Besicovitch pseudodistances with respect to non-Følner sequences

Silvio Capobianco, Pierre Guillon, Camille Noûs

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Silvio Capobianco*† Pierre Guillon‡ Camille Noûs§

Abstract

The Besicovitch pseudodistance defined in [BFK99] for one-dimensional configurations is invariant by translations. We generalize the definition to arbitrary groups and study how properties of the pseudodistance, including invariance by translations, are determined by those of the sequence of finite sets used to define it. In particular, we recover that if the Besicovitch pseudodistance comes from a nondecreasing exhaustive Følner sequence, then every shift is an isometry. For non-Følner sequences, we prove that some shifts are not isometries, and the Besicovitch pseudodistance with respect to some subsequence even makes them non-continuous.

Keywords: Besicovitch distance, Følner sequences, submeasures, amenability, non-compact space, symbolic dynamics.

1 Introduction

The Besicovitch pseudodistance was proposed by Blanchard, Formenti and Kůrka in [BFK99] as an "antidote" to sensitivity of the shift map in the prodiscrete (Cantor) topology of the space of 1D configurations over a finite alphabet. The idea is to take a window on the integer line, which gets larger and larger, and compute the probability that in a point under the window, chosen uniformly at random, two configurations will take different values. The upper limit of this sequence of probabilities behaves like a distance, except for taking value zero only on pairs of equal configurations: this defines an equivalence relation, and the resulting quotient space is a metric space on which the shift is an isometry, or equivalently, the distance is shift-invariant.

The original choice of windows is $X_n = [-n : n]$, the set of integers from $-n$ to n included. This notion can be easily extended to arbitrary dimension $d \geq 1$, taking a sequence of hypercubic windows. If we allow arbitrary shapes, the notion of Besicovitch space can be extended to configurations over arbitrary groups; in this case, however, the properties of the group and the choice of the windows can affect the the distance being or not being shift-invariant. An example of a Besicovitch pseudodistance which is not shift-invariant is given in [Cap09], where it is also proved that, if a countable group is *amenable* (cf. [CGK13] and [CSC10, Chapter 4]), then the Besicovitch distance with respect to any nondecreasing exhaustive *Følner sequence* is shift-invariant. The class of amenable groups is of great interest and importance in group theory, symbolic dynamics, and cellular automata theory.

In this paper, we explore the relation between the properties of Besicovitch pseudodistances over configuration spaces with countable base group and those of the sequence of finite sets

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used to define it. In Section 3, we give the main definition and prove that if a sequence of finite subsets is increasing, then the corresponding Besicovitch space is pathwise connected: this generalizes [BFK99, Proposition 1]. In Section 4, we introduce a notion of *synchronous Følner equivalence* between sequences, and a related order relation where one sequence comes before another sequence if it is synchronously Følner-equivalent to a subsequence of the latter. This, on the one hand, generalizes that of Følner sequences, and on the other hand, allows us to compare the Besicovitch distances and submeasures associated to different sequences. In particular, we prove that an increasing sequence of finite sets is Følner if and only if every shift is an isometry for the corresponding Besicovitch distance: this provides the converse of [Cap09, Theorem 3.5]. Finally, we give conditions for absolute continuity and Lipschitz continuity of Besicovitch submeasures with respect to each other.

2 Background

We use the notation $X \Subset Y$ to mean that X is a finite subset of Y. We denote the *symmetric difference* of two sets X and Y as $X \Delta Y$.

Given $\alpha \in \mathbb{R}$, its *integer part* $\lfloor \alpha \rfloor$ is the largest $m \in \mathbb{Z}$ such that $m \leq \alpha$.

If (α_n) and (β_n) are nonzero number sequences, we write $\alpha_n \sim_{n\to\infty} \beta_n$ if $\lim_{n\to\infty} \alpha_n/\beta_n = 1$, and $\alpha_n = o_{n\to\infty}\beta_n$ if $\lim_{n\to\infty} \alpha_n/\beta_n = 0$.

2.1 Submeasures

The following definition appears for instance in [Sab06].

Definition 2.1. A *submeasure* over a set G is a map $\mu : 2^G \to \mathbb{R} \sqcup \{+\infty\}$ such that:

1. $\mu(\emptyset) = 0$;
2. $\mu(W) < \infty$ if W is finite;
3. $\mu(V \cup W) \leq \mu(V) + \mu(W)$ for every $V, W \subset G$.

If G and A are two sets, the *difference set* of two functions $x, y : G \to A$ is the set $\Delta(x, y) = \{i \in G |\, x(i) \neq y(i)\}$.

