Computational Methods in Applied Mathematics
Vol. 11 (2011), No. 4, pp. 510–539

Mixed Variable Order h-finite Element Method for Linear Elasticity with Weakly Imposed Symmetry. Curvilinear Elements in 2D

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Abstract — We continue our study on variable order Arnold-Falk-Winther elements reported in [W. Qiu, L. Demkowicz, *Mixed hp-finite element method for linear elasticity with weakly imposed symmetry.* Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 3682-3701] and [W. Qiu, L. Demkowicz, *Mixed hp-finite element method for linear elasticity with weakly imposed symmetry: stability analysis.* SIAM J. Numer. Anal., 49 (2011), pp. 619-641] for 2D elasticity in context of parametric curvilinear elements. We present an asymptotic h-stability result.

2010 Mathematical subject classification: 65N30; 65L12.

Keywords: plane elasticity; mixed formulation; hp elements; curvilinear meshes.

1. Introduction

1.1. Dual–mixed formulation of linear elasticity

Linear elasticity is a classical subject and it has been studied for a long time. Continuing our study [17, 18], we focus on the dual–mixed formulation with weakly imposed symmetry that may be derived by considering stationary points of the generalized Hellinger-Reissner functional [16]. We consider the static case only and, for the sake of simplicity, we assume that the body is fixed on the whole boundary. This formulation is to find stress tensor $\sigma \in H(\mathrm{div}, \Omega; \mathbb{M})$, displacement vector $u \in L^2(\Omega; \mathbb{V})$, and infinitesimal rotation $p \in L^2(\Omega; \mathbb{K})$ satisfying

$$\begin{aligned} \int_\Omega (A\sigma : \tau + \mathrm{div}\tau \cdot u + \tau : p) d\boldsymbol{x} = 0, \quad \tau \in H(\mathrm{div}, \Omega; \mathbb{M}), \\ \int_\Omega \mathrm{div}\sigma \cdot v d\boldsymbol{x} = \int_\Omega f \cdot v d\boldsymbol{x}, \quad v \in L^2(\Omega; \mathbb{V}), \\ \int_\Omega \sigma : q d\boldsymbol{x} = 0, \quad q \in L^2(\Omega; \mathbb{K}). \end{aligned} \tag{1.1}$$

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The first equation represents a relaxed form of the constitutive equations combined with the definition of strain, the second one represents the equilibrium equations, and the third one enforces the symmetry of the stress tensor. We refer to the next section for a detailed description of energy spaces: $H(\mathrm{div},\Omega;\mathbb{M})$, $L^2(\Omega;\mathbb{V})$ and $L^2(\Omega;\mathbb{K})$. The operator A denotes the generalized compliance tensor mapping stress tensor into strain tensor. The operator is bounded, symmetric, uniformly positive definite, and it preserves the symmetry of the tensor. The usual motivation for using formulation (1.1) is to handle nearly incompressible materials. We refer to [13] for a detailed explanation.

In two space dimensions, the skew-symmetric tensors involve a single non-zero component p,

$$\begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}.$$

Formulation (1.1) reduces to seek $\sigma \in H(\mathrm{div},\Omega;\mathbb{M})$, $u \in L^2(\Omega;\mathbb{V})$, and $p \in L^2(\Omega)$ satisfying

$$\begin{aligned} \int_\Omega (A\sigma : \tau + \mathrm{div}\tau \cdot u - S_1\tau\, p)\, dx = 0, &\quad \tau \in H(\mathrm{div},\Omega;\mathbb{M}), \\ \int_\Omega \mathrm{div}\sigma \cdot v\, dx = \int_\Omega f \cdot v\, dx, &\quad v \in L^2(\Omega;\mathbb{V}), \\ \int_\Omega S_1\sigma\, q\, dx = 0, &\quad q \in L^2(\Omega), \end{aligned} \tag{1.2}$$

where operator S_1 maps a real 2×2 matrix to a real number. For any $\sigma \in \mathbb{R}^{2\times 2}$,

$$S_1\sigma = \sigma_{12} - \sigma_{21}. \tag{1.3}$$

1.2. Purpose of the paper

There exists a large number of numerical schemes based on (1.1), see [4, 5, 9, 13, 1, 2, 3, 14, 10, 15, 19, 20, 21, 22]. In [17, 18], we generalized the mixed finite element method of Arnold, Falk and Winther [5] to meshes using variable order elements. The motivation for using variable order meshes is twofold. From the mathematical point of view, it is the first step towards general hp-adaptive meshes where the element size h and polynomial order p are varied locally to achieve superior (exponential) rates of convergence. From the engineering point of view, variable order elements appear naturally when approximating complex geometries. Meshing of regions with boundaries or interfaces with high curvatures requires small elements and leads to very non-uniform meshes in element size. This in turn motivates one to use variable order elements with high polynomial degree for large elements and lower degree for the small ones.

All contributions mentioned above assume the domains are polygonal (polyhedral). In contrast to the mathematical work, almost all practical engineering problems involve complex curvilinear geometries. It is thus of a utmost importance to examine how the performace of the mixed methods extends to meshes with curvilinear elements. In the paper, we generalize the mixed finite element method studied in [17, 18] to a class of curvilinear meshes described precisely in Section 3. For the sake of simplicity, we consider only the two dimensional case using formulation (1.2). We utilize the concept of exact geometry element [11], i.e. the exact and computational domains coincide with each other.

The main contribution of this paper is an h-asymptotic stability result. At first, it seems that the stability analysis will be a simple variant of those in [17, 18]. However, the projection operators introduced therein may not be well-defined when the jacobian matrix of an element map is not constant, i.e. when we deal with non-affine elements. In order to overcome the difficulty, we perform a delicate analysis in Section 6 and the Appendixes. We manage to show that all projections we need, are well-defined when the element size is small enough. This leads to the asymptotic stability result.

1.3. Scope of the paper

An outline of the paper is as follows. Section 2 introduces notations. In Section 3, we discuss curvilinear meshes and geometry assumptions. Section 4 reviews definitions of finite element spaces on a reference triangle, on a (physical) curved triangle, and on curved meshes. In Section 5, we return to the mixed formulation for plane elasticity with weakly imposed symmetry, and recall Brezzi's conditions for stability. In Section 6, we establish necessary results for proving the asymptotic stability for curvilinear meshes. In Section 7 we prove that the Brezzi conditions are satisfied asymptotically for curvilinear meshes, i.e. for meshes that are sufficiently fine.

2. Notation

We denote by T an arbitrary triangle in $\mathbb{R}^2$. And let $\hat{T}$ be the reference triangle. We denote the set of subsimplexes of dimension k of T by $\triangle_k(T)$, $k = 0, 1, 2$, and the set of all subsimplexes of T by $\triangle(T)$. $\triangle_0(T)$ consists of all vertices of T, $\triangle_1(T)$ consists of all edges of T, and $\triangle_2(T) = T$.

For U, a bounded open subset in $\mathbb{R}^n$, we define:

$$C^k(\overline{U}) = \{u \in C^k(U) : D^\alpha \text{ is uniformly continuous on } U, \forall |\alpha| \leqslant k\}.$$

We also define

$$C^{k,1}(\overline{U}) = \{u \in C^k(\overline{U}) : D^\alpha \text{ is Lipschitz on } U, \forall |\alpha| = k\}.$$

For any vector space $\mathbf{X}$, we denote by $L^2(\Omega; \mathbf{X})$ the space of square-integrable vector fields on Ω with values in $\mathbf{X}$. In the paper, $\mathbf{X}$ will be $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{M}$, or $\mathbb{K}$. Here, $\mathbb{M}$ and $\mathbb{K}$ represent matrices and skew-symmetric matrices in $\mathbb{R}^{n\times n}$, respectively. When $\mathbf{X} = \mathbb{R}$, we will write $L^2(\Omega)$. The corresponding norms will be denoted with the same symbol $L^2(\Omega; \mathbb{M})$. The corresponding Sobolev space of order m, denoted $H^m(\Omega; \mathbf{X})$, is a subspace of $L^2(\Omega; \mathbf{X})$ consisting of functions with all partial derivatives of order less than or equal to m in $L^2(\Omega; \mathbf{X})$. The corresponding norm will be denoted by $\|\cdot\|_{H^m(\Omega)}$. The space $H(\mathrm{div}, \Omega; \mathbb{M})$ is defined by

$$H(\mathrm{div}, \Omega; \mathbb{M}) = \{\sigma \in L^2(\Omega; \mathbb{M}) : \mathrm{div}\sigma \in L^2(\Omega; \mathbb{R}^2)\},$$

where divergence of a matrix field is the vector field obtained by applying operator div row-wise, i.e.,

$$\mathrm{div}\sigma = (\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2}, \quad \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2})^\top.$$

We introduce also a special space

$$H(\Omega) = \{\omega \in H(\mathrm{div}, \Omega) : \omega|_{\partial\Omega} \in L^2(\partial\Omega; \mathbb{R}^2)\}$$

with the norm

$$\|\omega\|_{H(\Omega)} = \|\omega\|_{H(\mathrm{div},\Omega)} + \|\omega\|_{L^2(\partial\Omega)} \quad \forall \omega \in H(\Omega).$$

Notice that the space $H(\Omega)$ is just a normed linear space with the norm $\|\cdot\|_{H(\Omega)}$. It is not a Hilbert space. It is only used to define projection in definition 6.4 on the reference triangle.

For any scalar function u and any vector function ω with values in $\mathbb{R}^2$, we denote

$$\operatorname{curl} u = (\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1})^\top, \quad \operatorname{curl} \omega = \begin{bmatrix} \frac{\partial \omega_1}{\partial x_2}, & -\frac{\partial \omega_1}{\partial x_1} \\ \frac{\partial \omega_2}{\partial x_2}, & -\frac{\partial \omega_2}{\partial x_1} \end{bmatrix}.$$

Finally, by $\|\cdot\|$ we denote the standard 2-norm for vectors and matrices.

3. Curvilinear Meshes

In practice, meshes generated by CAD software are usually curvilinear. In this section, we will present our setting of curvilinear meshes and some properties which are needed in the asymptotic h-stability analysis.

3.1. Mesh regularity assumptions

Definition 3.1. (curved triangle) A closed set $T \subset \mathbb{R}^2$ is a curved triangle if there exists a C^1-diffeomorphism G_T from reference triangle $\hat{T}$ onto T. This means that G_T is a bijection from $\hat{T}$ to T such that $G_T \in C^1(\hat{T})$ and $G_T^{-1} \in C^1(T)$. We assume additionally that $\det(DG_T(\hat{\mathbf{x}})) > 0$ for any $\hat{\mathbf{x}} \in \hat{T}$.

We represent G_T in the form

$$G_T = \tilde{G}_T + \Phi_T, \tag{3.1}$$

where $\tilde{G}_T : \hat{\boldsymbol{x}} \to B_T\hat{\boldsymbol{x}} + b_T$, $B_T = DG_T(\hat{\boldsymbol{p}})$ with $\hat{\boldsymbol{p}}$ being the centroid of $\hat{T}$, and Φ_T a C^1-mapping from $\hat{T}$ into $\mathbb{R}^2$. The images of edges and vertices of $\hat{T}$ by G_T are edges and vertices of T respectively. We denote $\triangle_i(T) = G_T(\triangle_i(\hat{T})), i = 0, 1, 2$, and $\triangle(T) = G_T(\triangle(\hat{T}))$.

Definition 3.1 is practically identical with the definition of curved finite elements introduced in [6].

Definition 3.2. A curved triangle T is of class $C^k, k \geqslant 1$, if the mapping $G_T \in C^k(\hat{T})$. Similarly, a curved triangle T is of class $C^{k,1}, k \geqslant 1$, if the mapping $G_T \in C^{k,1}(\hat{T})$.

We define $\mathcal{T}_h$ to be a finite set of curved triangles T, where h denotes the maximal distance between two vertices of $T \in \mathcal{T}_h$. We define vertices of $\mathcal{T}_h$ to be vertices of $T \in \mathcal{T}_h$, and we define curves of $\mathcal{T}_h$ to be edges of $T \in \mathcal{T}_h$. We assume that any edge of $T \in \mathcal{T}_h$ is either an edge of another curved triangle in $\mathcal{T}_h$, or part of the boundary of $\mathcal{T}_h$.

Each curve of $\mathcal{T}_h$ is parametrized with a map from the reference unit interval into $\mathbb{R}^2$,

$$[0, 1] \ni s \to \boldsymbol{x}_e(s) \in \mathbb{R}^2$$

The parametrization determines the orientation of the curve.

Let $\zeta(s)$ be the local parametrization for a particular edge of a curved triangle $T \in \mathcal{T}_h$, occupied by a curve e of $\mathcal{T}_h$. This means that $\zeta(s)$ is an affine mapping from the reference interval onto an edge of the reference triangle $\hat{T}$, whose image under the mapping G_T is exactly the particular edge of T. We can choose $\zeta(s)$ so that $G_T(\zeta(s))$ has the same orientation as $\mathbf{x}_e(s)$.

Definition 3.3. (C^0-compatible mesh) $\mathcal{T}_h$ is called a C^0-compatible mesh if, for any curve e and any curved triangle T which contains e as an edge, there is a local parametrization $\zeta(s)$ of e satisfying

$$G_T(\zeta(s)) = \mathbf{x}_e(s).$$

The concept is illustrated in Fig 3.1.

