# A Language for Particle Interactions in Rule 54 and Other Cellular Automata 

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#### Abstract

This is a study of localised structures in one-dimensional cellular automata, with the elementary cellular automaton Rule 54 as a guiding example.

A formalism for particles on a periodic background is derived, applicable to all one-dimensional cellular automata. One can compute which particles collide and in how many ways. One can also compute the fate of a particle after an unlimited number of collisions - whether they only produce other particles, or the result is a growing structure that destroys the background pattern.

For Rule 54, formulas for the four most common particles are given and all two-particle collisions are found. We show that no other particles arise, which particles are stable and which can be created, provided that only two particles interact at a time. More complex behaviour of Rule 54 requires therefore multi-particle collisions.


## 1. Introduction

This article is part of a project to develop a higher-level language for the dynamical behaviour of cellular automata. In the current investigation we search for an intermediate-level description of the elementary cellular automaton Rule 54, in order to learn how to handle periodic background structures and simple particle interactions. The investigation leads to further streamlining and an extension of the existing formalism 1

The formalism is called Flexible Time. It was introduced in 18 and further developed in [20]. Flexible Time makes it possible to "calculate" with the localised structures in a cellular automaton and to determine their development over time. The structures in Flexible Time resemble the way in which a human observer views an evolution diagram of a cellular automaton (like Figure 1): by grouping the states of cells from different times and places to a single pattern in space-time.

Rule 54 is an elementary cellular automaton that was first inves-

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Figure 1. Development of a random initial configuration under Rule 54. Time runs from bottom to top.
tigated in detail by Boccara et al. [1. When evolving from random initial configurations, it develops a simple background pattern with a small number of interacting particles that move on this background. While it has not been shown to be computationally universal, it can at least evaluate Boolean expressions [10]. So it is a rather simple system (but not too simple) and therefore an ideal test object for a formalism that is still under development.

The right methods to handle large complex structures must still be found. I ask here new questions about the behaviour of Rule 54, and Flexible Time must "learn" how to handle them. As a result, there are differences and extensions of the formalism in this article that were not present in [20]. I will point them out and review them at the end.

Context Researchers on cellular automata have developed a number of concepts to describe the localised structures that arise in a cellular automaton.

The oldest named structures must be the particles (also called gliders or signals) and their collisions. This goes back at least to Zuse [23], whose cellular automaton simulates idealised physical particles. Particle-based research has continued since then, with Cook's construction of a universal computer in Rule 110 as its most spectacular result [2].

This rule, and Rule 54, belonged also to those rules in which a stable periodic background pattern occured; it was called "ether" by Cook.

For Rule 54, the starting point was the work by Boccara et al. [1] they identified the most common particles that arise from random initial configurations, described their interactions and gave them
the names that are still used. This research was later continued by the group around McIntosh [10, 11, 13], who found more complex particles and interactions.

The descriptions of these particles were mostly given by pictures and by a simple symbolism that showed which particle collides with which. But, especially to find general theorems about cellular automata, more abstract representations were developed too.

There is a more detailed investigation of particles and what they can achieve [3, 14]. For Rule 110 there is an approach for the systematic specification of initial configurations with interacting gliders [13], and to express the behaviour of the cellular automaton through a block substitution system 21.

There are also the approaches by Hordijk et al. [6] and by Martin [9, who use properties of the background and draw conclusions about the particles and particle interactions that are possible. More general, the cellular automaton is subdivided in "regular" regions and the boundaries between them [4, 5, 7, 8, 17; the boundaries move, often in an almost random fashion, and are thus a generalisation of the more straight-moving particles.

Other approaches view the evolution of the cellular automaton as two-dimensional, with one space and one time dimension. The cellular space-time is then subdivided into finite patches that represent e.g. a piece of the background or a collision between particles. The theory of cellular automata then becomes a special tiling problem. One can do this in a more informal way, like McIntosh and Martínez [12], or develop a complex formal theory around it, as Ollinger and Richard [15, 16] do it. (This approach is closest to the work described here.)

Overview After an introductory section about cellular automata and Rule 54, Section 3 recapitulates the work in [20], as far as it is relevant for the present work. At its end, a representation of Rule 54 as a "reaction system" (defined below) is shown, the same that was derived in [20]. In Section 4 we then find a way to compress this and similar systems, and we use the compressed reaction system to understand the local behaviour of Rule 54 better. Section 5 then turns to larger patterns and describes the triangular structures in Rule 54 and the stable background pattern that is formed by them. Then, in Section 6 , the four kinds of particles found by Boccara et al. [1] are represented in Flexible Time, together with the collision between the particles. A summary follows in Section 7

## 2. Cellular automata and Rule 54

## - 2.1 Elementary cellular automata

Rule 54 is an one-dimensional cellular automaton, more specifically an elementary cellular automaton. This kind of cellular automata was made popular by Stephen Wolfram [22], who also introduced the system of code numbers from which Rule 54 got its name.

One-dimensional cellular automata are dynamical systems with discrete time. The state of such an automaton is called a configuration. It consists of an infinite sequence of simpler objects, the cells. The state of each cell is an element of a finite set $\Sigma$; the configuration at time $t$ is therefore a function $c_{t}: \mathbb{Z} \rightarrow \Sigma$. We write $\Sigma^{\mathbb{Z}}$ for the set of configurations; $c_{t}(x)$ is then the state of the cell at position $x$ at time $t$.

The evolution of the automaton is then a sequence $\left(c_{0}, c_{1}, c_{2}, \ldots\right)$ of configurations that follow a common rule that is described below in (2). While the sequence here starts at time 0 , we will also accept other starting times.)

An elementary cellular automaton is a one-dimensional cellular automaton with two states and a three-cell neighbourhood. The set of states is $\Sigma=\{0,1\}$, and its behaviour is given by its local transition rule

$$
\begin{equation*}
\varphi: \Sigma^{3} \rightarrow \Sigma \tag{1}
\end{equation*}
$$

This is the function with which the configuration $c_{t+1}$ is computed from its predecessor $c_{t}$. To do this, we apply $\varphi$ to every three-cell neighbourhood of $c_{t}$, and the result is the next state of the middle cell:

$$
\begin{equation*}
c_{t+1}(x)=\varphi\left(c_{t}(x-1), c_{t}(x), c_{t}(x+1)\right) \quad \text { for all } t, x \in \mathbb{Z} \tag{2}
\end{equation*}
$$

The function $\varphi$ defines then a global transition rule $\varphi_{\text {global }}$ : It is the function that maps the configuration $c_{t}$ to its successor $c_{t+1}$ according to (2).

The transition rule (22) is also called a rule of radius 1 , because only the $c_{t}(y)$ with $|x-y| \leq 1$ contribute to $c_{t+1}(x)$. Rules with other radii are defined similarly.

## - 2.2 Rule 54

Rule 54 has a left-right symmetric transition rule,

$$
\varphi(s)= \begin{cases}1 & \text { for } s \in\{(0,0,1),(1,0,0),(0,1,0),(1,0,1)\}  \tag{3}\\ 0 & \text { otherwise }\end{cases}
$$

The rule is easier to remember in form of the following slogan [20],
$" \varphi(s)=1$ if $s$ contains at least one 1 , except if the cells in state 1 touch."

Here we say that two cells "touch" if they are direct neighbours. Thus the two cells in state 1 touch in the neighbourhood $(1,1,0)$, but not in the neighbourhood $(1,0,1)$.


Figure 2. Rule icon for Rule 54.
Figure 2 shows how the neighbourhoods influence the next state of the central cell. White squares are in state 0 , black squares are in state 1 , and the time runs upwards. This is also our convention in the other diagrams, even if white and black may also become dark and bright grey in the parts of the diagram that are less emphasised.

## 3. Flexible Time

## I 3.1 Situations

We need a means to describe and understand the interactions of gliders and other patterns under Rule 54. Flexible Time was developed in [20] for this task. The motivation was that it is easier to find patterns in the evolution of cellular automata if one works with structures that involve the states of cells at different times. These structures are called here situations.

They generalise the finite sequences of cells that are part of the configurations $c_{t}$ described above. In order to express e.g. that $c_{t}(0)=$ $c_{t}(1)=0$ and $c_{t}(2)=1$, one would often write that the subsequence of $c_{t}$ that begins at cell position 0 is 001 . Situations generalise this notation. They may extend not only over space but also over time. To write them, we use additional symbols that express a jump in spacetime.

Under Rule 54, situations are written as sequences of the symbols $0,1, \ominus_{i}$ and $\oplus_{i}$, for $i \in\{1,2\}$. The intended interpretation can most easily be described in terms of instructions to write symbols on squares in a grid. The squares are labelled by pairs $(t, x) \in \mathbb{Z}^{2} ; x$ is the position of a cell and $t$ a time step in its evolution. The writing rules are:

- Start reading at the first symbol. For writing, place the cursor at square $(0,0)$ of the grid.
- If the cursor is at $(t, x)$ and the current symbol is
- an element of $\Sigma$, write it down and move the cursor one square to the right, to $(t, x+1)$,
- $\ominus_{i}$, move the cursor to $(t-1, x-i)$,
- $\oplus_{i}$, move the cursor to $(t+1, x-i)$.

Then continue with the next symbol.

- No overwriting: One cannot write different symbols on the same square.