Any submeasure over G gives rise to an associated pseudodistance over A^G:

$$d_\mu(x, y) = \mu(\Delta(x, y)) \;\forall x, y \in A^G \; .$$

Remark 2.2. The topological space corresponding to such a pseudodistance is homogeneous in the following sense: the balls around every two points y and z are isometric. Indeed, identify A with the additive group $\mathbb{Z}/|A|\,\mathbb{Z}$. Then for every $y, z \in A^G$ the map $\psi_{y,z} : A^G \to A^G$ defined by $\psi_{y,z}(x)(i) = x(i) - y(i) + z(i)$ for every $x \in A^G$ and $i \in G$ is an isometry between any ball around y and the corresponding one around z.

We say that submeasure μ is *absolutely continuous* (resp. α-Lipschitz, for some $\alpha > 0$) with respect to submeasure ν if $\nu(W) = 0 \implies \mu(W) = 0$ (resp. $\mu(W) \leq \alpha\nu(W)$) for any $W \subset G$.

Remark 2.3. Let $\varepsilon, \delta > 0$, μ, ν two submeasures on G, and $z \in A^G$. The following are equivalent.

1. For every set $W \subset G$, $\mu(W) \geq \varepsilon \implies \nu(W) \geq \delta$.
2. For every $x, y \in A^G$, $d_\mu(x, y) \geq \varepsilon \implies d_\nu(x, y) \geq \delta$.

3. For every $x \in A^G$, $d_\mu(x,z) \geq \varepsilon \implies d_\nu(x,z) \geq \delta$.

Consequently, the identity map, from space A^G endowed with d_μ onto space A^G endowed with d_ν, is continuous (resp. α-Lispchitz) if and only if μ is absolutely continuous (resp. α-Lispchitz) with respect to ν. In that case the identity is even absolutely continuous.

2.2 Shifts and translations

If A is an alphabet, G is a group, and $g \in G$, the *shift* by g is the function $\sigma^g : A^G \to A^G$ defined by $\sigma^g(x)(i) = x(gi)$ for every $x \in A^G$ and $i \in G$. A map ψ from A^G to itself is *shift-invariant* if $\psi\sigma^g = \sigma^g\psi$ for every $g \in G$. Note that $\Delta(\sigma^g(x), \sigma^g(y)) = g^{-1}\Delta(x,y)$ for every $x,y \in A^G$ and $g \in G$, as:

$$\begin{aligned}
\Delta(\sigma^g(x), \sigma^g(y)) &= \{i \in G |\, x(gi) \neq y(gi)\} \\
&= \left\{g^{-1}j \middle|\, j \in G,\, x(j) \neq y(j)\right\} \\
&= g^{-1}\{j \in G |\, x(j) \neq y(j)\} \\
&= g^{-1}\Delta(x,y)\,.
\end{aligned}$$

Since the maps $\psi_{y,z}$ from Remark 2.2 are shift-invariant, one can see that the shift is continuous, Lipschitz, etc in every x if and only if it is in one x.

The shift by g, within space A^G endowed with d_μ, is topologically the same as the identity map, from A^G endowed with d_μ onto space A^G endowed with d_ν, where $\nu(W) = g^{-1}\mu(W) = \mu(g^{-1}W)$ for any set $W \subset G$. Remark 2.2 can then be rephrased into the following.

Remark 2.4. If G is a group, $g \in G$, and A^G is endowed with d_μ, then the shift map by g is continuous (resp. α-Lispchitz) if and only if μ is absolutely continuous (resp. α-Lispchitz) with respect to $g^{-1}\mu$. In that case, the shift by g is even absolutely continuous.

3 Besicovitch submeasure and pseudodistance

Among classical examples of submeasures are the ones that induce the Cantor topology, or shift-invariant Besicovitch, or Weyl pseudodistance (see [HM17, Def 4.1.1]. We will focus on the Besicovitch topology.

3.1 Definition

Let X and Y be nonempty sets and let (X_n) be a nondecreasing sequence of finite subsets of X. We may or may not require that (X_n) be *exhaustive*, that is, $\bigcup_n X_n = X$.

Let us denote $\mathfrak{P}(W|V) = \frac{|W \cap V|}{|V|}$ (by convention, this is $+\infty$ if $V = \emptyset$).

Remark 3.1.