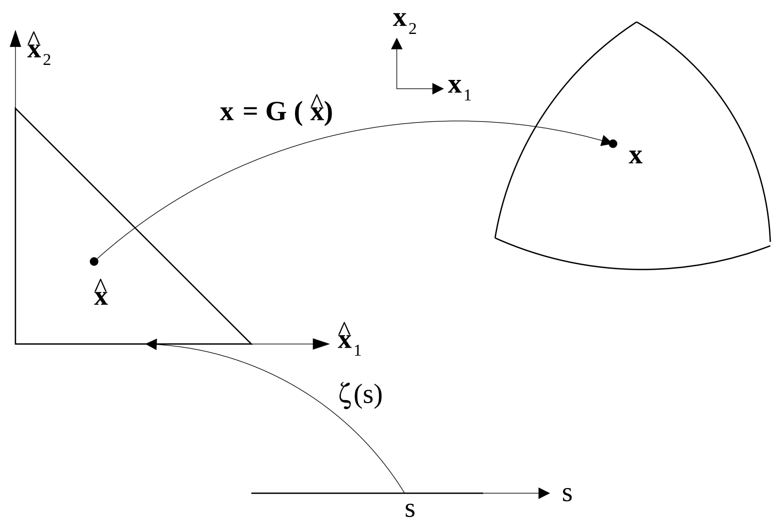

Figure 3.1. Compatibility of edge and triangle parametrizations

We denote

$$c_h := \sup_{T\in\mathcal{T}_h} \big((\sup_{\hat{\mathbf{x}}\in\hat{T}} \|D\Phi_T(\hat{\mathbf{x}})\|)\|B_T^{-1}\|\big). \tag{3.2}$$

For each $T \in \mathcal{T}_h$, we define $\tilde{T} = \tilde{G}_T(\hat{T})$. We denote by $\tilde{h}_T$ the diameter of $\tilde{T}$ and by $\tilde{\rho}_T$ the diameter of the sphere inscribed in $\tilde{T}$.

We define $\triangle_i(\mathcal{T}_h) = \bigcup_{T\in\mathcal{T}_h} \triangle_i T$, and $\triangle(\mathcal{T}_h) = \bigcup_{T\in\mathcal{T}_h} \triangle T$.

Remark 3.1. In order to simplify analysis, compared with [6], our definition of (3.2) replaces $\|D\Phi_T \cdot B_T^{-1}\|$ with the upper bound $\|D\Phi_T\| \cdot \|B_T^{-1}\|$.

Definition 3.4. The family $(\mathcal{T}_h)_h$ of C^0-compatible meshes is said to be regular if

$$\sup_h \sup_{T\in\mathcal{T}_h} \tilde{h}_T/\tilde{\rho}_T = \sigma < \infty, \text{ and } \lim_{h\to 0} c_h = 0.$$

We show the construction of $(\mathcal{T}_h)_h$ of C^0-compatible meshes in Appendix A.

Lemma 3.1. *There exist $c_1, c_2 > 0$ such that, for any triangle T,*

$$c_1\|B_T\| \cdot \|B_T^{-1}\| \leqslant \tilde{h}_T/\tilde{\rho}_T \leqslant c_2\|B_T\| \cdot \|B_T^{-1}\|,$$

where $\mathbf{x} = B_T\hat{\mathbf{x}} + b_T$ is the affine homeomorphism from $\hat{T}$ to T.

Proof. $\tilde{h}_T/\tilde{\rho}_T \leqslant c_2\|B_T\| \cdot \|B_T^{-1}\|$ comes from the geometric meaning of singular values of matrix B_T. $c_1\|B_T\| \cdot \|B_T^{-1}\| \leqslant \tilde{h}_T/\tilde{\rho}_T$ is a consequence of Theorem 3.1.3 in [8]. □

Lemma 3.2. *Let family $(\mathcal{T}_h)_h$ be regular. Then, for any indices $i,j,k,l \in \{1,2,3\}$, we have*

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = 0.$$

Proof. For any $T \in \mathcal{T}_h$ and any $\hat{\mathbf{x}} \in \hat{T}$, we have

$$\left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = |(D\Phi_T(\hat{\mathbf{x}}))_{kl}|\|B_T^{-1}\| \cdot |(B_T)_{ij}/\det(B_T)|\frac{1}{\|B_T^{-1}\|}.$$

Since $\|B_T^{-1}\| \cdot \|B_T\| \geqslant 1$, $\dfrac{1}{\|B_T^{-1}\|} \leqslant \|B_T\|$. Consequently,

$$\left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| \leqslant (\|D\Phi_T(\hat{\mathbf{x}})\| \cdot \|B_T^{-1}\|)\|B_T\|^2/|\det(B_T)|.$$

Since $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{h}_T/\tilde{\rho}_T = \sigma < \infty$, $\|B_T\|^2/|\det(B_T)| \leqslant c\sigma^2$ with $c > 0$.
Since $\lim_{h\to 0} c_h = 0$, we have $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\left|\dfrac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = 0$. □

Lemma 3.3. *If a family $(\mathcal{T}_h)_h$ is regular, then we have*

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = \lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|(DG_T(\hat{\mathbf{x}}))^{-1}B_T - I\| = 0.$$

Proof. For any $T \in \mathcal{T}_h$ and any $\hat{\mathbf{x}} \in \hat{T}$, we have

$$\|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = \|B_T(B_T + D\Phi_T(\hat{\mathbf{x}}))^{-1} - I\| = \|(I + D\Phi_T(\hat{\mathbf{x}})B_T^{-1})^{-1} - I\|.$$

Since $c_h \to 0$ as $h \to 0$, we have $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = 0$. Proof of the second property is fully analogous. □

Lemma 3.4. *If a family $(\mathcal{T}_h)_h$ is regular, then we have*

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = \lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(B_T(DG_T(\hat{\mathbf{x}}))^{-1}) - 1| = 0.$$

Proof. Since $\lim_{h\to 0} c_h = 0$, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = 0$. By Lemma 3.3, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(B_T(DG_T(\hat{\mathbf{x}}))^{-1}) - 1| = 0$. □

4. Finite element spaces

We begin by introducing the relevant finite element spaces on the reference triangle. Then, for any curved triangle T, we define the corresponding finite element spaces on T by the pull back mappings associated with the inverse of G_T. Finally, we define the finite element spaces on a whole mesh $\mathcal{T}_h$ by "gluing" the finite element spaces on curved triangles.

4.1. Finite element spaces on the reference triangle

For any $r \in \mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geqslant 0\}$, we introduce

$$\begin{aligned}
&\mathcal{P}_r(\hat{T}) := \{\text{space of polynomials of order } r \text{ on } \hat{T}\}, \\
&\mathcal{P}_r\Lambda^0(\hat{T}) = \mathcal{P}_r\Lambda^2(\hat{T}) := \mathcal{P}_r(\hat{T}), \mathcal{P}_r\Lambda^1(\hat{T}) := [\mathcal{P}_r(\hat{T})]^2, \\
&\mathring{\mathcal{P}}_r\Lambda^0(\hat{T}) := \{\hat{w} \in \mathcal{P}_r\Lambda^0(\hat{T}) : \hat{w}|_{\hat{e}} = 0, \forall \hat{e} \in \triangle_1(\hat{T})\}, \\
&\mathcal{P}_r^-\Lambda^1(\hat{T}) := [\mathcal{P}_{r-1}(\hat{T})]^2 + (\hat{x}_1, \hat{x}_2)^\top \mathcal{P}_{r-1}(\hat{T}), \\
&\mathcal{P}_r\Lambda^0(\hat{T}; \mathbb{R}^2) = \mathcal{P}_r\Lambda^2(\hat{T}; \mathbb{R}^2) := [\mathcal{P}_r(\hat{T})]^2, \\
&\mathcal{P}_r\Lambda^1(\hat{T}; \mathbb{R}^2) := \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11}, \sigma_{12})^\top, (\sigma_{21}, \sigma_{22})^\top \in \mathcal{P}_r\Lambda^1(\hat{T}) \right\}.
\end{aligned} \tag{4.1}$$

In [4, 13], spaces in (4.1) are defined in the language of exterior calculus. Here we just rewrite them using the language of calculus. We refer to [4] and [13] for a detailed correspondence before the exterior and classical calculus notations.

We denote by $\tilde{r}$ a mapping from $\triangle(\hat{T})$ to $\mathbb{Z}_+$ such that if $\hat{e}, \hat{f} \in \Delta(T)$ and $\hat{e} \subset \hat{f}$ then $\tilde{r}(\hat{e}) \leqslant \tilde{r}(\hat{f})$. We introduce now formally the FE spaces of variable order.

Definition 4.1.

$$\mathcal{P}_{\tilde{r}}\Lambda^0(\hat{T}) := \{\hat{u} \in \mathcal{P}_{\tilde{r}(\hat{T})}\Lambda^0(\hat{T}) : \forall \hat{e} \in \triangle_1(\hat{T}),\, \hat{u}|_{\hat{e}} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e})\},$$

$$\mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T}) := \mathcal{P}_{\tilde{r}(\hat{T})}\Lambda^2(\hat{T}) = \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T}),$$

$$\mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}) := \{\hat{\omega} \in \mathcal{P}_{\tilde{r}(\hat{T})}\Lambda^1(\hat{T}) : \forall \hat{e} \in \triangle_1(\hat{T}),\, \hat{\omega} \cdot \hat{\mathbf{n}}|_{\hat{e}} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e})\},$$

$$\mathcal{P}_{\tilde{r}}^-\Lambda^1(\hat{T}) := \{\hat{\omega} \in \mathcal{P}_{\tilde{r}(\hat{T})}^-\Lambda^1(\hat{T}) : \forall \hat{e} \in \triangle_1(\hat{T}),\, \hat{\omega} \cdot \hat{\mathbf{n}}|_{\hat{e}} \in \mathcal{P}_{\tilde{r}(\hat{e})-1}(\hat{e})\},$$

$$\mathcal{P}_{\tilde{r}}\Lambda^0(\hat{T}; \mathbb{R}^2) = \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T}; \mathbb{R}^2) := [\mathcal{P}_{\tilde{r}}(\hat{T})]^2,$$

$$\mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}; \mathbb{R}^2) := \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11}, \sigma_{12})^\top, (\sigma_{21}, \sigma_{22})^\top \in \mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}) \right\}.$$

Here $\hat{\mathbf{n}}$ is the outward normal unit vector along $\partial\hat{T}$.

Remark 4.1. According to [4], for any $\hat{e} \in \triangle_1(\hat{T})$, we have

$$\mathcal{P}_{\tilde{r}}\Lambda^0(\hat{T})|_{\hat{e}} = \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T})|_{\hat{e}} \cdot \hat{\mathbf{n}} = \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \mathcal{P}_{\tilde{r}}^-\Lambda^1(\hat{T})|_{\hat{e}} \cdot \hat{\mathbf{n}} = \mathcal{P}_{\tilde{r}(\hat{e})-1}(\hat{e}).$$

Lemma 4.1. *It holds*

$$\begin{aligned}
&\mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}) \subset \mathcal{P}_{\tilde{r}+1}^-\Lambda^1(\hat{T}) \subset \mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T}), \\
&div_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T}) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T}), curl_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}+1}\Lambda^0(\hat{T}) \subset \mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}).
\end{aligned}$$

Proof. The inclusions are a straightforward consequence of Definition 4.1. □

4.2. Finite element spaces on a curved triangle

Let T be a curved triangle from Definition 3.1 with G_T denoting the corresponding C^1-diffeomorphism from $\hat{T}$ to T, $\mathbf{x} = G_T(\hat{\mathbf{x}})$. We begin by introducing formally the mapping $\tilde{r}$ from $\triangle(T)$ to $\mathbb{Z}_+$ specifying the local order of discretization.

Definition 4.2. We denote by $\tilde{r}$ a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \triangle(T)$ and $e \subset f$ then $\tilde{r}(e) \leqslant \tilde{r}(f)$. With the same symbol $\tilde{r}$ we denote the corresponding mapping from $\triangle(\hat{T})$ to $\mathbb{Z}_+$, $\tilde{r}(\hat{f}) := \tilde{r}(f)$ for any $\hat{f} \in \triangle(\hat{T})$, where $f = G_T(\hat{f})$.

We define now the following FE spaces on T.

Definition 4.3. Let T be an curved triangle and $\hat{T}$ the reference triangle.

$$\begin{aligned}
\mathcal{P}_{\tilde{r}}\Lambda^0(T) &:= \{u(\mathbf{x}) : \hat{u}(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde{r}}\Lambda^0(\hat{T}) \text{ where } u(\mathbf{x}) = \hat{u}(\hat{\mathbf{x}})\},\\
\mathcal{P}_{\tilde{r}}\Lambda^1(T) &:= \{\omega(\mathbf{x}) : \hat{\omega}(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde{r}}\Lambda^1(\hat{T}) \text{ where } \omega(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})\},\\
\mathcal{P}^-_{\tilde{r}}\Lambda^1(T) &:= \{\omega(\mathbf{x}) : \hat{\omega}(\hat{\mathbf{x}}) \in \mathcal{P}^-_{\tilde{r}}\Lambda^1(\hat{T}) \text{ where } \omega(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})\},\\
\mathcal{P}_{\tilde{r}}\Lambda^2(T) &:= \{u(\mathbf{x}) : \hat{u}(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T}) \text{ where } u(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\hat{u}(\hat{\mathbf{x}})\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{\tilde{r}}\Lambda^0(T;\mathbb{R}^2) &:= \{(u_1,u_2) : u_1,u_2 \in \mathcal{P}_{\tilde{r}}\Lambda^0(T)\},\\
\mathcal{P}_{\tilde{r}}\Lambda^2(T;\mathbb{R}^2) &:= \{(u_1,u_2) : u_1,u_2 \in \mathcal{P}_{\tilde{r}}\Lambda^2(T)\},\\
\mathcal{P}_{\tilde{r}}\Lambda^1(T;\mathbb{R}^2) &:= \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde{r}}\Lambda^1(T) \right\}.
\end{aligned}$$

Remark 4.2. Since $G_T : \hat{T} \to T$ is a C^1-diffeomorphism with $\det(DG_T(\hat{\mathbf{x}})) \neq 0$, for any $\hat{\mathbf{x}} \in \hat{T}$, the formulae in Definition 4.3 are well-defined. The mappings used in the definition are the standard pull back mappings for differential forms $\Lambda^0, \Lambda^1, \Lambda^2$, see, e.g., formulas $(2.24), (2.26), (2.27)$ in [12].

