exact and the estimated error due to the theoretical results in [18, Thm. 5.2].

One should note the difference between the polynomial and fitted elements. In norms, Verfürth's [21] and our estimate behave the same, however, in the computations, the adaptive algorithm will not work properly on the polynomial spaces (Fig. 6.6) since in this case the residual will be small only if h is sufficiently small. This is illustrated by the fact that the adaptive error curves follow the a priori error curves that do not show convergence in the considered range of h.

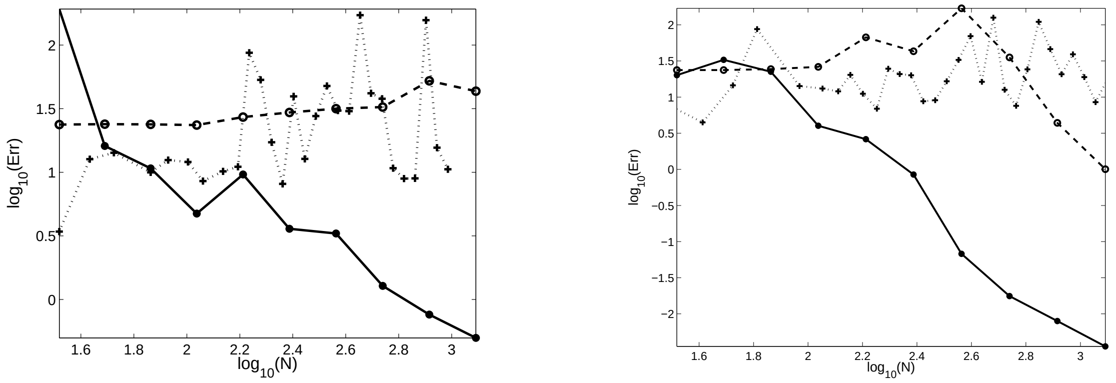

Figure 6.6. The adaptive finite element method in $\mathbb{P}_1(\mathcal{K})$ (left) and $\mathbb{P}_2(\mathcal{K})$ (right) for $[c_a, f_a] = [1.0e6, 2.0e3]$ and its error estimates (dotted). The others curves are those from Fig. 6.3 and 6.4.

Conclusion

For boundary value problems like (2.1) with large (negative) γ the finite element approach with polynomial ansatz functions leads to reasonable approximations only if $h\sqrt{|\gamma|} < 1$ which is a strong condition in applications. This deficiency is also visible in the adaptive finite element approach. We proposed a finite element method where the basis functions have been replaced by local solutions of the constant coefficient problem. These functions are well defined if the grids are corrected suitably. Using adapted interpolation estimates we derive a priori and a posteriori error estimates that do not have such a strong demand on the grid refinement. The results show that these new basis functions are beneficial for uniform as well as adaptive meshes.

This method can be implemented in existing finite element codes, if one replaces the subroutines that evaluate basis functions and the routines that do quadrature accordingly. However, these routines have to be implemented very carefully considering the various parameter ranges in their arguments. While provided explicit formulas here, one might in general use quadrature methods for oscillating integrals [12] [9].

Here, we restricted ourselves to the first and second order approach. A generalisation to higher order is possible, for example with the iterative technique developed in [19].

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