1. $\mathfrak{P}(W \cup U|V) \leq \mathfrak{P}(W|V) + \mathfrak{P}(U|V)$, and the equality holds if the union is disjoint.
2. If $V \subset U$, then $\mathfrak{P}(V|U)\,\mathfrak{P}(W|V) = \mathfrak{P}(V \cap W|U) \leq \mathfrak{P}(W|U)$

The *Besicovitch submeasure* $\mu_{(X_n)} : 2^X \to [0,1]$ is defined by:

$$\mu_{(X_n)}(W) = \limsup_n \mathfrak{P}(W|X_n)\ .$$

The *Besicovitch pseudodistance* is $d_{(X_n)} = d_{\mu_{(X_n)}}$. For example, if $X = \mathbb{N}$, $Y = \{0,1\}$, and $X_n = [0 : n-1]$, $x(i) = 0$ for every $i \in \mathbb{N}$ and $y \in \{0,1\}^{\mathbb{N}}$ is the characteristic function of the prime numbers, then $d_{(X_n)}(x,y) = 0$.

The topology of the quotient space is very different from the prodiscrete (Cantor) topology.

Remark 3.2. Remark obviously two dual cases (in general we will be in the first case, but not the second one):

1. If every $U \Subset G$ appears finitely many times in (X_n), then $\mu_{(X_n)}(W) = 0$ if W is finite.

2. If all $U \Subset G$ appear (or, more generally, if for every n, cofinitely many $U \Subset G$ of cardinality n appear) in (X_n), then $\mu_{(X_n)}(W) = 1$ if W is infinite.

We will now concentrate on the nondecreasing case.

3.2 Connectedness

Theorem 3.3. *If (X_n) is nondecreasing and has unbounded cardinality, then the Besicovitch space is pathwise-connected.*

Proof. It is enough to build, for every $W \subset G$ and for every $\alpha \in [0, 1]$, a set $V(\alpha)$ such that:

1. if $0 \leq \alpha < \beta \leq 1$ then $V(\alpha) \subset V(\beta)$,

2. $V(0) = \emptyset$ and $V(1) = W \cap \bigcup_n X_n$ (the submeasure does not account for what is outside the union), and

3. $\mu(V(\alpha)) = \alpha\mu(W)$.

Let us assume that there is a total order on each $Y_n = X_n \setminus \bigcup_{i<n} X_i$ (independent of our α). Now define, inductively on $n \in \mathbb{N}$, the set $U_n \subset W \cap Y_n$ by taking the minimal $\lfloor \alpha \,|W \cap X_n|\rfloor - \sum_{i<n} |U_i|$ elements in $W \cap Y_n$. Provided that this definition is valid, it maintains, for all $n \in \mathbb{N}$, the property that $\sum_{i\leq n} |U_i| = \lfloor \alpha \,|W \cap X_n|\rfloor$. It is actually valid because, by induction,

$$\begin{aligned}
\lfloor \alpha \,|W \cap X_{n+1}|\rfloor - \sum_{i\leq n} |U_i| &= \lfloor \alpha \,|W \cap X_{n+1}|\rfloor - \lfloor \alpha \,|W \cap X_n|\rfloor \\
&< \alpha \left(|W \cap X_{n+1}| - |W \cap X_n|\right) + 1 \\
&< \alpha \left(|W \cap Y_{n+1}|\right) + 1 \\
&< |W \cap Y_{n+1}| + 1\,.
\end{aligned}$$

We now define $V(\alpha) = \bigsqcup_{n\in\mathbb{N}} U_n$. By construction, thanks to the common total order, we immediately get that $\alpha < \beta \implies V(\alpha) \subset V(\beta)$. Moreover,

$$\begin{aligned}
\mu(V(\alpha)) &= \limsup_{n\in\mathbb{N}} \mathfrak{P}\left(V(\alpha)|X_n\right) \\
&= \limsup_{n\in\mathbb{N}} \frac{\left|\bigsqcup_{i\leq n} U_i\right|}{|X_n|} \\
&= \limsup_{n\in\mathbb{N}} \frac{\lfloor \alpha \,|W \cap X_n|\rfloor}{|X_n|}
\end{aligned}$$

Since $\alpha \,|W \cap X_n| - 1 < \lfloor \alpha \,|W \cap X_n|\rfloor \leq \alpha \,|W \cap X_n|$ and $|X_n|$ is unbounded, we get that:

$$\mu(V(\alpha)) = \alpha\mu(W)\,. \quad \square$$

4 Følner equivalence and Besicovitch submeasures

4.1 Følner equivalence

Let (X_n) and (Y_n) be nondecreasing sequences of finite subsets of G. We say that they are *synchronously Følner-equivalent* if

$$\lim_{n\to\infty} \frac{|X_n \Delta Y_n|}{|X_n|} = 0 \ .$$

Proposition 4.1. *Consider nondecreasing sequences (X_n) and (Y_n). The following are equivalent.*

1. *(X_n) and (Y_n) are synchronously Følner-equivalent.*
2. *$|X_n \cap Y_n| \sim_{n\to\infty} |X_n| \sim_{n\to\infty} |Y|_n$.*
3. *$|X_n| \sim_{n\to\infty} |Y_n|$ and $|X_n \setminus Y_n| = o_{n\to\infty}(|X_n|)$.*

Proof.