Lemma 4.2. *For any edge $e \in \triangle_1(T)$, let $\zeta(s)$ be the local parametrization for e discussed in Definition 3.3, i.e., the affine mapping from the reference interval onto $\hat{e} \in \triangle_1(\hat{T})$. Then we have,*

$$\begin{aligned}
\mathcal{P}_{\tilde{r}}\Lambda^0(T)|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s))) \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e})\},\\
\mathcal{P}_{\tilde{r}}\Lambda^1(T)\cdot\mathbf{n}|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s)))\|D(G_T\circ\zeta)(s)\| \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e})\},\\
\mathcal{P}^-_{\tilde{r}}\Lambda^1(T)\cdot\mathbf{n}|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s)))\|D(G_T\circ\zeta)(s)\| \in \mathcal{P}_{\tilde{r}(\hat{e})-1}(\hat{e})\}.
\end{aligned}$$

In addition, the above equalities do not depend on the choice of the orientation of the local parametrization $\zeta(s)$.

Proof. These are trivial observations on pull back mappings and their restrictions to edges. □

Lemma 4.3. *It holds*

$$\begin{aligned}
&\mathcal{P}_{\tilde{r}}\Lambda^1(T) \subset \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T) \subset \mathcal{P}_{\tilde{r}+1}\Lambda^1(T),\\
&\mathit{div}\mathcal{P}_{\tilde{r}+1}\Lambda^1(T) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(T),\ \mathit{curl}\mathcal{P}_{\tilde{r}+1}\Lambda^0(T) \subset \mathcal{P}_{\tilde{r}}\Lambda^1(T).
\end{aligned}$$

Proof. The embeddings are a straightforward consequence of Definition 4.3, Lemma 4.1, and the commutativity of pull back mappings with exterior derivatives $(\mathrm{curl}, \mathrm{div})$. □

4.3. Finite element spaces on a C^0-compatible mesh

Let $\mathcal{T}_h$ be a C^0-compatible mesh from Definition 3.3. We extend the $\tilde{r}$ mapping to a global map defined on $\triangle(\mathcal{T}_h)$ with values in $\mathbb{Z}_+$ such that if $e \subset f$, then $\tilde{r}(e) \leqslant \tilde{r}(f)$.

Definition 4.4. We put $\Omega_h := \bigcup_{T\in\mathcal{T}_h} T$ and define

$$\begin{aligned}C\Lambda^0(\mathcal{T}_h) &:= \{u \in H^1(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},\\ C\Lambda^1(\mathcal{T}_h) &:= \{\omega \in H(\mathrm{div}, \Omega_h) : \omega \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},\\ C\Lambda^2(\mathcal{T}_h) &:= \{u \in L^2(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\}.\end{aligned}$$

We define

$$\begin{aligned}\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h) &:= \{u \in C\Lambda^0(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde{r}}\Lambda^0(T), \forall T \in \mathcal{T}_h\},\\ \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h) &:= \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}_{\tilde{r}}\Lambda^1(T), \forall T \in \mathcal{T}_h\},\\ \mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h) &:= \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}^-_{\tilde{r}}\Lambda^1(T), \forall T \in \mathcal{T}_h\},\\ \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) &:= \{u \in C\Lambda^2(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde{r}}\Lambda^2(T), \forall T \in \mathcal{T}_h\},\\ \mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2) &:= [\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h)]^2, \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2) := [\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)]^2,\\ \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2) &:= \left\{\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h,\mathbb{R}^2)\right\}.\end{aligned}$$

Remark 4.3. According to Lemma 4.2 and the fact that $\mathcal{T}_h$ is C^0-compatible, we can conclude that $\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^0(T)$, $\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^1(T)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h)|_T = \mathcal{P}^-_{\tilde{r}}\Lambda^1(T)$, $\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^2(T)$, for any $T \in \mathcal{T}_h$. For standard (not curved) triangulations $\mathcal{T}_h$, spaces $\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h)$, $\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h)$, $\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$ coincide with those analyzed in [17].

Lemma 4.4. *It holds*

$$\begin{aligned}&\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h),\\ &div\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h), curl\mathcal{P}_{\tilde{r}+1}\Lambda^0(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h).\end{aligned}$$

Proof. This is an immediate consequence of Lemma 4.3. □

5. Mixed formulation for elasticity with weakly imposed symmetry

We assume that there is $r_{\max} \in \mathbb{N}$ such that, for any $h > 0$ and $f \in \Delta(\mathcal{T}_h)$, $\tilde{r}(f) \leqslant r_{\max}$.

We recall the mixed formulation (1.1): Find $(\sigma, u, p) \in H(\mathrm{div}, \Omega; \mathbb{M}) \times L^2(\Omega; \mathbb{R}^2) \times L^2(\Omega)$ such that

$$\begin{aligned}\langle A\sigma, \tau\rangle + \langle \mathrm{div}\tau, u\rangle - \langle S_1\tau, p\rangle = 0, \quad \tau \in H(\mathrm{div}, \Omega; \mathbb{M}),\\ \langle \mathrm{div}\sigma, v\rangle = \langle f, v\rangle, \quad v \in L^2(\Omega;\mathbb{R}^2),\\ \langle S_1\sigma, q\rangle = 0, \quad q \in L^2(\Omega).\end{aligned} \tag{5.1}$$

Here $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product on Ω. This problem is well-posed. See [4] and [13] for the proof.

We consider now a finite element discretization of (5.1). For this, we choose families of finite-dimensional subspaces

$$\Lambda^1_h(\mathbb{M}) \subset H(\mathrm{div}, \Omega; \mathbb{M}), \Lambda^2_h(\mathbb{R}^2) \subset L^2(\Omega;\mathbb{R}^2), \Lambda^2_h \subset L^2(\Omega),$$

indexed by h, and seek a discrete solution $(\sigma_h, u_h, p_h) \in \Lambda_h^1(\mathbb{M}) \times \Lambda_h^2(\mathbb{R}^2) \times \Lambda_h^2$ such that

$$\begin{aligned} \langle A\sigma_h, \tau\rangle + \langle \mathrm{div}\tau, u_h\rangle - \langle S_1\tau, p_h\rangle = 0, \quad \tau \in \Lambda_h^1(\mathbb{M}), \\ \langle \mathrm{div}\sigma_h, v\rangle = \langle f, v\rangle\,, \quad v \in \Lambda_h^2(\mathbb{R}^2), \\ \langle S_1\sigma_h, q\rangle = 0, \quad q \in \Lambda_h^2. \end{aligned} \tag{5.2}$$

The stability of (5.2) will be ensured by the Brezzi stability conditions

$$\text{(S1)}\ \ \|\tau\|^2_{H(\mathrm{div},\Omega;\mathbb{M})} \leqslant c_1(A\tau, \tau) \text{ whenever } \tau \in \Lambda_h^1(\mathbb{R}^2) \text{ satisfies } \langle \mathrm{div}\tau, v\rangle = 0 \\ \forall v \in \Lambda_h^2(\mathbb{R}^2) \text{ and } \langle S_1\tau, q\rangle = 0\ \forall q \in \Lambda_h^2, \tag{5.3}$$

$$\text{(S2) for all nonzero } (v, q) \in \Lambda_h^2(\mathbb{R}^2) \times \Lambda_h^2, \text{ there exists nonzero} \\ \tau \in \Lambda_h^1(\mathbb{R}^2) \text{ with } \langle \mathrm{div}\tau, v\rangle - \langle S_1\tau, q\rangle \geqslant c_2\, \|\tau\|_{H(\mathrm{div},\Omega;\mathbb{M})} \left(\|v\|_{L^2(\Omega;\mathbb{R}^2)} + \|q\|_{L^2(\Omega)}\right), \tag{5.4}$$

where the constants c_1 and c_2 are independent of h.

For meshes of arbitrary but uniform order, conditions (5.3) and (5.4) have been proved in [4] and [13]. In what follows, we will demonstrate that they are also satisfied for (2D) meshes with elements of variable (but limited) order.

Before presenting our proof, we would like to comment shortly on difficulties encountered in proving stability for generalizing AFW elements with variable order. As we have shown in Section 3, it is rather straightforward to generalize AFW elements to the variable order case. The following commuting diagrams are essential in the stability proof from [5, 4, 13],

$$\begin{array}{ccc} H^1(\Omega;\mathbb{R}^2) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \Pi_h^{1,-}\downarrow & & \Pi_h^2\downarrow \\ \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \end{array} \tag{5.5a}$$

$$\begin{array}{ccc} H^1(\Omega;\mathbb{R}^2) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \Pi_h^1\downarrow & & \Pi_h^2\downarrow \\ \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \end{array} \tag{5.5b}$$

$$\begin{array}{ccc} H^1(\Omega;\mathbb{R}^2) & \xrightarrow{Id} & H^1(\Omega;\mathbb{R}^2) \\ \Pi_h^0\downarrow & & \Pi_h^{1,-}\downarrow \\ \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2) & \xrightarrow{\Pi_h^{1,-}\circ Id} & \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}_h) \end{array} \tag{5.5c}$$

When uniform order r is replaced by variable order $\tilde{r}$, then the diagrams (5.5a) and (5.5b) do not commute if $\Pi_h^{1,-}$ and Π_h^1 are natural generalizations of canonical projection operators introduced in [5, 4, 13]. A counterexample is given in the appendix of [17]. This technical difficulty was overcome in [18] for affine meshes by constructing a new family of *new* projection based interpolation $\Pi_h^{1,-}, \Pi_h^1$ operators, and a new operator W_h (W_h takes place of Π_h^0) so that all three diagrams commute. In the following sections, we will show asymptotic h-stability can be achieved on curvilinear mesh, which is an analogue of the stability analysis in [18]. But the analysis in this paper is much more technical.

6. Preliminaries for the proof of stability for curvilinear meshes

We begin by recalling our assumptions on the domain and meshes: Ω is a (curvilinear) polygon and it is meshed with a family $(\mathcal{T}_h)_h$ of C^0-compatible meshes of class $C^{1,1}$. For any mesh $\mathcal{T}_h$, mapping $\tilde{r} : \triangle(\mathcal{T}_h) \to \mathbb{Z}_+$ defines a locally variable order of discretization that satisfies the minimum rule. The maximum order is limited, i.e. $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T) < \infty$. In order to make this paper more readable, we will put most proofs in this section into Appendix C.

Definition 6.1. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^2_{\tilde{r},T} : L^2(T) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(T)$ by the relations

$$\int_T (\Pi^2_{\tilde{r},T} u(\mathbf{x}) - u(\mathbf{x}))\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))\, d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T}) \tag{6.1}$$

Above, $\hat{\mathbf{x}}(\mathbf{x})$ signifies the inverse of the element map $\mathbf{x} = G_T(\hat{\mathbf{x}})$.

Definition 6.2. Operator $\Pi^2_{\tilde{r},\hat{T}} : L^2(\hat{T}) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T})$ will denote the L^2-projection in the reference space,

$$\int_T (\Pi^2_{\tilde{r},\hat{T}} \hat{u}(\hat{\mathbf{x}}) - \hat{u}(\hat{\mathbf{x}}))\hat{\psi}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T}) \tag{6.2}$$

Remark 6.1. Operator $\Pi^2_{\tilde{r},T}$ is a weighted L^2-projection in the physical space. For a regular triangle (affine element map), the jacobian is constant, and $\Pi^2_{\tilde{r},T}$ reduces to the standard L^2-projection in the physical space.

Lemma 6.1. *For any $T \in \mathcal{T}_h$, and arbitrary $u(\mathbf{x}) \in L^2(T)$, we define $\hat{u}(\hat{\mathbf{x}})$ by the relation*

$$u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{\hat{u}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}$$

Then $\hat{u}(\hat{\mathbf{x}}) \in L^2(\hat{T})$ and

$$\Pi^2_{\tilde{r},T} u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{\Pi^2_{\tilde{r},\hat{T}} \hat{u}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}})))}$$

Above, $\mathbf{x}(\hat{\mathbf{x}})$ signifies the element map $\mathbf{x} = G_T(\hat{\mathbf{x}})$.

Proof. Proof follows immediately from Lemma B.6 and the definitions of the two projections. □

Lemma 6.2. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\Pi^2_{\tilde{r},T} u - P_{\tilde{r},T} u\|_{L^2(T)} \leqslant \varepsilon \|u\|_{L^2(T)}, \forall u \in L^2(T).$$

Here $P_{\tilde{r},T}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$.

Proof. Please see Appendix C. □

6.1. Projection-Based Interpolation onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h)$

Definition 6.3. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^1_{\tilde{r}+1,T} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \operatorname{div}(\Pi^1_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}, \tag{6.3}$$

$$\int_T (\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}})^{-\top}\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\varphi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall \hat{\varphi} \in \mathring{\mathcal{P}}_{\tilde{r}(T)+2}(\hat{T}), \tag{6.4}$$

$$\int_{[0,1]} [(\Pi^1_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)+1}([0,1])\, \forall e \in \triangle_1(T). \tag{6.5}$$

Here $\mathbf{x} = G_T(\hat{\mathbf{x}})$ for any $\hat{\mathbf{x}} \in \hat{T}$, $\mathbf{x}_e(s) : [0,1] \to e$ is the parametrization of e, and $\mathbf{n}$ is a unit normal vector along e (the choice of its direction does not matter).