To get an example for such a writing process, let us set for a moment $\Sigma=\{0,1,2,3\}$ and look at the situation $01 \oplus_{1} 23$. First, the cell states 0 and 1 are written to the squares $(0,0)$ and $(0,1)$. The cursor is then at square $(0,2)$. Now the symbol $\oplus_{1}$ moves the cursor to $(1,1)$. The following symbols 2 and 3 are then written to the squares $(1,1)$ and $(1,2)$, leaving the cursor at $(1,3)$. The result is then the following grid:

| $t=1$ |  |  |  | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |

The horizontal rules mark the beginning and end of the symbol sequence, or, more exactly, the squares left of the starting point and right of the end point of the state sequence. Similar lines will later appear in the illustrations.

Now we need to express this construction in a mathematical form. We will use two-dimensional coordinates and call a coordinate pair $(t, x) \in \mathbb{Z}^{2}$ a space-time point. A pair $(p, \sigma) \in \mathbb{Z}^{2} \times \Sigma$ is a cellular event. The event $((t, x), \sigma)$ provides the information "At time $t$, the cell at position $x$ is in state $\sigma$ ". We will usually write them $[t, x] \sigma$ or $[p] \sigma$ for better readability. A situation is then a sequence of cellular events, together with the final cursor position: $s=\left(\left(\left[p_{0}\right] \sigma_{0}, \ldots,\left[p_{n-1}\right] \sigma_{n-1}\right), p_{n}\right)$. For the final cursor position of $s$ we write $\delta(s)$, the size of $s$. This means that we have in our example $\delta(s)=p_{n}$.

In a situation, the sequence of the cellular events is significant, and the size too, since they make algebraic operations possible. In many cases, we want to ignore however this information: Then we will use the cellular process that belongs to a situation; it is simply the set of its cellular events. The cellular process of a situation $s$ is written $\operatorname{pr}(s)$. In our example, with $s=01 \oplus_{1} 23$, we have therefore

$$
s=(([0,0] 0,[0,1] 1,[1,1] 2,[1,2] 3),(1,3)) .
$$

This means that $\delta(s)=(1,3)$ and $\operatorname{pr}(s)=\{[0,0] 0,[0,1] 1,[1,1] 2,[1,2] 3\}$.
Usually we will not need this explicit form, since situations are meant to make this unnecessary. It helps us however to explain the "no overwriting" rule above. This rule concerns expressions like $01 \oplus_{1} 2 \ominus_{1} 3$, where the cursor reaches the same point twice. If it were a situation, its cellular process would be $\{[0,0] 0,[0,1] 1,[1,1] 2,[0,1] 3\}$. This would provide contradicting information about the space-time point $(0,1)$ : At time 0 , is the cell at position 1 in state 0 or 3 ? The overwriting rule prevents this problem.

The most important algebraic property of situations is that they can be multiplied. The product of $s_{1}$ and $s_{2}$ is found by first writing $s_{1}$ and then, with the cursor at $\delta\left(s_{1}\right)$, writing $s_{2}$. The resulting product is written $s_{1} s_{2}$, but due to the overwrite rule, it may not always exist.

More complex terms of situations are defined in the usual way: $s^{2}$ is the result of writing $s$ twice, and so on. The Kleene closure of a situation $s$ is the set

$$
\begin{equation*}
s^{*}=\left\{s^{k}: k \geq 0\right\} \tag{4}
\end{equation*}
$$

The Kleene closure always contains the empty situation, which is written [0].

In Flexible Time, situations are used to express the evolution of a cellular automaton. But in order to understand how this is done, we first have to look at the way in which the evolution of a cellular automaton is expressed by cellular processes.

## | 3.2 Evolution Expressed with Cellular Processes

In a similar way to that in which a configuration $c_{0} \in \Sigma^{\mathbb{Z}}$ can be the starting point of an evolution $\left(c_{0}, c_{1}, c_{2}, \ldots\right)$, a cellular process $\pi$ can be extended to a larger process $\mathrm{cl} \pi$, its closure. Figure 3 shows how this is


Figure 3. A process and its closure.
meant for the initial configuration $\pi=\operatorname{pr}\left(10^{13} 1\right)$. The cellular events of the original process $\pi$ are displayed in black and white; together with the the events in grey they form the process $\mathrm{cl} \pi$. Each horizontal row in the diagram contains the events that belong to a specific time step. We see that the diagram becomes smaller at the top; this means that as time progresses, fewer cell states can be deduced from the information given by the initial process $\pi$.

To motivate the exact definition of the closure, we first express the configurations of the cellular automaton and their evolution in terms of cellular processes. This will then allow us to generalise the definition of evolution to processes that do not correspond to configurations.

Let now $c$ be the configuration of a cellular automaton. We define the embedding of $c$ at time $t$ to be the process

$$
\begin{equation*}
\eta_{t}(c)=\{[t, x] c(x): x \in \mathbb{Z}\} . \tag{5}
\end{equation*}
$$

A kind of inverse of the function $\eta_{t}$ is the concept of time slices. The time slice at time $t$ of a process $\pi$ is the process

$$
\begin{equation*}
\pi^{(t)}=\{[t, x] \sigma: x \in \mathbb{Z}\} \tag{6}
\end{equation*}
$$

The time slice is a process and not a configuration because $\pi^{(t)}$ must exist for all processes, not just for embeddings of configurations.

With these concepts, the cellular process that belongs to the evolution sequence $\left(c_{o}, c_{1}, c_{2} \ldots\right)$ is $\gamma=\bigcup_{t \geq 0} \eta_{t}\left(c_{t}\right)$. It has the time slices $\gamma^{(t)}=\eta_{t}\left(c_{t}\right)$, which represent the configurations $c_{t}$. The process $\gamma$ must then be the closure of $\eta_{0}\left(c_{0}\right)$.

A time slice $\pi^{(t)}$ of an arbitrary process is then understood as partial knowledge about the state of a cellular automaton at time $t$. In order to determine the state of the automaton at time $t+1$, we take all configurations that are compatible with this knowledge, evolve them for one time step and accept only the states of those cells about which all configurations agree. The result is the cellular process

$$
\begin{equation*}
\Delta_{t}(\pi)=\bigcap\left\{\eta_{t+1}\left(\varphi_{\text {global }}(c)\right): \eta_{t}(c) \supseteq \pi^{(t)}\right\} \tag{7}
\end{equation*}
$$

of those events that are determined by $\pi(t)$. The cellular events of which it consists all belong to time $t+1$.

We can now easily check that the process $\gamma$ has the property that $\gamma^{(t)}=\Delta_{t}(\gamma)$ for all $t>0$. Every time slice, except the first, can be computed from the previous one. Only $\gamma^{(0)}$, which represents the initial configuration, must still be handled separately.

This inconvenience in resolved in the full definition of the closure. In it, the initial process no longer needs to be the embedding of a configuration. This is possible because it is now split into time slices and then added piece-wise to the partial results of the computation.

Definition 1 (Closure [20, Def. 3.10]) Let $\pi$ be a cellular process for which there is a time $t_{0} \in \mathbb{Z}$ such that $\pi^{(t)}=\emptyset$ for all $t<t_{0}$.

If there is a process $\gamma$ with the property that

$$
\gamma^{(t)}= \begin{cases}\Delta_{t}(\gamma) \cup \pi^{(t)} & \text { for } t \geq t_{0}  \tag{8}\\ \emptyset & \text { for } t<t_{0}\end{cases}
$$

then we write $\gamma=\mathrm{cl} \pi$ and say that it is the closure of $\pi$.
It is easy to see that the choice of $t_{0}$ has no influence on $\mathrm{cl} \pi$.
We can now see that the set $\gamma$ that was defined above satisfies (8) if we set $t_{0}=0$ and $\pi=\eta_{0}\left(c_{0}\right)$ : Then we have $\gamma^{(t)}=\emptyset$ for $t<0$, $\gamma^{(0)}=\eta_{0}\left(c_{0}\right)$, and $\gamma^{(t)}=\Delta_{t}(\gamma)$ for $t>0$, and indeed $\gamma=\operatorname{cl} \eta_{0}\left(c_{0}\right)$. Definition 1 is thus a generalisation of the transition rule 2 to cellular processes.

Not to all cellular processes, however. One of the requirements of Definition 1 is that $\gamma$ must be a process, and this can easily be broken. All we need is conflicting information in $\Delta_{t}(\gamma)$ and $\pi^{(t)}$ : If there is a time step $t$ at which there is an event $[t, x] \sigma \in \Delta_{t}(\gamma)$ and another event

[^1]$[t, x] \tau \in \gamma^{(t)}$ with $\sigma \neq \tau$, then $\gamma^{(t)}$ is no cellular process, and neither is $\gamma$.

We will however introduce in the next section a class of situations whose cellular processes all have a closure: They will then be used to describe the evolution of cellular automata in an economical way.

## I 3.3 Reactions

The evolution of a cellular automaton is represented in Flexible Time by reactions. We will say that there is a reaction between two situations $a$ and $b$ if the situation $b$ consists only of events that are determined by of the events of $a$. They belong to the future of $a$, so to speak.


Figure 4. A reaction under Rule 54.
Figure 4 shows a reaction. On the left side we see the process of the situation $a=10^{13} 1$, together with its closure. As in Figure 3, the events of $\operatorname{pr}(a)$ are highlighted while the remaining cells of the closure are displayed in grey. On the right side we see the same closure, but with different events highlighted. This time they belong to the situation $b=(10 \oplus)^{7} 1(\ominus 01)^{7}$. With these diagrams we therefore see that the events of the process $b$ are determined by the process $a$.