Definition 6.4. (Projection-Based Interpolation operator onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$) We define a linear operator $\Pi^1_{\tilde{r}+1,\hat{T}} : H(\hat{T}) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$ by the relations

$$\int_{\hat{T}} \operatorname{div}_{\hat{\mathbf{x}}}(\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega})\hat{\psi}d\hat{\mathbf{x}} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}, \tag{6.6}$$

$$\int_{\hat{T}} (\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega}) \cdot \operatorname{curl}_{\hat{\mathbf{x}}}\hat{\varphi}d\hat{\mathbf{x}} = 0 \quad \forall \hat{\varphi} \in \mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+2}(\hat{T}), \tag{6.7}$$

$$\int_{\hat{e}} (\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega}) \cdot \hat{\mathbf{n}}\hat{\eta}d\hat{s} = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})+1}(\hat{e})\, \forall \hat{e} \in \triangle_1(\hat{T}). \tag{6.8}$$

Remark 6.2. The operator $\Pi^1_{\tilde{r}+1,\hat{T}}$ is the Projection-Based-Interpolation operator onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$ defined in [17]. The operator $\Pi^1_{\tilde{r}+1,T}$ is defined by the pull-back mapping from $\hat{T}$ to T.

Lemma 6.3. *For any $T \in \mathcal{T}_h$, any $\omega \in [H^1(T)]^2$, we define $\hat{\omega}$ by the relation*

$$\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\omega}(\hat{x}).$$

Then $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$, and

$$\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega}(\hat{\mathbf{x}}).$$

Proof. Please see Appendix C. □

Lemma 6.4. *For any $T \in \mathcal{T}_h$, and any $\omega \in [H^1(T)]^2$, we have $\Pi^2_{\tilde{r},T} div\omega = div\Pi^1_{\tilde{r}+1,T}\omega$.*

Proof. Please see Appendix C. □

Lemma 6.5. *There exists $\delta > 0$ and $C > 0$ such that, for any $h < \delta$, we have*

$$\|\Pi^1_{\tilde{r},T}\omega\|_{L^2(T)} \leqslant C\|\omega\|_{H^1(T)} \quad \forall T \in \mathcal{T}_h, \omega \in H^1(T;\mathbb{R}^2)$$

For affine meshes, the inequality above holds for any $h > 0$.

Proof. Please see Appendix C. □

6.2. Modified Projection Based Interpolation onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ and modified operator W onto $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T)$

Definition 6.5. Let $\tilde{r}\,:\,\triangle(\hat{T})\to\mathbb{Z}_+$ be a mapping that prescribes the local order of discretization and satisfies the minimum rule, i.e. if $\hat{e},\hat{f}\in\triangle(\hat{T})$ and $\hat{e}\subset\hat{f}$ then $\tilde{r}(\hat{e})\leqslant\tilde{r}(\hat{f})$. We put $k_{\tilde{r}}=\dim\mathrm{curl}_{\hat{\mathbf{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}(\hat{T})$. Let $\{\hat{\mathbf{f}}_{\tilde{r},1}(\hat{\mathbf{x}}),\cdots,\hat{\mathbf{f}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\}$ be a basis of $\mathrm{curl}_{\hat{\mathbf{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}(\hat{T})$. Let $\{\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}}),\cdots,\hat{\mathbf{g}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\}$ be a linearly independent subset of $\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{R}^2)$ such that

$$\mathrm{span}\{\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}}),\cdots,\hat{\mathbf{g}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\}\oplus\mathrm{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})=[\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T})]^2.$$

For $t\in[0,1]$, we define $\hat{\mathbf{h}}_{\tilde{r},i}(\hat{\mathbf{x}},t)=(1-t)\hat{\mathbf{f}}_{\tilde{r},1}(\hat{\mathbf{x}})+t\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}})$, $1\leqslant i\leqslant k_{\tilde{r}}$.

Remark 6.3. It is easy to check that $k_{\tilde{r}}=\dim\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{R}^2)-\dim\mathrm{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})$.

Definition 6.6. (One-parameter family of PB interpolation operators onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$) For any $t\in[0,1]$, we define a linear operator $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\,:\,H^1(\hat{T};\mathbb{R}^2)\longrightarrow\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ by the relations

$$\int_{\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}(\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}-\hat{\omega})(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}}=0\quad\forall\hat{\psi}\in\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R},\tag{6.9}$$

$$\int_{\hat{T}}(\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}(\hat{\mathbf{x}})-\hat{\omega}(\hat{\mathbf{x}}))^\top\hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}}=0\quad 1\leqslant i\leqslant k_{\tilde{r}},\tag{6.10}$$

$$\int_{\hat{e}}[(\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}-\hat{\omega})\cdot\hat{\mathbf{n}}]\hat{\eta}d\hat{s}=0\quad\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}),\forall\hat{e}\in\triangle_1(\hat{T}).\tag{6.11}$$

Definition 6.7. We define a linear operator $C_t\,:\,[H^1(\hat{T})]^2\longrightarrow\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ by the following relations

$$\int_{\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}C_t\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}}=\int_{\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}}\quad\forall\hat{\psi}\in\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}\tag{6.12}$$

$$\int_{\hat{T}}(C_t\hat{\omega}(\hat{\mathbf{x}}))^\top\hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}}=\int_{\hat{T}}(\hat{\omega}(\hat{\mathbf{x}}))^\top\hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}}\quad 1\leqslant i\leqslant k_{\tilde{r}}\tag{6.13}$$

$$\int_{\hat{e}}[(C_t\hat{\omega})\cdot\hat{\mathbf{n}}]\hat{\eta}d\hat{s}=\int_{\hat{e}}[\hat{\omega}\cdot\hat{\mathbf{n}}]\hat{\eta}d\hat{s}\quad\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}),\forall\hat{e}\in\triangle_1(\hat{T})\tag{6.14}$$

$$\int_{\hat{e}}[(C_t\hat{\omega})\cdot\hat{\mathbf{t}}]\hat{\eta}d\hat{s}=\int_{\hat{e}}[\hat{\omega}\cdot\hat{\mathbf{t}}]\hat{\eta}d\hat{s}\quad\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}),\forall\hat{e}\in\triangle_1(\hat{T})\tag{6.15}$$

$$C_t\hat{\omega}=0\quad\text{at all vertices of }\hat{T}\tag{6.16}$$

Here $\hat{\mathbf{n}},\hat{\mathbf{t}}$ denote the normal and tangent unit vectors along $\partial\hat{T}$.

According to Lemma 10.7 and Lemma 10.8 (the operator $\Pi^0_{\tilde{r},\hat{T},t}$ in this lemma is exactly the same as the operator $C_{t_{\tilde{r}(\hat{T})}}$ in this paper) in [18], we can have the following definition.

Definition 6.8. Let $\tilde{r}\,:\,\triangle(\hat{T})\to\mathbb{Z}_+$ be a locally variable order of discretization that satisfies the minimum rule. Let $t_{\tilde{r}(\hat{T})}\in[0,1]$ which depends on $\tilde{r}(\hat{T})$ only such that both $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t_{\tilde{r}(\hat{T})}}$ and $C_{t_{\tilde{r}(\hat{T})}}$ are well-defined (See Lemma 10.7 and Lemma 10.8 in [18]).

Definition 6.9. (PB interpolation operator onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$) For $t_{\tilde{r}(\hat{T})}$ given in Definition 6.8, and for any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^{1,-}_{\tilde{r}+1,T} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \operatorname{div}(\Pi^{1,-}_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))\, d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}, \tag{6.17}$$

$$\int_T (\Pi^{1,-}_{\tilde{r}+1,T}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}), t_{\tilde{r}(\hat{T})})d\mathbf{x} = 0 \quad 1 \leqslant i \leqslant k_{\tilde{r}}, \tag{6.18}$$

$$\int_{[0,1]} [(\Pi^{1,-}_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T). \tag{6.19}$$

In the above, $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{x})$ signifies the inverse of the element map.

Theorem 6.1. *The operator $\Pi^{1,-}_{\tilde{r}+1,T} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ is well-defined, and we have*

$$div\Pi^{1,-}_{\tilde{r}+1,T}\omega = \Pi^2_{\tilde{r},T} div\omega \quad \forall \omega \in H^1(T;\mathbb{R}^2).$$

Moreover, there exist $\delta > 0$ and $C > 0$ such that, for $h \leqslant \delta$, $T \in \mathcal{T}_h$, and $\omega \in H^1(T;\mathbb{R}^2)$,

$$\|\Pi^{1,-}_{\tilde{r}+1,T}\omega\|_{L^2(T)} \leqslant C\|\omega\|_{H^1(T)}.$$

Proof. The proof is analogous to that of Lemma 6.3, Lemma 6.4, and Lemma 6.5. □

Definition 6.10. For $t_{\tilde{r}(\hat{T})}$ given in Definition 6.8, and for any $T \in \mathcal{T}_h$, we define a linear operator $W_T : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ by the following relations

$$\int_T \operatorname{div}(W_T\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}, \tag{6.20}$$

$$\int_T (W_T\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}), t_{\tilde{r}(\hat{T})})d\mathbf{x} = 0 \quad 1 \leqslant i \leqslant k_{\tilde{r}}, \tag{6.21}$$

$$\int_{[0,1]} [(W_{T,t}\omega - \omega)(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T), \tag{6.22}$$

$$\int_{[0,1]} [(W_{T,t}\omega - \omega)(\mathbf{x}_e(s)) \cdot \mathbf{t}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T), \tag{6.23}$$

$$W_{T,t}\omega = 0 \text{ at all vertices of } T. \tag{6.24}$$

Here $\mathbf{n}, \mathbf{t}$ denote the normal and tangent unit vectors along ∂T.

Theorem 6.2. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$, $T \in \mathcal{T}_h$, the operator $W_T : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ is well-defined and,*

$$\|curlW_T\omega\|_{L^2(T)} \leqslant C(\tilde{h}_T^{-1}\|\omega\|_{L^2(T)} + \|\omega\|_{H^1(T)}) \quad \forall \omega \in H^1(T;\mathbb{R}^2)$$

Moreover, for affine meshes, W_T is well-defined and the above inequality holds for any $h > 0$.

Proof. Please see Appendix C. □

6.3. Projection operators on the whole curvilinear meshes

Definition 6.11. We define the following global interpolation operators,

$$\Pi^2_{\tilde r,\mathcal{T}_h} : L^2(\Omega) \longrightarrow \mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h), \quad (\Pi^2_{\tilde r,\mathcal{T}_h} u)|_T = \Pi^2_{\tilde r,T}(u|_T)$$

$$\tilde\Pi^2_{\tilde r,\mathcal{T}_h} : L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde\Pi^2_{\tilde r,\mathcal{T}_h}(u_1,u_2)^\top)|_T = (\Pi^2_{\tilde r,T}(u_1|_T), \Pi^2_{\tilde r,T}(u_2|_T))^\top$$

$$\Pi^{1,-}_{\tilde r+1,\mathcal{T}_h} : H^1(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}^{-1}_{\tilde r+1}\Lambda^1(\mathcal{T}_h), \quad (\Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}\omega)|_T = \Pi^{1,-}_{\tilde r+1,T}(\omega|_T)$$

$$\tilde\Pi^1_{\tilde r+1,\mathcal{T}_h} : H^1(\Omega;\mathbb{M}) \longrightarrow \mathcal{P}_{\tilde r+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde\Pi^1_{\tilde r+1,\mathcal{T}_h}\sigma)|_T = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}$$

where,

$$\begin{bmatrix} \tau_{11} \\ \tau_{12} \end{bmatrix} = \Pi^1_{\tilde r+1,\mathcal{T}_h} \begin{bmatrix} \sigma_{11}|_T \\ \sigma_{12}|_T \end{bmatrix}, \quad \begin{bmatrix} \tau_{21} \\ \tau_{22} \end{bmatrix} = \Pi^1_{\tilde r+1,\mathcal{T}_h} \begin{bmatrix} \sigma_{21}|_T \\ \sigma_{22}|_T \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$W_{\mathcal{T}_h} : H^1(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde r+2}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2), \quad (W_{\mathcal{T}_h}\omega)|_T = W_T(\omega|_T)$$

for all $T \in \mathcal{T}_h$.

Remark 6.4. Since $(\mathcal{T}_h)_h$ is C^0-compatible, operators $\Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}$, $\tilde\Pi^1_{\tilde r+1,\mathcal{T}_h}$ and $W_{\mathcal{T}_h}$ are well-defined.

Theorem 6.3. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$,*

$$\|\Pi^2_{\tilde r,\mathcal{T}_h} u - P_{\tilde r,\mathcal{T}_h} u\|_{L^2(\Omega)} \leqslant \varepsilon \|u\|_{L^2(\Omega)} \quad \forall u \in L^2(\Omega).$$

Here $P_{\tilde r,\mathcal{T}_h}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h)$.