The formal definition of reactions is then:
Definition 2 (Reactions [20, Def. 4.8]) Let $a$ and $b$ be two situations with

$$
\begin{equation*}
\operatorname{cl} \operatorname{pr}(a) \supseteq \operatorname{pr}(b) \quad \text { and } \quad \delta(a)=\delta(b) \tag{9}
\end{equation*}
$$

Then the pair $(a, b)$ is the reaction from $a$ to $b$. It is usually written $a \rightarrow b$.

We will use the expression $a \rightarrow b$ also as a proposition, meaning that there is a reaction from $a$ to $b$. The symbol " $\rightarrow$ " then specifies a relation, and as it is normal for relations, we can also write longer chains of reactions, like $a \rightarrow b \rightarrow c$. One can verify that if such a chain exists, then there is also a reaction $a \rightarrow c$.

Reactions are useful because they can be applied to situations. It can be shown [20, Th. 4.11] that if there are situations $x, y$ and $a$ for which $\mathrm{cl} \operatorname{pr}(x a y)$ exists and if there is a reaction $a \rightarrow b$, then there is also a reaction xay $\rightarrow x b y$. This reaction is then called the application of $a \rightarrow b$ to xay.

Table 1. The local reaction system for Rule 54, long form.
Generating Slopes:

| Reactions: | $\ominus_{1} 000 \rightarrow 0 \ominus_{1} 00$ | $000 \oplus_{1} \rightarrow 00 \oplus_{1} 0$ |
| :---: | :---: | :---: |
|  | $\ominus_{1} 001 \rightarrow 1 \ominus_{1} 01$ | $100 \oplus_{1} \rightarrow 10 \oplus_{1} 1$ |
|  | $1 \ominus_{1} 010 \rightarrow 111 \ominus_{2} 10$ | $010 \oplus_{1} 1 \rightarrow 01 \oplus_{2} 111$ |
|  | $1 \ominus_{1} 011 \rightarrow 100 \ominus_{2} 11$ | $110 \oplus_{1} 1 \rightarrow 11 \oplus_{2} 001$ |
|  | $1 \ominus_{2} 100 \rightarrow 1 \ominus_{1} 00$ | $001 \oplus_{2} 1 \rightarrow 00 \oplus_{1} 1$ |
|  | $1 \ominus_{2} 101 \rightarrow 1 \ominus_{1} 01$ | $101 \oplus_{2} 1 \rightarrow 10 \oplus_{1} 1$ |
|  | $00 \ominus_{2} 110 \rightarrow 001 \ominus_{2} 10$ | $011 \oplus_{2} 00 \rightarrow 01 \oplus_{2} 100$ |
|  | $00 \ominus_{2} 111 \rightarrow 000 \ominus_{2} 11$ | $111 \oplus_{2} 00 \rightarrow 11 \oplus_{2} 000$ |
|  | $00 \rightarrow 00 \oplus_{1} \ominus_{1} 00$ | $\ominus_{1} 00 \oplus_{1} \rightarrow[0]$ |
|  | $01 \rightarrow 01 \oplus_{2} 1 \ominus_{1} 01$ | $1 \ominus_{1} 01 \oplus_{2} 1 \rightarrow 1$ |
|  | $10 \rightarrow 10 \oplus_{1} 1 \ominus_{2} 10$ | $1 \ominus_{2} 10 \oplus_{1} 1 \rightarrow 1$ |
|  | $11 \rightarrow 11 \oplus_{2} 00 \ominus_{2} 11$ | $00 \ominus_{2} 11 \oplus_{2} 00 \rightarrow 00$ |

Now it is possible that there is also a reaction that can be applied to xby. We would then have a reaction $b^{\prime} \rightarrow c$ and two processes $x^{\prime}$ and $y^{\prime}$ such that $x b y=x^{\prime} b^{\prime} y^{\prime} \rightarrow x^{\prime} c y^{\prime}$ and therefore, by transitivity, also a reaction $x a y \rightarrow x^{\prime} c y^{\prime}$. This way application allows one to specify a large set of reactions by a small set of "generator reactions", provided only that there is a large enough set of situations to which they can be applied.

The result is a reaction system. It is the foundation of all calculations in Flexible Time.

Definition 3 (Reaction System [20, Def. 4.13]) Let $D$ be a set of situations and $R$ a set of reactions between them. We say that $R$ is a reaction system with domain $D$ if the following is true:

1. If $a \in D$, then $a \rightarrow a$ is in $R$.
2. If $a \rightarrow b$ and $b \rightarrow c$ are in $R$, then $a \rightarrow c$ is in $R$.
3. $R$ is closed under application of reactions to the situations in $D$.

We will now define a reaction system by specifying $D$ and a set $G \subseteq R$ of generators; it is then extended by repeated application and concatenation of reactions, as described above. The system describes Rule 54; its derivation is described in detail in Chapters 6 and 7 of [20].

The reaction system is summarised in Table 1. The top of the table, entitled "Generating Slopes", specifies the domain $D$ of $\Phi$. More specifically, it lists the neighbourhoods that a $\ominus$ or $\oplus$ operator may have if it is is part of a situation $s \in D$. The first entry, $\ominus_{1} 00$, specifies that a $\ominus_{1}$ may occur in $s$ at the left of the term 00 , the second entry

[^2]$1 \ominus_{1} 01$, that it may occur between a 1 (at its left) and a 01 (at its right). No other possibilities exist since the remaining entries refer to other operators. One can prove [20, Theorem 6.10] that all situations in $D$ have a closure.

The bottom of Table 1 contains the generating reactions of $\Phi$. Its upper part (i.e. the middle of the whole table) contains the reactions that involve a single $\ominus$ or $\oplus$ operator. If we had only them, no reaction could have an element of $\Sigma^{*}$ at its left side: Therefore we have at the bottom left of the table a set of reactions that create a $\ominus$ and a $\oplus$ operator from an element of $\Sigma^{*}$. Their converses are listed at the bottom right: reactions that destroy a $\ominus$ and a $\oplus$ operator. All reactions of $\Phi$ are the results of repeated applications of these four types of generators.

The arrangement of the reactions in Table 1 has also another purpose. It allows one to read off two important subsystems of $\Phi$.

Definition 4 (Slopes) Let $R$ be a reaction system with domain $D$.
The system $R_{+}$(with domain $D_{+}$) of positive slopes consists of the situations of $D$ that only contain $\oplus$ operators and the reactions between these situations.

The system $R_{-}$(with domain $D_{-}$) of negative slopes consists of the situations of $D$ that only contain $\ominus$ operators and the reactions between these situations.

In case of Rule 54, we can find the generators of $\Phi_{\text {_ }}$ if we take only the generating slopes at the right and the generator reactions at the top right of the middle section in Table 1. Similarly, $\Phi_{+}$is represented by the slopes and reactions at the top right of the table.

Details of the reaction system We will now have a closer look at the way in which the reaction system $\Phi$ represents Rule 54.

We begin with the slopes. Figure 5 displays the generating slopes for $\Phi$, first the negative slopes and then their mirror images, the positive slopes.


Figure 5. Generating slopes.

In this and in later diagrams, the endpoints of the situations are marked by horizontal lines. They represent the places where the surrounding events would be expected if the slopes were parts of larger situations. Or, in the interpretation of Section 3.1, the square at which the left horizontal line ends is always one point left of the coordinate origin, while the right horizontal line always begins at $\delta(s)$. The beginning of the situation is also marked by the small vertical bar, which is located at the left boundary of the square at the coordinate origin.

An important property of the generating slopes is that they trace the boundaries of the closure. We can see in Figure 6 what this means.


Figure 6. Generating slopes as boundaries of the closure.
It shows a situation, 110101000, together with two generations of its closure. We see at its left the slope $00 \oplus_{2} 11$ (the mirror image of $11 \ominus_{2} 00$ in Figure 5), and at its right, the term $\ominus_{1}$, both in bolder colours. Note that the situation $\ominus_{1} 00$ reaches over two time steps and has its starting point directly at the right end of the second time step of the closure. This is the way the slope terms trace the boundary of a closure.

The generator reactions of $\Phi_{-}$are designed with the goal that the reaction result consists of events near the right boundary of the closure of the initial situation. (For $\Phi_{+}$it is similar, with left and right exchanged.) How this is done is shown in Figures 7 and 8. They contain reactions of the form $a \rightarrow b$ and display $\operatorname{pr}(a)$ and $\operatorname{pr}(b)$ in relation to the closure of $\operatorname{pr}(a)$. Figure 7 shows the generator reactions of $\Phi_{-}$. In it, we see that the process of $b$ is always located more to the right than $\operatorname{pr}(a)$ and that it touches the right boundary of $\operatorname{cl} \operatorname{pr}(a)$. The reactions involve only two time steps, and one of the $\ominus$ operators must always be present. To get the system started from situations in $\Sigma^{*}$, we need the reactions at the right side of Figure 8. Here we see reactions in which $\operatorname{pr}(b)$ completely fills the closure of $\operatorname{pr}(a)$ and $b$ is a situation with both $\mathrm{a} \oplus$ and $\mathrm{a} \ominus$ operator.

The converses of the reactions at the left side of Figure 8 are shown at its right side: Reactions $a \rightarrow b$ in which $a$ contains one $\ominus$ and one $\oplus$, while $b$ contains none. We can use them for cleanup, since they remove pairs of neighbouring $\ominus$ and $\oplus$ operators. The same manoeuvre is also possible in all other cases where $\mathrm{a} \ominus$ is left of $\mathrm{a} \oplus$, and we get a result that for every situation $a$ there is a reaction $a \rightarrow b_{+} b_{-}$with $b_{+} \in D_{+}$ and $b_{-} \in D_{-}$. If we start from $a$ and continue to apply the generator reactions as long as possible, we can even enforce that $b_{+}$and $b_{-}$trace the boundaries of $\mathrm{cl} \operatorname{pr}(a)$.