Proof. This is an immediate result of Lemma 6.2. □

Theorem 6.4. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$, we have*

$$\|\tilde\Pi^1_{\tilde r+1,\mathcal{T}_h}\sigma\|_{L^2(\Omega)} \leqslant C\|\sigma\|_{H^1(\Omega)}, \quad \|\Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}\omega\|_{L^2(\Omega)} \leqslant C\|\omega\|_{H^1(\Omega)},$$

for any $\sigma \in H^1(\Omega;\mathbb{M})$ and $\omega \in H^1(\Omega;\mathbb{R}^2)$. For affine meshes, the inequalities above hold for any $h > 0$. Moreover,

$$div\ \tilde\Pi^1_{\tilde r+1,\mathcal{T}_h}\sigma = \tilde\Pi^2_{\tilde r,\mathcal{T}_h} div\ \sigma, \quad div\ \Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}\omega = \Pi^2_{\tilde r,\mathcal{T}_h} div\ \omega$$

Proof. This is an immediate result of Lemma 6.5 and Theorem 6.1. □

Definition 6.12. Let R_h denote the generalized Clement interpolant operator from Theorem 5.1 in [6], which maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(\mathcal{T}_h;\mathbb{R}^2)$. And we define

$$\tilde W_h = W_h(I - R_h) + R_h.$$

Theorem 6.5. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$,*

$$\|curl \tilde W_h \omega\|_{L^2(\Omega)} \leqslant C\|\omega\|_{H^1(\Omega)} \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2).$$

For affine meshes, the inequality holds for any $h > 0$. The operator $\tilde W_h$ maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_{\tilde r+2}\Lambda^0(T;\mathbb{R}^2)$ and satisfies the condition

$$\Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}\omega = \Pi^{1,-}_{\tilde r+1,\mathcal{T}_h}\tilde W_h \omega \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2).$$

Proof. We utilize Example 2 from [6] with uniform order equal 1 to construct operator R_h. Since $(\mathcal{T}_h)_h$ is C^0-compatible, and $c_h \to 0$ as $h \to 0$, operator R_h maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(T;\mathbb{R}^2) \subset \mathcal{P}_{\tilde r+2}\Lambda^0(T;\mathbb{R}^2)$. Then the proof will be straightforward. □

7. Asymptotic stability of the finite element discretization on curvilinear meshes

Lemma 7.1. *There exist $\delta > 0$ and $c > 0$ such that, for any $h < \delta$ and any $(\omega, \mu) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) \times \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h; \mathbb{R}^2)$ such that*

$$div\sigma = \mu, \quad -\Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma = \omega$$

and

$$\|\sigma\|_{H(div,\Omega)} \leqslant c(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)})$$

Here, the constant c depends on $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T)$. For affine meshes, the inequality above holds for any $h > 0$.

Proof. The proof is the same as that of Theorem 33 in [17]. ☐

Theorem 7.1. *There exist $\delta > 0$ and $c > 0$ such that, for solution (σ, u, p) of elasticity system (1.1), and corresponding solution (σ_h, u_h, p_h) of discrete system (5.1), we have*

$$\begin{aligned}&\|\sigma - \sigma_h\|_{H(div,\Omega)} + \|u - u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}\\ &\leqslant c \inf[\|\sigma - \tau\|_{H(div,\Omega)} + \|u - v\|_{L^2(\Omega)} + \|p - q\|_{L^2(\Omega)}],\end{aligned}$$

where the infimum is taken over all $\tau \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2)$, $v \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h, \mathbb{R}^2)$, and $q \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$, for $h < \delta$. For affine meshes, the inequality holds for any $h > 0$.

Proof. We need to show that conditions (5.3) and (5.4) are satisfied asymptotically in h. Condition (5.3) follows from the fact that, by construction, $\mathrm{div}\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, and the fact that A is coercive.

We turn now to condition (5.4). According to Lemma 7.1, there exist $\delta > 0$ and $c > 0$ such that, for $h < \delta$ and $(\omega, \mu) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) \times \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h; \mathbb{R}^2)$ such that $\mathrm{div}\sigma = \mu$, $-\Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma = \omega$, and $\|\sigma\|_{H(\mathrm{div},\Omega)} \leqslant c(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)})$. We have then

$$\begin{aligned}\langle \mathrm{div}\sigma, \mu\rangle - \langle S_1\sigma, \omega\rangle =&\langle \mathrm{div}\sigma, \mu\rangle - \langle \Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma, \omega\rangle + \langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h})S_1\sigma, \omega\rangle\\ \geqslant& c\|\sigma\|_{H(\mathrm{div},\Omega)}(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}) + \langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h})S_1\sigma, \omega\rangle.\end{aligned}$$

According to Theorem 6.3, for sufficiently small h,

$$|\langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h})S_1\sigma, \omega\rangle| \leqslant \frac{c}{2}\|\sigma\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega)}$$

So, asymptotically in h, we have

$$\langle \mathrm{div}\sigma, \mu\rangle - \langle S_1\sigma, \omega\rangle \geqslant \frac{c}{2}\|\sigma\|_{H(\mathrm{div},\Omega)}(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}).$$

For affine meshes, $\Pi^2_{\tilde{r},\mathcal{T}_h}$ reduces to the standard L^2-projection. The inequality above holds then for any $h > 0$. This finishes the proof. ☐

8. Conclusions

We have presented a complete asymptotic h-stability analysis for a generalization of Arnold-Falk-Winther elements to curvilinear meshes of variable order in two space dimensions. The asymptotic stability analysis for curvilinear elements has proved to be rather non-trivial.

The analysis of curvilinear meshes for elasticity differs considerably from that for problems involving only grad-curl-div operators. Piola maps transform gradients, curls and divergence in the physical domain into the corresponding gradients, curls and divergence in the reference domain. Consequently, problems involving the grad, curl or div operators only (e.g. Maxwell equations or the mixed formulation for a scalar elliptic problem) can be reformulated in the parametric domain at the expense of introducing material anisotropies reflecting the geometric parametrizations. This is not the case for elasticity where the strain tensor (symmetric part of the displacement gradient) in the physical domain does not transform into the symmetric part of the displacement gradient in the reference domain. Consequently, the analysis for affine meshes cannot be simply reproduced for curvilinear ones, and new interpolation operators have to be carefully drafted. We have managed to prove only the asymptotic stability for the curvilinear meshes.

A. Mesh generation

We assume that the domain Ω is a (curvilinear) polygon, and that it can be meshed with a regular family $(\mathcal{T}_h)_h$ of C^0-compatible meshes (i.e. $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$, for all h) that satisfy the regularity assumptions discussed in the previous section.

We will outline now shortly how one can generate such meshes in practice. Suppose the domain Ω has been meshed with a C^0-compatible initial mesh $\mathcal{T}_{\text{int}}$, $\overline{\Omega} = \bigcup_{i=1}^m T_i$, where $\{T_i\}_{i=1}^m$ are curved triangles of class $C^{1,1}$. We denote by $\{G_1, \cdots, G_m\}$ the mappings from $\hat{T}$ to $\{T_1, \cdots, T_m\}$. Then $G_i \in C^{1,1}(\hat{T})$ and $G_i^{-1} \in C^1(T_i)$ for any $1 \leqslant i \leqslant m$. For examples of techniques to generate an initial mesh satisfying the assumptions above, see [11].

Lemma A.1. *Let $T = G_T(\hat{T})$ be a closed triangle in $\mathbb{R}^2$ with $G_T \in C^{1,1}(\hat{T})$. Let $\check{\sigma} > 0$ be a positive constant. For any $\check{h} > 0$, we denote by $\check{T}_{\check{h}}$ any triangle contained in $\hat{T}$ such that the diameter of $\check{T}_{\check{h}}$ is $\check{h}$, and $\check{h}/\check{\rho} \leqslant \check{\sigma}$ where $\check{\rho}$ is the diameter of the sphere inscribed in $\check{T}_{\check{h}}$. Let $\hat{T} \ni \hat{\mathbf{x}} \to H\hat{\mathbf{x}} = \check{B}\hat{\mathbf{x}} + \check{b}$ be an affine mapping from $\hat{T}$ onto $\check{T}_{\check{h}}$. Let $\hat{\mathbf{p}}$ be the centroid of $\hat{T}$. We put $B = D(G_T \circ H)(\hat{\mathbf{p}})$, $b = (G_T \circ H)(\hat{\mathbf{p}})$, and $\Psi(\hat{\mathbf{x}}) = (G_T \circ H)(\hat{\mathbf{x}}) - \check{B}(\hat{\mathbf{x}} - \hat{\mathbf{p}}) - b$.*

Then, we have

$$(\sup_{\hat{\mathbf{x}}\in\hat{T}} \|D\Psi(\hat{\mathbf{x}})\|)\|B^{-1}\| \leqslant C\check{h},$$

where C is a constant independent of $\check{h}$, and $\check{T}_{\check{h}}$.

Proof. Obviously, $B = DG_T(H(\hat{\mathbf{p}}))\check{B}$. So $B^{-1} = \check{B}^{-1}(DG_T(H(\hat{\mathbf{p}})))^{-1}$. We have

$$D\Psi(\hat{\mathbf{x}}) = (DG_T(H(\hat{\mathbf{x}})) - DG_T(H(\hat{\mathbf{p}})))\check{B}.$$

Since G_T is a C^1-diffeomorphism from $\hat{T}$ onto T, we have $\sup_{\hat{\mathbf{x}}\in\hat{T}} \|(DG_T(\hat{\mathbf{x}}))^{-1}\| < \infty$. With $G_T \in C^{1,1}(\hat{T})$, we also have

$$\|DG_T(H(\hat{\mathbf{x}})) - DG_T(H(\hat{\mathbf{p}}))\| \leqslant M\|H(\hat{\mathbf{x}}) - H(\hat{\mathbf{p}})\| \quad \text{for any } \hat{\mathbf{x}} \in \hat{T}$$

where M is the Lipschitz constant for all first order derivatives of G_T on $\hat{T}$.

With $\check{h}/\check{\rho} \leqslant \check{\sigma}$, we obtain that $\|\check{B}\| \cdot \|\check{B}^{-1}\| \leqslant \dfrac{c}{\check{\sigma}}$, for some $c > 0$. With $\check{h}$ denoting the diameter of $\check{T}_{\check{h}}$, we have $\|H(\hat{\mathbf{x}}) - H(\hat{\mathbf{p}})\| \leqslant \check{h}$ for any $\hat{\mathbf{x}} \in \check{T}_{\check{h}}$. The definition of Ψ implies then

$$(\sup_{\hat{\mathbf{x}} \in \hat{T}} \|D\Psi(\hat{\mathbf{x}})\|)\|B^{-1}\| \leqslant \frac{c}{\check{\sigma}} M(\sup_{\hat{\mathbf{y}} \in \hat{T}} \|(DG_T(\hat{\mathbf{y}}))^{-1}\|)\check{h}.$$

Setting $C = \dfrac{c}{\check{\sigma}} M(\sup_{\hat{\mathbf{y}} \in \hat{T}} \|(DG_T(\hat{\mathbf{y}}))^{-1}\|)$ finishes the proof. □

Let $\hat{T}_i$ be now multiple copies of the reference triangle corresponding to the initial mesh, $T_i = G_i(\hat{T}_i)$, $i = 1, \dots, m$.

Definition A.1. (Regular triangular meshes in the reference space) A family of triangulations $(\hat{\mathcal{T}}_{i,\check{h}})_{\check{h}}$ of reference triangles $\hat{T}_i$ is said to be regular provided two conditions are satisfied:

(i) Partitions of edges of $\hat{T}_i$ mapped into the same edge in the physical space are identical.

(ii) $\sup_{\check{h}} \sup_{\check{T} \in \hat{\mathcal{T}}_{i,\check{h}}} \check{h}/\check{\rho} < \infty$, where $\check{h}$ and $\check{\rho}$ are the outer and inner diameters of $\check{T}$.

Obviously, uniform refinements of reference triangles are regular. A number of adaptive refinement algorithms produces regular meshes as well. To this class belong, e.g., Rivara's algorithm (bisection by the longest edge), Arnold's algorithm (bisection by the newest edge), the Delaunay triangulation (see [7]).

Using Lemma A.1 and the fact that G_i is C^1-diffeomorphism from $\hat{T}$ to T_i, $1 \leqslant i \leqslant m$, we easily conclude that any regular refinements in the reference space produce curvilinear meshes that satisfy our mesh regularity assumptions.

B. Properties of Sobolev spaces on curved and reference triangles

Lemma B.1. *Let T be a curved triangle. For any $\omega \in H^1(T; \mathbb{R}^2)$, we define $\hat{\omega}(\hat{\mathbf{x}})$ on $\hat{T}$ by*

$$\omega(\mathbf{x}) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\omega}(\hat{\mathbf{x}}) \quad \hat{\mathbf{x}} \in \hat{T}.$$

Then $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$. Divergence transforms by the classical Piola's rule:

$$div\omega(\mathbf{x}) = (\det(DG_T(\hat{\mathbf{x}})))^{-1} div_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})$$

for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere.

Proof. Notice that $\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. It is straightforward to see that $\det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}$ is a matrix whose entries contain first order partial derivatives of $G_T(\hat{x})$. Notice that

$$\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(\mathbf{x})$$

is the standard pull back mapping from $H(\mathrm{div}, T)$ to $H(\mathrm{div}_{\hat{\mathbf{x}}}, \hat{T})$. So we immediately have $\mathrm{div}\omega(\mathbf{x}) = \dfrac{1}{\det(DG_T(\hat{\mathbf{x}}))}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})$ for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere. Since G_T is a C^1-diffeomorphism from $\hat{T}$ to T, we can conclude that $\hat{\omega}(\hat{\mathbf{x}}) \in H(\mathrm{div}, \hat{T})$.

Let $e \in \triangle_1(T)$. We denote by $\zeta(s) : [0,1] \to \hat{e}$, the local affine parametrization of $\hat{e}$. We have then

$$\begin{aligned}
\|\hat{\omega}\|^2_{L^2(\hat{e})} &= \int_{[0,1]} (\hat{\omega}(\zeta(s)))^\top \hat{\omega}(\zeta(s)) \|\dot{\zeta}(s)\| ds \\
&= \int_{[0,1]} (\det(DG_T(\zeta(s))))^2 \omega(G_T(\zeta(s)))^\top [DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1}] \\
&\qquad \omega(G_T(\zeta(s))) \|(DG_T(\zeta(s)))^{-1} [DG_T(\zeta(s)) \dot{\zeta}(s)]\| ds.
\end{aligned}$$

Since G_T is a C^1-diffeomorphism from $\hat{T}$ to T, we can conclude that $\hat{\omega}|_{\partial\hat{T}} \in L^2(\partial\hat{T}; \mathbb{R}^2)$. So $\hat{\omega} \in H(\hat{T})$. $\square$

Lemma B.2. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\omega\|_{L^2(T)} \leqslant C \|\hat{\omega}\|_{L^2(\hat{T})}, \forall \omega \in L^2(T; \mathbb{R}^2),$$

where $\dfrac{DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} = \omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.

Proof.

$$\begin{aligned}
\|\omega\|_{L^2(T)} &= \int_T \frac{1}{\det(DG_T(\hat{\mathbf{x}}))^2} [DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})]^\top DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}}) d\mathbf{x} \\
&= \int_{\hat{T}} \det((DG_T)^{-1}) \hat{\omega}^\top [DG_T^\top DG_T] \hat{\omega} d\hat{\mathbf{x}} \\
&= \int_{\hat{T}} \det(DG_T^{-1} B_T) \det(B_T^{-1}) \hat{\omega}^\top B_T^\top [B_T^{-\top} DG_T^\top DG_T B_T^{-1}] B_T \hat{\omega} d\hat{\mathbf{x}}.
\end{aligned}$$

Since $c_h \to 0$ as $h \to 0$, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|B_T^{-\top} DG_T^\top DG_T B_T^{-1} - I\| = 0$. By Lemma 3.4, we have $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T^{-1} B_T) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there is a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leqslant \sigma$ for any $T \in \mathcal{T}_h$, where σ_1 and σ_2 denote the biggest and smallest singular value of the corresponding matrix B_T. Then $\|B_T^\top\| = \|B_T\| = \sigma_1$ and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So $\|B_T^\top\| \cdot \|B_T\| \det(B_T^{-1}) = \sigma_1/\sigma_2 \leqslant \sigma$ for any $h > 0$ and any $T \in \mathcal{T}_h$.

We can conclude thus that there exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\omega\|_{L^2(T)} \leqslant C \|\hat{\omega}\|_{L^2(\hat{T})} \quad \forall \omega \in L^2(T; \mathbb{R}^2)$$

If, for all h, $T \in \mathcal{T}_h$ are (regular) triangles, the asymptotic argument is not necessary, and the above inequality holds for any $h > 0$. $\square$

Lemma B.3. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\hat{\omega}\|^2_{H(div_{\hat{\mathbf{x}}}, \hat{T})} + \|\hat{\omega}\|^2_{L^2(\partial\hat{T})} \leqslant C \|\omega\|^2_{H^1(T)} \quad \forall \omega \in H^1(T; \mathbb{R}^2),$$

where $\dfrac{DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} = \omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.

Proof.

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} (\det(DG_T(\hat{\mathbf{x}})))^2 \omega(\mathbf{x})^\top [DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1}] \omega(\mathbf{x}) d\hat{\mathbf{x}} \\ &= \int_T \det(DG_T(\hat{\mathbf{x}})) \omega(\mathbf{x})^\top [DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1}] \omega(\mathbf{x}) d\mathbf{x} \\ &= \int_T \det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) \det(B_T) \omega(\mathbf{x})^\top B_T^{-\top} \\ &\qquad [B_T^\top DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1} B_T] B_T^{-1} \omega(\mathbf{x}) d\mathbf{x}.\end{aligned}$$

According to Lemma 3.3, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|B_T^\top DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1} B_T - I\| = 0$. By Lemma 3.4, we have $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leqslant \sigma$ for any $T \in \mathcal{T}_h$, where σ_1 and σ_2 denote the biggest and smallest singular value of the matrix B_T. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$ and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So $\|B_T^{-\top}\| \cdot \|B_T^{-1}\| \det(B_T) = \sigma_1/\sigma_2 \leqslant \sigma$. Consequently, there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}&\det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) \det(B_T) \omega(\mathbf{x})^\top B_T^{-\top} [B_T^\top DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1} B_T] B_T^{-1} \omega(\mathbf{x}) \\ &\leqslant C_1^2 \omega(\mathbf{x})^\top \omega(\mathbf{x})\end{aligned}$$

for all $\omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T$. We can conclude that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\|\hat{\omega}\|_{L^2(\hat{T})} \leqslant C_1 \|\omega\|_{L^2(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2).$$

According to Lemma B.1, we have

$$\begin{aligned}\|\mathrm{div}_{\hat{\mathbf{x}}} \hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} (\det(DG_T(\hat{\mathbf{x}})))^2 (\mathrm{div}\omega(\mathbf{x}))^2 d\hat{\mathbf{x}} = \int_T \det(DG_T(\hat{\mathbf{x}})) (\mathrm{div}\omega(\mathbf{x}))^2 d\mathbf{x} \\ &= \int_T \det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) \det(B_T) (\mathrm{div}\omega(\mathbf{x}))^2 d\mathbf{x}.\end{aligned}$$

By Lemma 3.4, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) - 1| = 0$. Obviously, $\det(B_T) \leqslant \tilde{h}_T^2$. There must exist then $\delta_2 > 0$ and $C_2 > 0$ such that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\det(DG_T(\hat{\mathbf{x}}) B_T^{-1}) \det(B_T) (\mathrm{div}\omega(\mathbf{x}))^2 \leqslant C_2^2 \tilde{h}_T^2 (\mathrm{div}\omega(\mathbf{x}))^2 \quad \forall \omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T.$$

We conclude that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\|\mathrm{div}_{\hat{\mathbf{x}}} \hat{\omega}\|_{L^2(\hat{T})} \leqslant C_2 \tilde{h}_T \|\mathrm{div}\omega\|_{L^2(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2).$$

We take now an arbitrary $e \in \triangle_1(T)$. We denote by $\zeta(s) : [0,1] \to \hat{e}$ the local affine parametrization of $\hat{e}$. We have then

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{e})} &= \int_{[0,1]} (\hat{\omega}(\zeta(s)))^\top \hat{\omega}(\zeta(s)) \|\dot{\zeta}(s)\| ds \\ &= \int_{[0,1]} (\det(DG_T(\zeta(s))))^2 \omega(G_T(\zeta(s)))^\top [DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1}] \\ &\qquad \omega(G_T(\zeta(s))) \|\dot{\zeta}(s)\| ds \\ &= \int_{[0,1]} (\det(DG_T(\zeta(s))) B_T^{-1})^2 (\det(B_T))^2 \omega(G_T(\zeta(s)))^\top B_T^{-\top} \\ &\qquad [B_T^\top DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1} B_T] B_T^{-1} \omega(G_T(\zeta(s))) \|\dot{\zeta}(s)\| ds.\end{aligned}$$

According to Lemma 3.3, we have that

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in T}\sup_{s\in[0,1]}\|B_T^\top DG_T(\zeta(s))^{-\top}DG_T(\zeta(s))^{-1}B_T - I\| = 0.$$

By Lemma 3.4, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in T}\sup_{s\in[0,1]}|(\det(DG_T(\zeta(s)))B_T^{-1})^2 - 1| = 0$.

Consider again the singular values of B_T, $\sigma_1 \geqslant \sigma_2$. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$, and $\det(B_T) = \sigma_1\cdot\sigma_2$. So $\|B_T^{-\top}\|\cdot\|B_T^{-1}\|(\det(B_T))^2 = (\sigma_1)^2 \leqslant \tilde{h}_T^2$. Consequently, there exist $\delta_3 > 0$ and $C_3 > 0$ such that, for any $h < \delta_3$ and $T\in\mathcal{T}_h$,

$$\begin{aligned}&(\det(DG_T(\zeta(s)))B_T^{-1})^2(\det(B_T))^2\omega(G_T(\zeta(s)))^\top B_T^{-\top}\\&[B_T^\top DG_T(\zeta(s))^{-\top}DG_T(\zeta(s))^{-1}B_T]B_T^{-1}\omega(G_T(\zeta(s)))\\&\leqslant C_3\tilde{h}_T^2\omega(\mathbf{x})^\top\omega(\mathbf{x})\quad\forall\omega\in H^1(T;\mathbb{R}^2), e\in\triangle_1(T), s\in[0,1]\end{aligned}$$

and

$$\begin{aligned}\|\dot{\zeta}(s)\| =&\|B_T^{-1}[B_T DG_T(\zeta(s))^{-1}]\cdot[DG_T(\zeta(s))\dot{\zeta}(s)]\|\\ \leqslant& C_3\tilde{h}_T^{-1}\|DG_T(\zeta(s))\dot{\zeta}(s)\|\quad\forall e\in\triangle_1(T), s\in[0,1].\end{aligned}$$

We can conclude that, for any $h < \delta_3$ and $T\in\mathcal{T}_h$,

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{e})} &\leqslant C_3^2\tilde{h}_T\int_{[0,1]}\omega(G_T(\zeta(s)))^\top\omega(G_T(\zeta(s)))\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds\\&= C_3^2\tilde{h}_T\|\omega\|^2_{L^2(e)}.\end{aligned}$$

Obviously, $\sup_h\sup_{T\in\mathcal{T}_h}\tilde{h}_T < \infty$. Since $\omega\in H^1(T;\mathbb{R}^2)$, we can use the Trace Theorem to conclude that there exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T\in\mathcal{T}_h$,

$$\|\hat{\omega}\|^2_{H(\mathrm{div}_{\hat{\mathbf{x}}},\hat{T})} + \|\hat{\omega}\|^2_{L^2(\partial\hat{T})} \leqslant C\|\omega\|^2_{H^1(T)}\quad\forall\omega\in H^1(T;\mathbb{R}^2).$$

It is easy to see that, if T is a (regular) triangle for any $T\in\mathcal{T}_h$ and $h > 0$, then the inequality above holds for any $h > 0$. $\square$

Lemma B.4. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T\in\mathcal{T}_h$,*

$$\|curl\omega\|^2_{L^2(T)} \leqslant C\tilde{h}_T^{-2}\|curl_{\hat{\mathbf{x}}}\hat{\omega}\|^2_{L^2(\hat{T})}\quad\forall\omega\in H^1(T;\mathbb{R}^2)$$

where $\dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)} = \omega(\mathbf{x})$ *for any* $\hat{\mathbf{x}}\in\hat{T}$. *If T is a triangle for any $T\in\mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.*

Proof. We have

$$\begin{aligned}\mathrm{curl}\omega(\mathbf{x}) =&\frac{B_T}{\det(B_T)}(\mathrm{curl}\hat{\omega})(\hat{\mathbf{x}}) = \frac{1}{\det(B_T DG_T(\hat{\mathbf{x}}))}B_T\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})(DG_T(\hat{\mathbf{x}}))^\top\\ =&\frac{(\det(B_T^{-1}))^2}{\det(B_T^{-1}DG_T(\hat{\mathbf{x}}))}B_T\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top.\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{curl}\omega\|^2_{L^2(T)} &= \int_T \frac{(\det(B_T^{-1}))^4}{(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2}(DG_T(\hat{\mathbf{x}})B_T^{-1})B_T(\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top \\ &\qquad [B_T^\top B_T]\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top d\mathbf{x} \\ &= \int_{\hat{T}} \frac{(\det(B_T^{-1}))^3}{(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2}(DG_T(\hat{\mathbf{x}})B_T^{-1})B_T(\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top \\ &\qquad [B_T^\top B_T]\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top d\hat{\mathbf{x}}.\end{aligned}$$

Since $c_h \to 0$ as $h \to 0$, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|DG_T(\hat{\mathbf{x}})B_T^{-1} - I\| = 0$.
By Lemma 3.4, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2 - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leqslant \sigma$ for any $T \in \mathcal{T}_h$, with $\sigma_1 \geqslant \sigma_2$ denoting the singular values of matrix B_T. Then $\|B_T^\top\| = \|B_T\| = \sigma_1$ and $\det(B_T^{-1}) = \sigma_1^{-1}\cdot\sigma_2^{-1}$. So $\|B_T^\top\|^2\cdot\|B_T\|^2(\det(B_T^{-1}))^3 = \sigma_1/\sigma_2^3 \leqslant \sigma/\sigma_2^2 \leqslant c'\tilde{h}_T^{-2}$ for some constant $c' > 0$.

Consequently, there exist $\delta_1 > 0$ and $C > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}&\int_{\hat{T}} \frac{(\det(B_T^{-1}))^3}{(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2}(DG_T(\hat{\mathbf{x}})B_T^{-1})B_T(\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top \\ &\quad [B_T^\top B_T]\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top d\hat{\mathbf{x}} \\ &\quad \leqslant C\tilde{h}_T^{-2}\int_{\hat{T}}\|(\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top(\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))\|d\hat{\mathbf{x}}.\end{aligned}$$

Again, it is easy to see that, if T is a (regular) triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$. This finishes the proof. ☐

Lemma B.5. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\hat{\omega}\|^2_{H(curl_{\hat{\mathbf{x}}},\hat{T})} \leqslant C(\|\omega\|^2_{L^2(T)} + \tilde{h}_T^2\|curl\omega\|^2_{L^2(T)}) \quad \forall \omega \in H^1(T;\mathbb{R}^2)$$

where $\dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)} = \omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.