$1 \ominus_{1} 010 \rightarrow 111 \ominus_{2} 10$

$1 \ominus_{2} 100 \rightarrow 1 \ominus_{1} 00$

$00 \ominus_{1} 110 \rightarrow 001 \ominus_{2} 10$

$1 \ominus_{1} 011 \rightarrow 100 \ominus_{2} 11$

$1 \ominus_{2} 101 \rightarrow 1 \ominus_{1} 01$

$00 \ominus_{2} 111 \rightarrow 000 \ominus_{2} 11$

Figure 7. Reactions of $\Phi_{-}$as motion towards the boundaries of the closure.

This was a summary of the content of [20] as far as it concerns Rule 54.

## 4. Understanding the Reaction System

Up to now, the representation of Rule 54 in Table 1 looks complex and does not provide much insight. This makes it difficult to do calculations about Rule 54 without always looking at the table. We will therefore develop a more compact representation of the reaction system. The goal is to find "slogans" for it that are easy to remember, analogous to the slogan for $\varphi$ on page 4

## - 4.1 A simpler Rule Table

As a first simplification, we omit the indices from the $\oplus$ and $\ominus$ operators. This is possible because the indices of the operators are always determined by the environment. We can see from the list of generating slopes in Table 1 that if $\ominus_{i}$ is followed by a 0 , then always $i=1$, and if it is followed by a 1 , then $i=2$. A similar law is valid for $\oplus_{i}$, and we can recover the indices of $\ominus$ and $\oplus$ from the equations

$$
\begin{equation*}
\ominus 0=\ominus_{1} 0, \quad \ominus 1=\ominus_{2} 1, \quad 0 \oplus=0 \oplus_{1}, \quad 1 \oplus=1 \oplus_{2} \tag{10}
\end{equation*}
$$

This kind of abbreviation is possible in every reaction system, because in a generating slope $u \ominus_{i} v$, the term $u \ominus_{i}$ is completely determined by $v$.


Figure 8. Reactions that generate and destroy slopes. The generator reactions are shown at the left, the destructors, right.

For the same reason we can shorten the generator reactions by removing common factors from their left and right sides. The generator reactions of $\Phi_{-}$all have the form $u \ominus v \sigma \rightarrow u x \ominus v^{\prime}$, with a generating slope $u \ominus v$. When such a reaction is applied to a situation $s$, there must be always a factor $u$ to the left of $\ominus v$ in $s$. Therefore we can shorten these generator reactions to the form $\ominus v \sigma \rightarrow x \ominus v^{\prime}$ and do not get new reactions when the shortened reactions are applied.

We then get four pairs of reactions as generators for $\Phi_{-}$:

$$
\begin{array}{ll}
\ominus 000 \rightarrow 0 \ominus 00, & \ominus 010 \rightarrow 11 \ominus 10, \\
\ominus 001 \rightarrow 0 \ominus 01, & \ominus 011 \rightarrow 00 \ominus 11, \\
\ominus 100 \rightarrow \ominus 00, & \ominus 110 \rightarrow 1 \ominus 00, \\
\ominus 101 \rightarrow \ominus 01, & \ominus 111 \rightarrow 0 \ominus 01 . \tag{11b}
\end{array}
$$

They can be compressed further with the help of a new notation. For a cell state $\sigma \in \Sigma$ we will write $\bar{\sigma}$ for the complementary state, such that $\overline{0}=1$ and $\overline{1}=0$. Then we can write the following reactions, valid

Table 2. The local reaction system for Rule 54, short form.

| Generating Slopes$\begin{aligned} & G_{-}=\{\ominus 00,1 \ominus 01,1 \ominus 10,00 \ominus 11\} \\ & G_{+}=\{00 \oplus, 01 \oplus 1,01 \oplus 1,11 \oplus 00\} \end{aligned}$ |  |
| :---: | :---: |
| Reactions |  |
| $\ominus 00 \sigma \rightarrow \sigma \ominus 0 \sigma$ | $\sigma 00 \oplus \rightarrow \sigma 0 \oplus$ |
| $\ominus 10 \rightarrow \ominus 0$ | $01 \oplus \rightarrow 0 \oplus$ |
| $\ominus 01 \sigma \rightarrow \bar{\sigma} \bar{\sigma} \ominus 1 \sigma$ | $\sigma 10 \oplus \rightarrow \sigma 1 \oplus \bar{\sigma} \bar{\sigma}$ |
| $\ominus 11 \sigma \rightarrow \bar{\sigma} \ominus 1 \sigma$ | $\sigma 11 \oplus \rightarrow \sigma 1 \oplus \bar{\sigma}$ |
| $u \ominus v \oplus u \rightarrow u$ |  |
| $v \rightarrow v \oplus u \ominus v$ | for $u \ominus v \in G_{-}$ |
| Abbreviations |  |
| $\ominus 0=\ominus_{1} 0$ | $0 \oplus=0 \oplus_{1}$ |
| $\ominus 1=\ominus_{2} 1$ | $1 \oplus=1 \oplus_{2}$ |

for all $\sigma{ }^{2}$

$$
\begin{align*}
\ominus 00 \sigma \rightarrow \sigma \ominus 0 \sigma, & \ominus 01 \sigma \rightarrow \bar{\sigma} \bar{\sigma} \ominus 1 \sigma  \tag{12a}\\
\ominus 10 \rightarrow \ominus 0, & \ominus 11 \sigma \rightarrow \bar{\sigma} \ominus 1 \sigma \tag{12b}
\end{align*}
$$

Written in this form we will analyse the reaction system and show what the generator reactions actually mean. But before we can do this, we must see how to simplify the rest of Table 1 .

The reactions at the bottom of the table can be brought easily to a common form, when we define the set $G_{-}=\{\ominus 00,1 \ominus 01,1 \ominus 10,00 \ominus 11\}$ of negative generating slopes. With this name at hand, we can see that the bottom reactions have the common form

$$
\begin{equation*}
v \rightarrow v \oplus u \ominus v \quad u \ominus v \oplus u \rightarrow u \tag{13}
\end{equation*}
$$

whenever $u, v \in \Sigma^{*}$ and $u \ominus v \in G_{-}$. This then completes the compression of Table 1. The result is Table 2,

Relation to the Transition Rule In order to understand this new form of the reaction system and to see how it is related to the transition rule $\varphi$, we write the reactions of $\Phi_{-}$in the following manner:

[^3]|  | $\tau_{0}$ | $\tau_{1}$ | $\tau_{2}$ |
| :--- | :--- | :--- | :--- |
| $\ominus_{1} 00 \sigma \rightarrow \tau_{0} \ominus_{1} 0 \sigma$ | $\varphi(0,0, \sigma)=\sigma$ | $\varphi(0, \sigma, \cdot) \uparrow$ |  |
| $\ominus_{2} 10 \rightarrow \ominus_{1} 0$ | $\varphi(0, \sigma, \cdot) \uparrow$ |  |  |
| $\ominus_{1} 01 \sigma \rightarrow \tau_{0} \tau_{1} \ominus_{2} 1 \sigma$ | $\varphi(0,1, \sigma)=\bar{\sigma}$ | $\varphi(1, \sigma, \cdot)=\bar{\sigma}$ | $\varphi(\sigma, \cdot, \cdot) \uparrow$ |
| $\ominus_{2} 11 \sigma \rightarrow \tau_{0} \ominus_{2} 1 \sigma$ | $\varphi(1, \sigma, \cdot)=\bar{\sigma}$ | $\varphi(\sigma, \cdot \cdot \cdot) \uparrow$ |  |

In the reactions at the leftmost column of the table, each variable $\tau_{i}$ stands for the state of the cell at position $(0, i)$. The other columns then show for each $\tau_{i}$ the computation that determines its value - or, if it cannot be computed, which application of $\varphi$ fails to have a determined value.

We can see e.g. in the first row that the state of the cell at $(0,0)$ can be computed from the information presented in the initial situation $\ominus_{1} 00 \sigma$. The cellular process of this situation consists of the events $[-1,-1] 0,[-1,0] 0$, and $[-1,1] \sigma$, and therefore the state $\tau_{0}$ of the cell at $(0,0)$ must be $\varphi(0,0, \sigma)$.

In the same way we can see that in the third row, $\tau_{0}$ is $\varphi(0,1, \sigma)$. The diagram contains however also entries for which not all arguments of $\varphi$ are known. The missing arguments are marked by a dot. When the value of $\varphi$ is independent of the missing argument, it is entered in the table, otherwise the entry is marked with an arrow.

We can see that the values of the $\tau_{i}$ only depend on three equations,

$$
\begin{equation*}
\varphi(0,0, \sigma)=\sigma, \quad \varphi(0,1, \sigma)=\bar{\sigma}, \quad \varphi(1, \sigma, \cdot)=\bar{\sigma} \tag{14}
\end{equation*}
$$

They all can be derived from the rule that a pair of touching 1's cause a $\varphi$ value of 0 , while one or more isolated 1's make the value equal to 1 . In the case of $\varphi(0,0, \sigma)$, a pair of touching 1 's cannot occur, therefore the value of $\varphi$ is one if and only if $\sigma=1$. In the other two cases, $\sigma=1$ creates a touching pair and $\sigma=0$ inhibits it, therefore the function value is $\bar{\sigma}$. In a similar way we can see that in the remaining entries of the table, the value of $\varphi$ is undefined. This is how $\varphi$ influences the reactions in $\Phi$.

In the table, the $\ominus$ have been written once again with indices-not just to ease the translation from situations to cellular processes, but also because with them we can see how many new events are generated in the reactions. One can thus see that in the first reaction one new event is generated because $\delta\left(\ominus_{1} 00 \sigma\right)$ must be equal to $\delta\left(\tau_{0} \ominus_{1} 0 \sigma\right)$, and so on. If the left side of a reaction has a $\ominus_{i}$ operator and the right side a $\ominus_{j}$, then $j-i$ new cell states must be generated in the reaction.

Slogans These considerations may help to understand the reactions of the system $\Phi$ a bit better. To help memorising them, I will introduce two slogans. Both refer to the left side of the reactions of $\Phi_{-}$. This side can always be written as $\ominus \alpha \beta \sigma$, with $\alpha, \beta, \sigma \in \Sigma$. The first slogan
tells in which cases the value of $\alpha \beta$ makes the reaction product longer or shorter than the initial term:
"01 causes growth, 10 shrinking, everything else no change."
The second slogan describes the influence of $\beta \sigma$ on the newly generated cell states. They can be either be a copy $(\sigma)$ or the inversion $(\bar{\sigma})$ of the variable $\sigma$, and the rule is:
" $0 \sigma$ copies and $1 \sigma$ inverts."

## 5. Triangles and Ether

In the rest of this article we will describe the behaviour of larger systems of cells under Rule 54. We want to describe the interaction of particles that move on a periodic background, the so-called ether. So we will now introduce, as a first step, reactions for the ether. Since it has been done already to some extent in [20, Ch. 8], we will do it here in a shorter form and from a higher point of view.

The first tool that we will use are reaction families, which allow to represent many similar reactions in a single formula. Reaction families appeared already in [20], but here we use a more streamlined notation.

Definition 5 (Reaction Families) If there is a reaction $a_{k} \rightarrow b_{k}$ for every $k \geq 0$, we will write this as

$$
\begin{equation*}
\left(a_{k} \rightarrow b_{k}\right)_{k} \tag{15}
\end{equation*}
$$

The notation will be extended in the usual way to expressions like $\left(a_{k} \rightarrow b_{k}\right)_{k \geq N}$ or $\left(a_{j, k} \rightarrow b_{j, k}\right)_{j, k}$. We will also speak of $\left(a_{k}\right)_{k}$ as a situation family.

## - 5.1 Triangle Reactions

We will first find general formulas for reactions that represent triangular structures like that in Figure 4 .

There are two general laws that we will use here. The first one makes it possible to iterate a reaction of a special form. This can be done in two ways,

$$
\begin{align*}
& \text { if } \quad a x \rightarrow y a, ~ t h e n ~  \tag{16a}\\
& \text { if } \left.\quad x a \rightarrow a x^{k} \rightarrow y^{k} b\right)_{k},  \tag{16b}\\
& \text { inen } \quad\left(x^{k} a \rightarrow b y^{k}\right)_{k} .
\end{align*}
$$

The second law "iterates" a specific reaction family; in it, $n$ is a constant:

$$
\begin{equation*}
\text { if } \quad\left(a_{k+n} \rightarrow x a_{k} y\right)_{k}, \quad \text { then } \quad\left(a_{k n+i} \rightarrow x^{k} a_{i} y^{k}\right)_{i, k} \tag{17}
\end{equation*}
$$

Both laws can easily be proved by induction [20, Ch. 8.1].
We now search for cases in which the first law can be applied and in which the left side is a generator reaction. There are two candidates, $\ominus 000 \rightarrow 0 \ominus 00$ and $\ominus 111 \rightarrow 0 \ominus 11$. The first one has $a=\ominus 00$ and $x=y=0$ and leads to

$$
\begin{equation*}
\left(\ominus 0^{k+2} \rightarrow 0^{k} \ominus 00\right)_{k} \tag{18a}
\end{equation*}
$$

while the second reaction has $a=\ominus 11, x=1$ and $y=0$ and leads to

$$
\begin{equation*}
\left(\ominus 1^{k+2} \rightarrow 0^{k} \ominus 11\right)_{k} \tag{18b}
\end{equation*}
$$

Family 18a is the more interesting one. It becomes the core of another reaction family,

$$
\begin{equation*}
\left(10^{k+2} 1 \rightarrow 10 \oplus 10^{k} 1\right)_{k} \tag{19}
\end{equation*}
$$

whose derivation I will show here in detail, as an example for calculation with reactions:

$$
\begin{aligned}
\underline{1000^{k} 1} & \rightarrow 10 \oplus 1 \underline{\ominus 1000^{k} 1} \\
& \rightarrow 10 \oplus 1 \underline{\ominus 000^{k}} 1 \\
& \rightarrow 10 \oplus 10^{k} \underline{\ominus 001} \rightarrow 10 \oplus 10^{k} 1 \ominus 01
\end{aligned}
$$

Parts of the situations are underlined; they are the places that change in the next reaction step. We will use this notation in later calculation without special notice.

Reaction family (19) can now be iterated by rule (17), with $a_{k}=$ $10^{k} 1$ and $n=2$. The result is

$$
\begin{equation*}
\left(10^{2 k+i} 1 \rightarrow(10 \oplus)^{k} 10^{i} 1(\ominus 01)^{k}\right)_{i, k} \tag{20}
\end{equation*}
$$

In families like these, the cases with $i<2$ are the most important ones, since the reactions in $\sqrt{19}$ have been applied in them for the highest number of times. For $i=1$, we can add one more step, since we have $\underline{101} \rightarrow 10 \oplus 1 \underline{\ominus 101} \rightarrow 10 \oplus 1 \ominus 01$. therefore $(20)$ can be written as two families,

$$
\begin{align*}
\left(10^{2 k} 1\right. & \left.\rightarrow(10 \oplus)^{k} 11(\ominus 01)^{k}\right)_{k}  \tag{21a}\\
\left(10^{2 k+1} 1\right. & \left.\rightarrow(10 \oplus)^{k+1} 1(\ominus 01)^{k+1}\right)_{k} \tag{21b}
\end{align*}
$$

They, and all reactions of the form $\left(a_{k+n} \rightarrow x^{k} a_{n} y^{k}\right)_{k}$, are called triangle reactions.

Diagrams for the reactions with $k=3$ are shown in Figure 9 ,
If we try the same manoeuvre with the other reaction family, 18 b , we get $\left(01^{k+2} 0 \rightarrow 01 \oplus 10^{k+2} 1 \ominus 10\right)_{k}$. This is a family to which (17)


Figure 9. Triangle reactions for $k=3$.
cannot be applied. Therefore we will now use the reaction families 21) as our base for the description of the ether.

## - 5.2 The Ether

We will find now represent the ether of Rule 54 by reactions. The reactions for Rule 54 will turn out to be a special case of a generic scheme that applies to periodic patterns in any one-dimensional cellular automaton.

In Rule 54 11, the ether is a periodic structure whose configurations consist alternatingly of the two patterns ... 100010001... and $\ldots 011101110 \ldots$. When one of them occurs again, it is shifted horizontally by two cells, so that the true time period is 4 .

Our starting point for representing them by reactions must be the configuration ...100010001..., since to it we can apply one reaction of type (21b),

$$
\begin{equation*}
10001 \rightarrow(10 \oplus)^{2} 1(\ominus 01)^{2} \tag{22}
\end{equation*}
$$

It would be therefore advantageous to decompose the initial configuration into components of the form 10001. With a small extension of our notation, this is actually possible.

Definition 6 (Overlapping Situations) Let $a x$ be a situation. The $a\langle x\rangle$ is also a situation, and $\langle x\rangle$ is the overlapping part. We have

$$
\begin{equation*}
\operatorname{pr}(a\langle x\rangle)=\operatorname{pr}(a x) \quad \text { and } \quad \delta(a\langle x\rangle)=\delta(a) . \tag{23}
\end{equation*}
$$

A product of situations with overlap, like $a\langle x\rangle b\langle y\rangle$, is only allowed if the situation by begins with $x$; then $a\langle x\rangle b\langle y\rangle=a b\langle y\rangle$.

A reaction that begins with $a\langle x\rangle$ must have the form

$$
\begin{equation*}
a\langle x\rangle \rightarrow a^{\prime}\langle x\rangle ; \tag{24}
\end{equation*}
$$

it exists if $a x \rightarrow a^{\prime} x$ is a reaction.
If we remind ourselves that the transitions of a cellular automaton are defined in terms of overlapping cell neighbourhoods, then the new extension looks quite natural.

We can now write a term like $(1000)^{k} 1$ as a product $(1000\langle 1\rangle)^{k} 1$ and apply the ether reactions in parallel to each factor, except for
the final 1. In this style, Reaction 22 is best written in the form $1000\langle 1\rangle \rightarrow(10 \oplus\langle 1\rangle)^{2}(1 \ominus 0\langle 1\rangle)^{2}$.

But now we should better introduce abbreviations. We will write,

$$
\begin{equation*}
\varepsilon_{+}=10 \oplus\langle 1\rangle \quad \text { and } \quad \varepsilon_{-}=1 \ominus 0\langle 1\rangle, \tag{25}
\end{equation*}
$$

such that 22 becomes

$$
\begin{equation*}
1000\langle 1\rangle \rightarrow \varepsilon_{+}^{2} \varepsilon_{-}^{2} \tag{26}
\end{equation*}
$$

The terms $\varepsilon_{+}$and $\varepsilon_{-}$are the simplest of the higher level structures in Rule 54 that we will identify.