Proof. We have

$$\|\hat{\omega}\|^2_{H(\mathrm{curl}_{\hat{\mathbf{x}}},\hat{T})} = \|\hat{\omega}\|^2_{L^2(\hat{T})} + \|\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}\|^2_{L^2(\hat{T})},$$

and

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}}(\det(B_T))^2\omega(\mathbf{x})^\top B_T^{-\top}B_T^{-1}\omega(\mathbf{x})d\hat{\mathbf{x}} \\ &= \int_T \det(B_T)\det(B_T DG_T(\hat{\mathbf{x}})^{-1})\omega(\mathbf{x})^\top B_T^{-\top}B_T^{-1}\omega(\mathbf{x})d\mathbf{x}.\end{aligned}$$

By Lemma 3.4, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(B_T DG_T(\hat{\mathbf{x}})^{-1}) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leqslant \sigma$ for any $T \in \mathcal{T}_h$, where $\sigma_1 \geqslant \sigma_2$ are the singular values of matrix B_T. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$ and $\det(B_T) = \sigma_1\cdot\sigma_2$. So, $\|B_T^{-\top}\|\cdot\|B_T^{-1}\|\det(B_T) = \sigma_1/\sigma_2 \leqslant \sigma$. Consequently, there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\det(B_T)\det(B_T DG_T(\hat{\mathbf{x}})^{-1})\omega(\mathbf{x})^\top B_T^{-\top}B_T^{-1}\omega(\mathbf{x}) \leqslant C_1^2\omega(\mathbf{x})^\top\omega(\mathbf{x})$$

for any $\omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T$. We can conclude that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\|\hat{\omega}\|_{L^2(\hat{T})} \leqslant C_1 \|\omega\|_{L^2(T)}, \forall \omega \in H^1(T;\mathbb{R}^2).$$

At the same time,

$$\begin{aligned}
\operatorname{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}) &= \det(B_T) B_T^{-1} \operatorname{curl}_{\hat{\mathbf{x}}} \omega(\mathbf{x}) \\
&= \det(B_T) B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) (DG_T(\hat{\mathbf{x}}))^{-\top} \det(DG_T(\hat{\mathbf{x}})) \\
&= \det(B_T) B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (B_T^{\top} DG_T(\hat{\mathbf{x}})^{-\top}) \det(DG_T(\hat{\mathbf{x}})) \\
&= \det(B_T) B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^{\top} \det(DG_T(\hat{\mathbf{x}})),
\end{aligned}$$

and

$$\begin{aligned}
\|\operatorname{curl}_{\hat{\mathbf{x}}} \hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} \det(B_T)^2 \det(DG_T(\hat{\mathbf{x}}))^2 (DG_T(\hat{\mathbf{x}})^{-1} B_T) B_T^{-1} (\operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}))^{\top} \\
&\qquad B_T^{-\top} B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^{\top} d\hat{\mathbf{x}} \\
&= \int_{\hat{T}} \det(B_T)^2 \det(DG_T(\hat{\mathbf{x}})) (DG_T(\hat{\mathbf{x}})^{-1} B_T) B_T^{-1} (\operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}))^{\top} \\
&\qquad B_T^{-\top} B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^{\top} d\mathbf{x} \\
&= \int_{\hat{T}} \det(B_T)^3 \det(B_T^{-1} DG_T(\hat{\mathbf{x}})) (DG_T(\hat{\mathbf{x}})^{-1} B_T) B_T^{-1} (\operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}))^{\top} \\
&\qquad B_T^{-\top} B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^{\top} d\mathbf{x}.
\end{aligned}$$

According to Lemma 3.3, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|DG_T(\hat{\mathbf{x}})^{-1} B_T - I\| = 0$.
By Lemma 3.4, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(B_T^{-1} DG_T(\hat{\mathbf{x}})) - 1| = 0$.
Since $(\mathcal{T}_h)_h$ is regular, $\det(B_T)^3 \|B_T^{-\top}\|^2 \cdot \|B_T^{-1}\|^2 \leqslant \sigma \tilde{h}_T$, for some constant $\sigma > 0$.

There exist thus $\delta_2 > 0$ and $C_2 > 0$ such that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}
&\det(B_T)^3 \det(B_T^{-1} DG_T(\hat{\mathbf{x}})) (DG_T(\hat{\mathbf{x}})^{-1} B_T) B_T^{-1} (\operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}))^{\top} \\
&\quad B_T^{-\top} B_T^{-1} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^{\top} d\mathbf{x} \\
&\quad \leqslant C_2^2 \tilde{h}_T^2 \|(\operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x}))^{\top} \operatorname{curl}_{\mathbf{x}} \omega(\mathbf{x})\| \quad \forall \omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T.
\end{aligned}$$

We conclude that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\|\operatorname{curl}_{\hat{\mathbf{x}}} \hat{\omega}\|_{L^2(\hat{T})} \leqslant C_2 \tilde{h}_T \|\operatorname{curl} \omega\|_{L^2(T)}, \forall \omega \in H^1(T;\mathbb{R}^2).$$

Again, it is easy to see that, if T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the inequality above holds for any $h > 0$. This ends the proof. ☐

Lemma B.6. *Let T be a curved triangle. Let $u(\mathbf{x})$ be defined on T and $\hat{u}(\hat{\mathbf{x}})$ be defined on $\hat{T}$ and*

$$u = \hat{u} \circ G_T^{-1} \quad \text{or} \quad \hat{u} = u \circ G_T$$

where G_T is the element map. Then $u(\mathbf{x}) \in L^2(T)$ if and only if $\hat{u}(\hat{x}) \in L^2(\hat{T})$.

Proof. This is an immediate consequence of the fact that G_T is a C^1-diffeomorphism. ☐

C. Proofs in Section 8

C.1. Proof of Lemma 6.2

Proof. We put $r = \tilde{r}(T)$. We assume $\{\hat{\xi}_1(\hat{\mathbf{x}}), \cdots, \hat{\xi}(\hat{\mathbf{x}})_{l_r}\}$ is a basis for $\mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T})$. Then

$$\Pi^2_{\tilde{r},T} u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \sum_{i=1}^{l_r} \alpha_i \hat{\xi}_i(\hat{\mathbf{x}})$$

and

$$P^2_{\tilde{r},T} u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \sum_{i=1}^{l_r} \beta_i \hat{\xi}_i(\hat{\mathbf{x}}).$$

Coefficients $(\alpha_1, \cdots, \alpha_{l_r})^\top$ and $(\beta_1, \cdots, \beta_{l_r})^\top$ are obtained by solving the following two linear systems,

$$A_1(\alpha_1, \cdots, \alpha_{l_r})^\top = \mathbf{b}_1 \quad \text{and} \quad A_2(\beta_1, \cdots, \beta_{l_r})^\top = \mathbf{b}_2$$

with

$$\begin{aligned} (A_1)_{ij} &= \int_T \frac{\hat{\xi}_i(\hat{\mathbf{x}}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{\det(DG_T(\hat{\mathbf{x}}(\mathbf{x})))}, \quad (\mathbf{b}_1)_j = \int_T u(\mathbf{x})\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}, \\ (A_2)_{ij} &= \int_T \frac{\hat{\xi}_i(\hat{\mathbf{x}}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{(\det(DG_T(\hat{\mathbf{x}}(\mathbf{x}))))^2}, \quad (\mathbf{b}_2)_j = \int_T \frac{\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{\det(DG_T(\hat{\mathbf{x}}(\mathbf{x})))} u(\mathbf{x}), \end{aligned}$$

for $1 \leqslant i, j \leqslant l_r$. By pulling back to $\hat{T}$ we obtain, for any $1 \leqslant i, j \leqslant l_r$,

$$\begin{aligned} (A_1)_{ij} &= \int_{\hat{T}} \hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}, \quad (\mathbf{b}_1)_j = \det(B_T)\int_{\hat{T}} \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))u(\mathbf{x})\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\hat{\mathbf{x}}, \\ (A_2)_{ij} &= \int_{\hat{T}} \frac{\hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}}{\det(DG_T(\hat{\mathbf{x}}))}, \quad (\mathbf{b}_2)_j = \int_{\hat{T}} u(\mathbf{x}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}. \end{aligned}$$

Since $\det(B_T)$ is a non-zero constant, $(\det(B_T)A_2)(\beta_1, \cdots, \beta_{l_r})^\top = \det(B_T)\mathbf{b}_2$. So we can redefine A_2 and $\mathbf{b}_2$ in the following way.

$$(A_2)_{ij} = \int_{\hat{T}} \frac{\det(B_T)}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad (\mathbf{b}_2)_j = \det(B_T)\int_{\hat{T}} u(\mathbf{x}(\hat{\mathbf{x}}))\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}.$$

According to Lemma 3.4, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\|A_1 - A_2\| = 0$. And, for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\|\mathbf{b}_1 - \mathbf{b}_2\|^2 \leqslant &(\varepsilon\det(B_T))^2\int_{\hat{T}} u^2(\mathbf{x}(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \\ =&\varepsilon^2\det(B_T)\int_T \det(B_T DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-1})u^2(\mathbf{x})d\mathbf{x} \\ \leqslant&4\varepsilon^2\det(B_T)\|u\|^2_{L^2(T)}. \end{aligned}$$

The last inequality holds when h is small enough. This implies that

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\|\mathbf{b}_1 - \mathbf{b}_2\| \leqslant 2\varepsilon\sqrt{\det(B_T)}\|u\|_{L^2(T)}.$$

So, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h \leqslant \delta$ and $T \in \mathcal{T}_h$, we have

$$\|(\alpha_1 - \beta_1, \cdots, \alpha_{l_r} - \beta_{l_r})\| \leqslant 3\varepsilon \sqrt{\det(B_T)} \|u\|_{L^2(T)}.$$

We have then

$$\begin{aligned}
\|\Pi^2_{\tilde{r},T} u - P_{\tilde{r},T} u\|^2_{L^2(T)} &= \int_T \frac{1}{(\det(DG_T(\hat{\mathbf{x}}(\mathbf{x}))))^2} \sum_{i=1}^{l_r} \hat{\xi}_i^2(\hat{\mathbf{x}}(\mathbf{x}))(\alpha_i - \beta_i)^2 d\mathbf{x} \\
&= \int_{\hat{T}} \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \sum_{i=1}^{l_r} \hat{\xi}_i^2(\hat{\mathbf{x}})(\alpha_i - \beta_i)^2 d\hat{\mathbf{x}} \\
&\leqslant c\varepsilon^2 \|u\|^2_{L^2(T)} \int_{\hat{T}} \det(B_T DG_T(\hat{\mathbf{x}})^{-1}) \sum_{i=1}^{l_r} \hat{\xi}_i^2(\hat{\mathbf{x}}) d\hat{\mathbf{x}}.
\end{aligned}$$

Here c is a positive constant which depends on l_r only. By Lemma 3.4, there exists $M > 0$ such that, for any h small enough and any $T \in \mathcal{T}_h$,

$$\int_{\hat{T}} \det(B_T DG_T(\hat{\mathbf{x}})^{-1}) \sum_{i=1}^{l_r} \hat{\xi}_i^2(\hat{\mathbf{x}}) d\hat{\mathbf{x}} \leqslant M^2.$$

We can conclude that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\Pi^2_{\tilde{r},T} u - P_{\tilde{r},T} u\|_{L^2(T)} \leqslant \varepsilon \|u\|_{L^2(T)}, \forall u \in L^2(T).$$

Here $P_{\tilde{r},T}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$. □

C.2. Proof of Lemma 6.3

Proof. Obviously, $\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(G_T(\hat{\mathbf{x}}))$. Using Lemma B.1, we can conclude that $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$.

By pulling back to $\hat{T}$ and using the definition of $\mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$, we can see that (6.4) is the same as (6.7), and (6.3) is the same as (6.6). Thus, we only need to show that (6.5) is the same as (6.8). Since $\mathcal{T}_h$ is C^0-compatible, then $G_T(\zeta(s)) = \mathbf{x}_e(s)$ for any $s \in [0,1]$ where $\zeta : [0,1] \to \hat{e}$ is an affine local parametrization of $\hat{e}$.
We have then

$$\begin{aligned}
&\int_{[0,1]} [\omega(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\| ds \\
&= \int_{[0,1]} [\omega(G_T(\zeta(s))) \cdot \mathbf{n}(G_T(\zeta(s)))]\hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\| ds \\
&= \int_{[0,1]} \left[\frac{DG_T(\zeta(s))\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|} \right] \hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\| ds \\
&= \int_{[0,1]} \left[\frac{\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|} \right] \hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\| ds.
\end{aligned}$$

Notice that $\zeta(s) = c\hat{t}$ where $\hat{t}$ is a unit tangent vector along $\hat{e}$, and c is a nonzero constant, $\hat{\mathbf{n}}(\zeta(s)) = (\hat{t}_2, -\hat{t}_1)^\top$, and $(DG_T(\zeta(s)))^{-\top} = \dfrac{A}{\det(DG_T(\zeta(s)))}$ with

$$A = \begin{bmatrix} (DG_T)_{22}(\zeta(s)) & -(DG_T)_{21}(\zeta(s)) \\ -(DG_T)_{12}(\zeta(s)) & (DG_T)_{11}(\zeta(s)) \end{bmatrix}.$$

Therefore, we have

$$\int_{[0,1]} \left[\frac{\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|} \right] \hat{\eta}(s) \|DG_T(\zeta(s))\dot{\zeta}(s)\| ds$$
$$= c \int_{[0,1]} [\hat{\omega}(\zeta(s)) \cdot \hat{\mathbf{n}}(\zeta(s))] \, \hat{\eta}(s) ds.$$

We conclude that (6.5) is equivalent with (6.8). This finishes the proof. □

C.3. Proof of Lemma 6.4

Proof. By Lemma B.1, we have $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$, and

$$\text{div } \omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \text{div}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}).$$

for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere, provided we define $\hat{\omega}(\hat{\mathbf{x}})$ on $\hat{T}$ by

$$\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} \hat{\omega}(\hat{\mathbf{x}})$$

for any $\hat{\mathbf{x}} \in \hat{T}$.