There is also a complementary reaction to (26),

$$
\begin{equation*}
\varepsilon_{-}^{2} \varepsilon_{+}^{2} \rightarrow 1000\langle 1\rangle \tag{27}
\end{equation*}
$$

In contrast to 26 , this reaction does not belong to a known family, and we will derive it by hand (see below). Together the two reactions form a type that naturally represents the periodic patterns of onedimensional cellular automata. Before a formal definition is given, we introduce the abbreviations

$$
\begin{equation*}
e_{-}=\varepsilon_{-}^{2}, \quad e_{+}=\varepsilon_{+}^{2}, \quad b=1000\langle 1\rangle \tag{28}
\end{equation*}
$$

Then we see that 26 and 27 are example of the following general pattern:

Definition 7 (Background Pairs) Two situations, $e_{-}, e_{+}$, form a background pair if there is a reaction

$$
\begin{equation*}
e_{-} e_{+} \rightarrow e_{+} e_{-} \tag{29a}
\end{equation*}
$$

If there is also a situation $b \in \Sigma^{*}$ with

$$
\begin{equation*}
e_{-} e_{+} \rightarrow b \rightarrow e_{+} e_{-} \tag{29b}
\end{equation*}
$$

then $b$ is the baseline of the background pair.
A background pairs represent the elementary region of a tiling of the two-dimensional space-time (Figure 10. If a background pair is present, we automatically get the reaction families

$$
\begin{align*}
\left(b^{k}\right. & \left.\rightarrow e_{+}^{k} e_{-}^{k}\right)_{k}  \tag{30a}\\
\left(e_{-}^{k} e_{+}^{\ell}\right. & \left.\rightarrow e_{+}^{\ell} e_{-}^{k}\right)_{k, \ell} \tag{30b}
\end{align*}
$$

which represent larger patches of the background. As we can see in Figure 10, the reactions of 30a represent the generation of a larger piece of ether from an initial configuration, while (30b) represents the development of a background fragment at a later time.

[^4]

Figure 10. An ether, represented by a background pair $e_{-}, e_{+}$with baseline $b$.

Derivation of the remaining ether reaction We have not yet proved equation (27), the reaction $e_{-} e_{+} \rightarrow 1000\langle 1\rangle$. This will be done now.

The computation is an example for a larger calculation with Flexible Time. We will prove (27) via the two reactions

$$
\begin{align*}
& \varepsilon_{-} \varepsilon_{+} \rightarrow 1^{2}\langle 1\rangle  \tag{31a}\\
& \varepsilon_{-}^{2} \varepsilon_{+}^{2} \rightarrow 1000\langle 1\rangle \tag{31b}
\end{align*}
$$

and the auxiliary step

$$
\begin{equation*}
01^{3} 0 \rightarrow 01 \oplus 10^{3} 1 \ominus 10 \tag{31c}
\end{equation*}
$$

The last reaction is an element of the reaction family

$$
\begin{equation*}
\left(01^{k+2} 0 \rightarrow 01 \oplus 10^{k+2} 1 \ominus 10\right)_{k} \tag{32}
\end{equation*}
$$

Its derivation uses the reaction family 18 b and is done in the following way:

$$
\begin{aligned}
\underline{0111^{k}} 0 & \rightarrow 01 \oplus 1 \underline{\ominus 0111^{k} 0} \\
& \rightarrow 01 \oplus 1 \underline{\ominus 111^{k}} 0 \\
& \rightarrow 01 \oplus 1000^{k} \underline{\ominus 110} \rightarrow 01 \oplus 10^{k+2} 1 \ominus 10
\end{aligned}
$$

Now we can derive the other two reactions of (31):

$$
\begin{aligned}
& \varepsilon_{-} \varepsilon_{+}=1 \ominus 0\langle 1\rangle 10 \oplus\langle 1\rangle \\
& =1 \underline{\ominus} 010 \oplus\langle 1\rangle \\
& \rightarrow 111 \ominus 10 \oplus\langle 1\rangle \rightarrow 11\langle 1\rangle, \\
& \varepsilon_{-} \underline{\varepsilon_{-}} \varepsilon_{+} \varepsilon_{+} \rightarrow \varepsilon_{-} 11\langle 1\rangle \varepsilon_{+} \\
& =1 \ominus 0\langle 1\rangle 11\langle 1\rangle 10 \oplus\langle 1\rangle \\
& =1 \ominus \underline{01^{3} 10} \oplus\langle 1\rangle \\
& \rightarrow \underline{1 \ominus 01 \oplus 10^{3}} \underline{1 \ominus 01 \oplus\langle 1\rangle} \rightarrow 10^{3}\langle 1\rangle .
\end{aligned}
$$

In the second computation we have used (31a) and (31c).

## 6. Particles

In the ether particles move. Boccara et al. 1 have found four of them and called them $\overleftarrow{w}, \vec{w}, g_{o}$ and $g_{e}$ (Figure 11). We will refer to the moving particles $\overleftarrow{w}$ and $\vec{w}$ sometimes as gliders, in contrast to the static particles $g_{o}$ and $g_{e}$.


Figure 11. Particles under Rule 54. The diagrams show the four types of gliders on an ether background.

Now we will represent these particles by situations and reactions. The characterisation of particles is a natural generalisation of that of a background:

Definition 8 (Particles) Let $\left(b_{-}, b_{+}\right)$be a background pair. A particle that moves in this background is a situation $p$ for which there is a reaction

$$
\begin{equation*}
b_{-}^{m} p b_{+}^{n} \rightarrow b_{+}^{n} p b_{-}^{m} \tag{33}
\end{equation*}
$$

The pair $(m, n)$ is the type of the particle.
The type of $p$ represents its speed relative to the background. To convert it to a more conventional form, we notice that in the initial situation of the reaction (33), the left side of $p$ is located at the spacetime point $m \delta\left(b_{-}\right)$, while in its final situation, it is at $n \delta\left(b_{+}\right)$. The period vector $(\Delta t, \Delta x)=n \delta\left(b_{+}\right)-m \delta\left(b_{-}\right)$is therefore the displacement that $p$ undergoes during one cycle of its existence. After $\Delta t$ time steps, the particle is in the same state, and it has $\Delta x$ positions to the right. The speed of $p$ is then $\frac{\Delta x}{\Delta t}$. (Figure 12.)

Often it is simpler to work with speeds relative to the background. For this we use the vectors $T=\delta\left(b_{+}\right)-\delta\left(b_{-}\right)$and $X=\delta\left(b_{+}\right)+\delta\left(b_{-}\right)$ as our base, the first one pointing to the future and the second one to the right. A particle of type $(m, n)$ has then a period vector of $\frac{n+m}{2} T+\frac{n-m}{2} X$ and we can say that its relative speed is $\frac{n-m}{n+m}$.


Figure 12. A particle of type $(2,3)$ as part of a periodic background. Its relative speed is $\frac{1}{5}$.

The particles of Rule $\mathbf{5 4}$ For Rule 54 we use the following definitions:

$$
\begin{array}{ll}
\overleftarrow{w}=\varepsilon_{-} 1^{2}\langle 1\rangle, & g_{o}=\varepsilon_{+} \varepsilon_{-} \\
\vec{w}=1^{2} \varepsilon_{+}, & g_{e}=\varepsilon_{+} 1 \varepsilon_{-} \tag{34}
\end{array}
$$

They have this specific form because we can then use a simple subset of our reaction system to represent their behaviour. This subset consists of two reaction families and one extra reaction,

$$
\begin{array}{rlr}
\left(\varepsilon_{-} 1^{2 k} \varepsilon_{+}\right. & \left.\rightarrow \varepsilon_{+}^{k+1} \varepsilon_{-}^{k+1}\right)_{k \geq 1}, & \varepsilon_{-} \varepsilon_{+} \rightarrow 1^{2}\langle 1\rangle \\
\left(\varepsilon_{-} 1^{2 k+1} \varepsilon_{+}\right. & \left.\rightarrow \varepsilon_{+}^{k+1} 1 \varepsilon_{-}^{k+1}\right)_{k}, \tag{35b}
\end{array}
$$

which transform situations that consist only of $\varepsilon_{-}, \varepsilon_{+}$and 1 into each other. They can easily be derived from the reaction families 32) and 21. With the reactions of (35a, the ether reaction $\varepsilon_{-} \varepsilon_{+}$can be proved, as we have seen on page 21 .

With these reactions we can now verify that the terms in (34) are indeed particles:

$$
\begin{align*}
\vec{w} e_{+} & =\varepsilon_{-} 1^{2} \varepsilon_{+}^{2} \rightarrow \varepsilon_{+}^{2} \varepsilon_{-}^{2} \varepsilon_{+} \rightarrow \varepsilon_{+}^{2} \varepsilon_{-} 1^{2}\langle 1\rangle & =e_{+} \vec{w},  \tag{36a}\\
e_{-} g_{o} e_{+} & =\varepsilon_{-}^{2} \varepsilon_{+} \varepsilon_{-} \varepsilon_{+}^{2} \rightarrow \varepsilon_{-} 1^{2} 1^{2} \varepsilon_{+} \rightarrow \varepsilon_{+}^{3} \varepsilon_{-}^{3} & =e_{+} g_{o} e_{-}  \tag{36b}\\
e_{-} g_{e} e_{+} & =\varepsilon_{-}^{2} \varepsilon_{+} 1 \varepsilon_{-} \varepsilon_{+}^{2} \rightarrow \varepsilon_{-} 1^{2} 11^{2} \varepsilon_{+} \rightarrow \varepsilon_{+}^{3} 1 \varepsilon_{-}^{3} & =e_{+} g_{e} e_{-} . \tag{36c}
\end{align*}
$$

The reaction $e_{-} \overleftarrow{w} \rightarrow \overleftarrow{w} e_{-}$has been omitted since the reactions in (34) are left-right symmetric. We see from these reactions that the types of $\vec{w}$ and $\overleftarrow{w}$ are $(0,1)$ and $(1,0)$, while $g_{o}$ and $g_{e}$ both have type $(1,1)$ Figure 13 contains diagrams of the reactions.