Using Definition 6.1, Definition 6.2, Lemma 6.1, Lemma 6.3, and Lemma 10 in [17], it is easy to see that

$$\Pi^2_{\tilde{r},T} \text{div}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \Pi^2_{\tilde{r},\hat{T}} \text{div}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \text{div}_{\hat{\mathbf{x}}} \Pi^1_{\tilde{r}+1,\hat{T}} \hat{\omega}(\hat{\mathbf{x}}),$$
$$\text{div}\Pi^1_{\tilde{r}+1,T} \omega(\mathbf{x}(\hat{\mathbf{x}})) = \text{div} \left[\frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} \Pi^1_{\tilde{r}+1,\hat{T}} \hat{\omega}(\hat{\mathbf{x}}) \right] = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} \text{div}_{\hat{\mathbf{x}}} \Pi^1_{\tilde{r}+1,\hat{T}} \hat{\omega}(\hat{\mathbf{x}}).$$

We have thus $\Pi^2_{\tilde{r},T} \text{div}\omega = \text{div}\Pi^1_{\tilde{r}+1,T}\omega$. □

C.4. Proof of Lemma 6.5

Proof. According to Lemma B.2 and Lemma B.3, there exist $\delta > 0$ and $C_1 > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\Pi^1_{\tilde{r},T}\omega\|_{L^2(T)} \leqslant C_1 \|\Pi^1_{\tilde{r},\hat{T}}\hat{\omega}\|_{L^2(\hat{T})} \quad \forall \omega \in L^2(T;\mathbb{R}^2)$$

$$\|\hat{\omega}\|^2_{H(\text{div}_{\hat{\mathbf{x}}},T)} + \|\hat{\omega}\|^2_{L^2(\partial\hat{T})} \leqslant C_1 \|\omega\|_{H^1(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2)$$

By Lemma 6.3, $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$ for any $\omega \in H^1(T;\mathbb{R}^2)$.

The definition of operator $\Pi_{\tilde{r},\hat{T}}$ implies that there exists a constant $C_2 > 0$ such that

$$\int_{\hat{T}} (\Pi^1_{\tilde{r},\hat{T}}\hat{\omega}(\hat{\mathbf{x}}))^\top \Pi^1_{\tilde{r},\hat{T}}\hat{\omega}(\hat{\mathbf{x}}) d\hat{\mathbf{x}} \leqslant C_2 (\|\hat{\omega}\|^2_{H(\text{div}_{\hat{\mathbf{x}}},\hat{T})} + \|\hat{\omega}\|^2_{L^2(\partial(\hat{T}))}), \forall \hat{\omega}(\hat{\mathbf{x}}) \in [H^1(\hat{T})]^2.$$

It is easy to see that for affine meshes the above inequality holds for any $h > 0$. This finishes the proof. □

C.5. Proof of Theorem 6.2

Proof. For any $h > 0$ and any $T \in \mathcal{T}_h$, we define a linear isomorphism A_T from $H^1(\hat{T};\mathbb{R}^2)$ to $H^1(T;\mathbb{R}^2)$ by $(A_T\hat{\omega})(\mathbf{x}(\hat{\mathbf{x}})) = \dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)}$. It is easy to see that A_T is a linear isomorphism from $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ to $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$.

We define an operator $E_T : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ by $W_T(A_T\hat{\omega}) = A_T(E_T\hat{\omega})$. Obviously, W_T is well-defined if and only if E_T is well-defined. We denote by $\{\hat{\xi}_1, \cdots, \hat{\xi}_{l_{\tilde{r}}}\}$ a particular basis of $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$.

According to Lemma B.4 and Lemma B.5, it is sufficient to show that there exist $\delta > 0$ and $C_1 > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$, E_T is well-defined, and $\|(z_1, \cdots, z_{l_{\tilde{r}}})\| \leqslant C_1\|\hat{\omega}\|_{H^1(\hat{T})}$ for any $\hat{\omega}$. Here $\sum_{k=1}^{l_{\tilde{r}}} z_k\hat{\xi}_k = E_T\hat{\omega}$.

According to the definition of W_T, E_T can be defined by relations

$$\int_T \operatorname{div}(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}, \tag{C.1}$$

$$\int_T ((A_TE_T\hat{\omega})(\mathbf{x}) - (A_T\hat{\omega})(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}), t_{\tilde{r}(\hat{T})})d\mathbf{x} = 0 \quad 1 \leqslant i \leqslant k_{\tilde{r}}, \tag{C.2}$$

$$\int_{[0,1]} [(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]) \quad \forall e \in \triangle_1(T), \tag{C.3}$$

$$\int_{[0,1]} [(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x}_e(s)) \cdot \mathbf{t}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]) \quad \forall e \in \triangle_1(T), \tag{C.4}$$

$$E_T\hat{\omega} = 0 \text{ at all vertices of } \hat{T}. \tag{C.5}$$

Denote

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B_T, J = \det(B_T), \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = E_T\hat{\omega}, \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \hat{\omega}, \hat{u}_{i,j} = \frac{\partial\hat{u}_i}{\partial\hat{x}_j}, \hat{w}_{i,j} = \frac{\partial\hat{w}_i}{\partial\hat{x}_j}.$$

By pulling back to $\hat{T}$, E_T can be defined by relations

$$\begin{aligned} &\int_{\hat{T}} J^{-1}[(b_{11}(DG_T)_{22} - b_{21}(DG_T)_{12})\hat{u}_{1,1} + (b_{12}(DG_T)_{22} - b_{22}(DG_T)_{12})\hat{u}_{2,1} \\ &\quad + (b_{21}(DG_T)_{11} - b_{11}(DG_T)_{21})\hat{u}_{1,2} + (b_{22}(DG_T)_{11} - b_{12}(DG_T)_{21})\hat{u}_{2,2}]\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \\ &= \int_{\hat{T}} J^{-1}[(b_{11}(DG_T)_{22} - b_{21}(DG_T)_{12})\hat{w}_{1,1} + (b_{12}(DG_T)_{22} - b_{22}(DG_T)_{12})\hat{w}_{2,1} \\ &\quad + (b_{21}(DG_T)_{11} - b_{11}(DG_T)_{21})\hat{w}_{1,2} + (b_{22}(DG_T)_{11} - b_{12}(DG_T)_{21})\hat{w}_{2,2}]\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}}, \\ &\forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}. \end{aligned} \tag{C.6}$$

$$\begin{aligned} &\int_{\hat{T}} (E_T\hat{\omega}(\hat{\mathbf{x}}))^\top B_T^\top DG_T(\hat{\mathbf{x}})^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}, t)\det(B_T^{-1}DG_T(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \\ &= \int_{\hat{T}} (\hat{\omega}(\hat{\mathbf{x}}))^\top B_T^\top DG_T(\hat{\mathbf{x}})^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}, t)\det(B_T^{-1}DG_T(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \quad 1 \leqslant i \leqslant k_{\tilde{r}}. \end{aligned} \tag{C.7}$$

$$\int_{[0,1]}[B_T E_T\hat{\omega}(\zeta(s))\cdot(DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}(\zeta(s)))]\hat{\eta}(s)\det(B_T^{-1}DG_T(\zeta(s))ds \tag{C.8}$$
$$=\int_{[0,1]}[B_T\hat{\omega}(\zeta(s))\cdot(DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}(\zeta(s)))]\hat{\eta}(s)\det(B_T^{-1}DG_T(\zeta(s))ds$$
$$\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T).$$

$$\int_{[0,1]}[B_T E_T\hat{\omega}(\zeta(s))\cdot(DG_T(\zeta(s))\dot{\zeta}(s))]\hat{\eta}(s)\det(B_T^{-1})ds \tag{C.9}$$
$$=\int_{[0,1]}[B_T\hat{\omega}(\zeta(s))\cdot(DG_T(\zeta(s))\dot{\zeta}(s))]\hat{\eta}(s)\det(B_T^{-1})ds\quad\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T).$$

$$E_T\hat{\omega}=0 \text{ at all vertices of } \hat{T}. \tag{C.10}$$

It is easy to see that (C.6) comes from (C.1), (C.7) comes from (C.2), (C.9) comes from (C.4), and (C.10) comes from (C.5). And (C.8) can be got from (C.3) by using the fact that $\|\dot{\mathbf{x}}_e(s)\|=\|DG_T(\zeta(s))\hat{\mathbf{t}}\|=c\|DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}\|$ for some non-zero constant c, which comes from direct calculation.

Notice that vector $\dot{\zeta}(s)$ is constant tangent vector along each edge of $\hat{T}$. Set

$$a=\hat{\mathbf{n}}^\top B_T^\top B_T\dot{\zeta}(s)\det(B_T^{-1}),\quad b=\frac{\det(B_T)\|\dot{\zeta}(s)\|}{\dot{\zeta}(s)^\top B_T^\top B_T\dot{\zeta}(s)}.$$

Obviously, $b\neq 0$.

Perform now the operation: $b\times[(C.9)-a\times(C.8)]$. We have,

$$\int_{[0,1]}[E_T\hat{\omega}(\zeta(s))\cdot F_T(s)]\hat{\eta}(s)ds=\int_{[0,1]}[\hat{\omega}(\zeta(s))\cdot F_T(s)]\hat{\eta}(s)ds \tag{C.11}$$
$$\forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T),$$

where

$$F_T(s)=\det(B_T^{-1})[B_T^\top(DG_T(\zeta(s)))\dot{\zeta}(s)$$
$$-\det(B_T^{-1}DG_T(\zeta(s)))(\hat{\mathbf{n}}^\top B_T^T B_T\dot{\zeta}(s))B_T^\top(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}})]\frac{\det(B_T)\|\dot{\zeta}(s)\|}{\dot{\zeta}(s)^\top B_T^\top B_T\dot{\zeta}(s)}.$$

Then the definition of operator E_T can be rewritten by using conditions (C.6),(C.7), (C.8), (C.11), and (C.10).

Using the fact that $\hat{\mathbf{n}}\perp\dot{\zeta}(s)$, Lemmas 3.2,3.3,3.4, and the assumption that $(\mathcal{T}_h)_h$ is regular, we obtain

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in\triangle_1(T)}\sup_{s\in[0,1]}\|F_T(s)\cdot\frac{\dot{\zeta}(s)}{\|\dot{\zeta}(s)\|}-1\|=\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in\triangle_1(T)}\sup_{s\in[0,1]}\|F_T(s)\cdot\hat{\mathbf{n}}\|=0.$$

Consequently,

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in\triangle_1(T)}\sup_{s\in[0,1]}\|F_T(s)-\hat{\mathbf{t}}\|=0.$$

We denote now by $E(T,\tilde{r})$ the matrix corresponding to the left-hand side of conditions (C.6),(C.7),(C.8), (C.11),(C.10), a particular basis of $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ (the solution space), some basis of $\mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}$, and some basis of $\mathcal{P}_{\tilde{r}(e)}([0,1])$ for each $e \in \triangle_1(T)$. We denote by $\{\hat{\xi}_1, \cdots, \hat{\xi}_{l_{\tilde{r}}}\}$ a basis for $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$. Finally, we denote by $C(\tilde{r})$ the matrix corresponding to the left-hand side of conditions (6.12), (6.13),(6.14), and (6.15), and the same bases as above.

Using the fact that

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in\triangle_1(T)} \sup_{s\in[0,1]} \|F_T(s) - \hat{\mathbf{t}}\| = 0$$

and Lemmas 3.2, 3.3, 3.4, we conclude that, for any $t \in [0,1]$,

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \|E(T,\tilde{r}) - C(\tilde{r})\| = 0.$$

Then, for any given $t \in [0,1]$, and any given $\tilde{r}$ with non-singular $C(\tilde{r})$, the matrix $E(T,\tilde{r})$ is non-singular for any $T \in \mathcal{T}_h$ when $h > 0$ small enough. Notice that the right-hand sides of conditions (C.6), (C.7),(C.8),(C.11), and (C.10) are continuous linear functionals of $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$. We can conclude thus that the operator $W_{T,t}$ is well-defined for any $T \in \mathcal{T}_h$ with small enough h.

Since for any $t \in [0,1]$, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \|E(T,\tilde{r}) - C(\tilde{r})\| = 0$, and the matrix $C(\tilde{r})$ depends only on $\tilde{r}$, we can conclude that there exists $C_1 > 0$ such that when $h > 0$ small enough, then $\|(z_1, \cdots, z_{l_{\tilde{r}}})\| \leqslant C_1 \|\hat{\omega}\|_{H^1(\hat{T})}$ for any $T \in \mathcal{T}_h$. Here $\sum_{k=1}^{l_{\tilde{r}}} z_k \hat{\xi}_k = E_T \hat{\omega}$.

Finally, it is easy to see that the operator W_T will be well-defined and the inequality in the statement of this theorem holds for any $h > 0$ for affine meshes. This finishes the proof. $\square$

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