Collisions of two particles With the reactions of (35) we can already find out simple facts about the particles and their interactions.


$$
\vec{w} e_{+} \rightarrow e_{+} \vec{w}
$$



$$
e_{-} g_{o} e_{+} \rightarrow \vec{w} \overleftarrow{w} \rightarrow e_{+} g_{o} e_{-}
$$



$$
e_{-} g_{e} e_{+} \rightarrow \vec{w} 1 \overleftarrow{w} \rightarrow e_{+} g_{e} e_{-}
$$

Figure 13. Evolution of the Rule 54 particles. The particles are shown in strong colours, and the outlined squares are ether.

One fact is hidden in 36b: the reaction

$$
\begin{equation*}
\vec{w} \overleftarrow{w} \rightarrow e_{+} g_{o} e_{-} \tag{37}
\end{equation*}
$$

can easily be recognised once we remember that $\vec{w} \overleftarrow{w}=\varepsilon_{-} 1^{2} 1^{2} \varepsilon_{+}$. This is the reaction in which two colliding $w$ particles create a $g_{o}$. It is in fact the only reaction that is possible between the two $w$ particles. To see this, we note that if $\vec{w}$ moves towards $\overleftarrow{w}$ with nothing else than ether between them, this must be represented by a situation $\vec{w} E \overleftarrow{w}$, where $E$ is a product of an arbitrary number of $e_{-}$and $e_{+}$terms. Then there must be a reaction $E \rightarrow e_{+}^{m} e_{-}^{n}$, where $m$ is the number of $e_{+}$factors in $E$ and $n$ the number of $e_{-}$factors. This leads to a reaction chain

$$
\begin{equation*}
\vec{w} E \overleftarrow{w} \rightarrow \vec{w} e_{+}^{m} e_{-}^{n} \overleftarrow{w} \rightarrow e_{+}^{m} \vec{w} \overleftarrow{w} e_{-}^{n} \tag{38}
\end{equation*}
$$

to which we can apply (37). We have thus seen that two $w$ gliders always move towards each other unchanged until they react to the position $\vec{w} \overleftarrow{w}$, and that therefore (37) is their only possible collision.

[^5]The same principle can be applied to any pair of colliding particles. We have then the following theorem:

Theorem 1 (Particle Collisions) Let $p$ and $p^{\prime}$ be two particles of types $(m, n)$ and $\left(m^{\prime}, n^{\prime}\right)$, with $p$ left of $p^{\prime}$. Then $p$ moves toward $p^{\prime}$ if $n m^{\prime}>m n^{\prime}$, away from $p^{\prime}$ if $n m^{\prime}<m n^{\prime}$, otherwise they keep the same distance.

If they collide, then there are $n m^{\prime}$ possible interactions between them.

Proof. If $p$ and $p^{\prime}$ collide, the relative speed of $p$ must be greater than that of $p^{\prime}$. This means that $\frac{n-m}{n+m}>\frac{n^{\prime}-m^{\prime}}{n^{\prime}+m^{\prime}}$, or equivalently that $n m^{\prime}>$ $m n^{\prime}$. The other two cases are similar.

For the second statement we represent the relative positions of $p$ and $p^{\prime}$ by a situation $a p b p^{\prime} c$ with $a, b, c \in\left\{b_{-}, b_{+}\right\}^{*}$. Here $a$ and $c$ represent the empty space left and right of the particles. We can make them arbitrarily large without changing the relative position of $p$ and $p^{\prime}$. (A change of $a$ changes the absolute position of $p$ and $p^{\prime}$, but that has no influence on their behaviour.) Especially we can assume that $a=b_{-}^{m}$ and $c=b_{+}^{n^{\prime}}$. The situation $b$ represents the space between $p$ and $p^{\prime}$, and we can always bring it by background reactions to the form $b_{+}^{i} b_{-}^{j}$.

So we can assume that the environment of the particles has the form $b_{-}^{m} p b_{+}^{i} b_{-}^{j} p^{\prime} b_{+}^{n^{\prime}}$. Since $p$ and $p^{\prime}$ collide, none of the reactions $b_{-}^{m} p b_{+}^{n} \rightarrow$ $b_{+}^{n} p b_{-}^{m}$ and $b_{-}^{m^{\prime}} p^{\prime} b_{+}^{n^{\prime}} \rightarrow b_{+}^{n^{\prime}} p^{\prime} b_{-}^{m^{\prime}}$ can be applied to this situation. This means that $i<n$ and $j<m^{\prime}$, for which there are $n m^{\prime}$ possibilities.

Interaction between the static particles and the $w$ gliders. When we start with a random initial configuration and let it evolve for a short time, we typically see some $g_{o}$ and $g_{e}$ particles on a background, with $\vec{w}$ and $\overleftarrow{w}$ moving between them (Figure 1). The formalism for Rule 54 is now developed far enough to describe with it the behaviour of these particles in reasonable detail.

Specifically, we can now describe the behaviour of isolated $g_{o}$ and $g_{e}$ particles, which never interact with each other, only with $\vec{w}$ and $\overleftarrow{w}$. In Flexible Time we can express this requirement by restricting ourselves to the reactions that start from a situation $x g y$ with $x \in\left\{e_{-}, \vec{w}\right\}^{*}$, $g \in\left\{g_{o}, g_{e}\right\}$ and $y \in\left\{e_{+}, \overleftarrow{w}\right\}^{*}$

The $g_{o}$ case is the simplest, since the collision with a $w$ always destroys this particle. Up to symmetry we have only the following reactions,

$$
\begin{equation*}
\vec{w} g_{o} e_{+} \rightarrow e_{+} \vec{w} e_{-}, \quad \vec{w} g_{o} \overleftarrow{w} \rightarrow e_{+}^{2} e_{-}^{2} \tag{39}
\end{equation*}
$$

They could be verified directly, but we will now compute them in a way that is also useful in the more complex case of $g_{e}$. For this we begin with $\vec{w} g_{o}$, a common factor of the two left sides in (39), and also the smallest situation that represents a collision of $\vec{w}$ and $g_{o}$. Their
reaction is $w g_{o}=\underline{\varepsilon_{-}} 1^{2} \varepsilon_{+} \varepsilon_{-} \rightarrow \varepsilon_{+}^{2} \varepsilon_{-}^{3}=e_{+} \varepsilon_{-} e_{-}$. The end result is here interpreted as an $\varepsilon_{-}$surrounded by two ether fragments. We can consider it as a short-lived intermediate stage, or a resonance, if we use once again the jargon of particle physics. In the next step we ignore the ether fragments and consider only the development of the $\varepsilon_{-}$. There are two ways in which it can interact with an ether fragment or a $w$ particle, namely through the reactions $\varepsilon_{-} e_{+}=\varepsilon_{-} \varepsilon_{+}^{2} \rightarrow 1^{2} \varepsilon_{+}=\overleftarrow{w}$ and $\varepsilon_{-} \vec{w}=\varepsilon_{-} 1^{2} \varepsilon_{+} \rightarrow \varepsilon_{+}^{2} \varepsilon_{-}^{2}=e_{+} e_{-}$. No further resonances arise from these reactions, so we can stop here.

The result is a scheme of three reactions; they describe the behaviour of $g_{o}$ in the same way as 39):

$$
\begin{align*}
\vec{w} g_{o} & \rightarrow e_{+} \varepsilon_{-} e_{-}  \tag{40a}\\
\varepsilon_{-} e_{+} & \rightarrow \vec{w} \\
\varepsilon_{-} \overleftarrow{w} & \rightarrow e_{+} e_{-} \tag{40b}
\end{align*}
$$

One can use them to derive the reactions of (40), e. g. with the reaction chain $\vec{w} g_{o} e_{+} \rightarrow e_{+} \varepsilon_{-} e_{-} e_{+} \rightarrow e_{+} \varepsilon_{-} e_{+} e_{-} \rightarrow e_{+} \overleftarrow{w} e_{-}$for the first reaction. But for most purposes, 40 can be interpreted directly as a two-step scheme that describes how an $\varepsilon_{-}$is created 40a and how it decays to $\vec{w}$ or ether 40 b . The ether particles at the right side of 40a can be thought as becoming part of the surrounding space, which is why they do not appear in 40b.

A similar but more complex scheme describes the collision of $g_{e}$ with one or more $w$ particles. Up to symmetry it has the intermediate states $1,1 \varepsilon_{-}$and $1^{5}\langle 1\rangle$ and can be written as follows:

$$
\begin{align*}
\vec{w} g_{e} & \rightarrow e_{+} e_{-} 1 \varepsilon_{-}  \tag{41a}\\
1 \varepsilon_{-} e_{+} & \rightarrow 1 \overleftarrow{w} \\
1 \varepsilon_{-} \overleftarrow{w} & \rightarrow 1 e_{+} e_{-}  \tag{41b}\\
e_{-} 1 e_{+} & \rightarrow 1^{5}\langle 1\rangle \\
e_{-} 1 \overleftarrow{w} & \rightarrow \overleftarrow{w} 1 e_{-} \\
\vec{w} 1 \overleftarrow{w} & \rightarrow e_{+} g_{e} e_{-}  \tag{41c}\\
e_{-} 1^{5} e_{+} & \rightarrow \overleftarrow{w} g_{e} \vec{w} \\
e_{-} 1^{5} \overleftarrow{w} & \rightarrow \overleftarrow{w} e_{+}^{2} 1 e_{-}^{2} \\
\vec{w} 1^{5} \overleftarrow{w} & \rightarrow e_{+}^{2} g_{e} e_{-}^{2} \tag{41d}
\end{align*}
$$

All these reactions are short and can be verified directly. They show that an isolated $g_{e}$ can neither be destroyed nor does it explode to a larger structure. (See [9] for the deeper reasons behind this.) The intermediate states can however persist for an indefinite time if the right pattern of incoming $w$ gliders is given. One can see this e.g.

[^6]from the reaction $e_{-} 1 \overleftarrow{w} \rightarrow \overleftarrow{w} 1 e_{-}$in (41c). It can be iterated to $\left(e_{-}^{k} 1 \overleftarrow{w}^{k} \rightarrow \overleftarrow{w}^{k} 1 e_{-}^{k}\right)_{k}$, which shows how the intermediate state 1 can be kept alive indefinitely by a sequence of incoming $\overleftarrow{w}$ gliders.

In summary we get a description of the behaviour not just of a single $g_{o}$ and $g_{e}$, but also of a whole system of particles, provided that the $g$ particles and their intermediate states all keep a distance from each other. The distance must be so large that next to each $g$ particle or intermediate state there is always a $w$ particle or an ether fragment. As long as this is true, the $g_{o}$ particles are created (37) and destroyed 40 by $w$ gliders, while the $g_{e}$ persist but go through intermediate states (41).

## 7. Summary

This text consists of two interleaving tracks, one with the goal of understanding Rule 54 better, the other to find concepts that are valid for all cellular automata.

After a recapitulation of the results derived in [20], we began with constructing a shorter representation of the local reaction system for Rule 54 (Table 2). We then described how the transition rule $\varphi$ influences the local reaction system $\Phi$ and at the end introduced two slogans to summarise the generator reactions of the local system.

With (16) and (17), we learned how to iterate reactions. This helped to derive expressions for the triangles under Rule 54 and to find a subsystem $\sqrt{35}$ of $\Phi$ that consists only of modified triangle reactions. It also introduced the situations $\varepsilon_{-}$and $\varepsilon_{+}$, which, together with the situation 1, were the building blocks of the following construction.

We introduced definitions for the background and for particles and explored particle collisions. A formula for the number of particle interactions was already found in [6] under a different framework, but the proof here seems more direct.

Expressions for the ether and the main particles of Rule 54 were found and the collisions of the particles computed. We could see that an isolated $g_{e}$ is stable under all collisions with incoming $w$ gliders. This extends in a way a result in [9], which already showed that a single $g_{e}$ could not be destroyed, but the current, more detailed investigation also shows that it could not "explode" either and become a steadily growing perturbation in the ether.

On the way to this result, we saw an efficient method to display all possible interactions of an isolated particle with all other particles and the background (41).

The track about Rule 54 lead therefore to results about the interaction of its particles, while the general track lead to generic definitions of triangles, background and particles and a theorem about glider collisions. Both show how Flexible Time helps to understand an automaton
like Rule 54 as a system of interacting particles.

Changes in the formalism One of the aims of this work was to extend the capabilities of Flexible Time by applying it to the understanding of a "naturally occuring" cellular automaton, i. e. one that was not constructed for a specific purpose. This resulted in the following changes with respect to the version in [20]:

1. The interpretation of $\ominus$ and $\oplus$ were changed silently in 10). In 20, they were abbreviations for $\ominus_{r}$ and $\oplus_{r}$, where $r$ was the radius of the cellular automaton. Now the horizontal offsets associated to $\ominus$ and $\oplus$ depend on the context in which the symbols occur.
2. Reaction families, which were already present in 20, got a shorter notation.
3. A short notation for overlapping situations was introduced in Definition 6. There was already an overlap notation in [20, but it was more clumsy. Now overlapping situations are part of the normal formalism.

The new interpretation of $\ominus$ and $\oplus$ allowed us to write the formulas of the local reaction system completely without indices and to make the similarities between the basic reactions more visible.

With overlaps, definitions like those of a background pair 29) could be written in a concise way.

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## References

[1] N. Boccara, J. Nasser, and M. Roger. Particlelike structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular-automaton rules. Physical Review A, 44:866-875, 1991.
[2] Matthew Cook. Universality in elementary cellular automata. Complex Systems, 15(1):1-40, 2004.
[3] Jérôme Durand-Lose. The signal point of view: from cellular automata to signal machines. In Bruno Durand, editor, First Symposium on Cellular Automata "Journées Automates Cellulaires" (JAC 2008), Uzès, France, April 21-25, 2008. Proceedings, pages 238-249. MCCME Publishing House, Moscow, 2008.
[4] Kari Eloranta. Random walks in cellular automata. Nonlinearity, 6:1025-1036, 1993.
[5] James E. Hanson and James P. Crutchfield. Computational mechanics of cellular automata: An example. Working Paper 95-10-095, Santa Fe Institute, 1995.
[6] Wim Hordijk, Cosma Rohilla Shalizi, and James P. Crutchfield. Upper bound on the products of particle interactions in cellular automata. Physica D, 154(3-4):240-258, 2001.
[7] Erica Jen. Exact solvability and quasiperiodicity of one-dimensional cellular automata. Nonlinearity, 4:251-276, 1990.
[8] Wentian Li and Mats G. Nordahl. Transient behavior of cellular automata rule 110. Working Paper 92-03-016, Santa Fe Institute, 1992.
[9] Bruno Martin. A group interpretation of particles generated by one dimensional cellular automaton, 54 Wolfram's rule. International Journal of Modern Physics C, 11(1):101-123, 2000.
[10] Genaro J. Martínez, Andrew Adamatzky, and Harold V. McIntosh. Phenomenology of glider collisions in cellular automaton rule 54 and associated logical gates. Chaos, Fractals and Solitons, 28:100-111, 2006.
[11] Genaro J. Martínez, Andrew Adamatzky, and Harold V. McIntosh. Complete characterization of structure of Rule 54. Complex Systems, 23(3):259-293, 2014.
[12] Genaro J. Martínez and Harold V. McIntosh. ATLAS: Collisions of gliders like phases of ether in rule 110, August 2001. http://uncomp.uwe.ac.uk/genaro/Papers/Papers_on_CA_ files/ATLAS/bookcollisions.html
[13] Genaro J. Martínez, Harold V. McIntosh, Juan Carlos Seck Tuoh Mora, and Sergio V. Chapa Vergara. Determining a regular language by gliderbased structures called phases $f_{i} 1$ in Rule 110. Journal of Cellular Automata, 3(3):231-270, 2008.
[14] Jacques Mazoyer and Véronique Terrier. Signals in one-dimensional cellular automata. Theoretical Computer Science, 217(1):53-80, 1999.
[15] Nicolas Ollinger and Gaétan Richard. Collisions and their catenations: Ultimately periodic tilings of the plane. In Giorgio Ausiello, Juhani Karhumäki, Giancarlo Mauri, and C.-H. Luke Ong, editors, Fifth IFIP International Conference on Theoretical Computer Science - TCS 2008, volume 273 of IFIP, pages 229-240. Springer, 2008.
[16] Nicolas Ollinger and Gaétan Richard. Automata on the plane vs particles and collisions. Theoretical Computer Science, 410:2767-2773, 2009.
[17] Marcus Pivato. Defect particle kinematics in one-dimensional cellular automata. Theoretical Computer Science, 377(1-3):205-228, 2007.
[18] Markus Redeker. Flexible time and the evolution of one-dimensional cellular automata. Journal of Cellular Automata, 5(4-5):273-287, 2010.
[19] Markus Redeker. Gliders and ether in Rule 54. In Nazim Fatès, Jarkko Kari, and Thomas Worsch, editors, Automata 2010: 16th International Workshop on Cellular Automata and Discrete Complex Systems, pages 299-308, 2010.
[20] Markus Redeker. Flexible Time and Ether in One-Dimensional Cellular Automata. PhD thesis, University of the West of England, Bristol, UK, 2013.
[21] Juan C. Seck Tuoh Mora, Genaro J. Martínez, Norberto HernándezRomero, and Joselito Medina-Marín. Elementary cellular automaton rule 110 explained as a block substitution system. Computing, 88:193205, 2010.
[22] Stephen Wolfram. Universality and complexity in cellular automata. Physica D, 10:1-35, 1984.
[23] Konrad Zuse. Rechnender Raum. Elektronische Datenverarbeitung, 8:336-344, 1967.


[^0]:    ${ }^{1}$ This article started as an extension of [19], but has now grown considerably and is completely rewritten.

[^1]:    Complex Systems, Volume (year) 1-1+

[^2]:    Complex Systems, Volume (year) 1-1+

[^3]:    ${ }^{2}$ The bottom left reaction has been shortened even more, it should have been $\ominus 10 \sigma \rightarrow \ominus 0 \sigma$.

[^4]:    Complex Systems, Volume (year) 1-1+

[^5]:    Complex Systems, Volume (year) 1-1+

[^6]:    Complex Systems, Volume (year) 1-1+