$1 \Longrightarrow 2$ This follows from the inequalities $|X| \geq |X \cap Y| \geq |X| - |X\Delta Y|$ and $||X| - |Y|| \leq |X\Delta Y|$ which hold for every finite X and Y.

$2 \Longrightarrow 3$ Just note that $|X_n \setminus Y_n| = |X_n| - |X_n \cap Y_n|$.

$3 \Longrightarrow 1$ Note that:

$$\begin{aligned} |X_n \Delta Y_n| &= |X_n \setminus Y_n| + |Y_n \setminus X_n| \\ &= |X_n \setminus Y_n| + |Y_n| - |X_n \cap Y_n| \\ &= 2\,|X_n \setminus Y_n| + |Y_n| - |X_n| \\ &= o_{n\to\infty}(|X_n|) \ . \end{aligned}$$

Corollary 4.2. *Synchronous Følner equivalence is an equivalence relation.*

Proof.

- From Proposition 4.1, note that if (X_n) and (Y_n) are synchronously Følner-equivalent, then $\frac{|X_n \Delta Y_n|}{|Y_n|} \sim_{n\to\infty} \frac{|X_n \Delta Y_n|}{|X_n|} \to_{n\to\infty} 0$. So the relation is symmetric.
- Transitivity follows from Proposition 4.1 and the inclusion $X\Delta Z \subseteq (X\Delta Y) \cup (Y\Delta Z)$, which holds for every X, Y and Z.
- Reflexivity is trivial. □

Since the definition involves a lim (and not a lim inf), we immediately note the following.

Remark 4.3. (X_n) and (Y_n) are synchronously Følner-equivalent if and only if (X_{k_n}) and (Y_{k_n}) are synchronously Følner-equivalent, for every increasing sequence (k_n).

We also denote $(X_n) \preceq (Y_n)$ if (X_n) is synchronously Følner-equivalent to a subsequence (Y_{m_n}). Equivalently,

$$\lim_{n\to\infty} \min_{m\in\mathbb{N}} \frac{|X_n \Delta Y_m|}{|X_n|} = 0 \ .$$

To be convinced of the equivalence, note that the minimum is reached by some m_n for each $n \in \mathbb{N}$, because (Y_m) is nondecreasing and X_n is finite. Thanks to symmetry of synchronous equivalence, we also have that $(X_n) \preceq (Y_n)$ if and only if $\lim_{n\to\infty} \min_{m\in\mathbb{N}} \frac{|X_n \Delta Y_m|}{|Y_m|} = 0$. We say that they are *Følner-equivalent*, and write $(X_n) \sim (Y_n)$, if both $(X_n) \preceq (Y_n)$ and $(Y_n) \preceq (X_n)$. This is the case if they are synchronously Følner equivalent, but the converse is false. As counterexamples, one can consider twice the same sequence, but with repetitions on both sides that are longer and longer, and not synchronized. If one wants to obtain strictly increasing sequences, repetitions can be replaced by very slowly increasing sequences (point by point).

Remark 4.4.

1. $\preceq$ is a preorder relation.

2. Følner-equivalence is an equivalence relation.

Proof.

1. Reflexivity is obvious, and transitivity is not difficult.

2. Følner-equivalence is defined as the equivalence corresponding to a preorder, which is classical. □

Proposition 4.5. *Assume that* $|X_n| \sim_{n\to\infty} |Y_n|$.
Then (X_n) *and* (Y_n) *are synchronously Følner-equivalent if and only if* $(X_n) \preceq (Y_n)$.

Proof. Assume $(X_n) \preceq (Y_n)$ (the converse implication is trivial). Let $n, m \in \mathbb{N}$. If $m \le n$, then $|X_n \setminus Y_n| \le |X_n \setminus Y_m|$ and $|Y_n \setminus X_n| \le |Y_n \setminus Y_m| + |Y_m \setminus X_n|$ since (Y_n) is nondecreasing. Summing up, $|X_n \Delta Y_n| \le |X_n \Delta Y_m| + |Y_n \setminus Y_m|$. Symmetrically, if $n \le m$, $|X_n \Delta Y_n| \le |X_n \Delta Y_m| + |Y_m \setminus Y_n|$. Overall for every $m \in \mathbb{N}$, we get $|X_n \Delta Y_n| \le |X_n \Delta Y_m| + ||Y_m| - |Y_n||$. If we apply this with (m_n) the subsequence from the definition of $\preceq$, which is such that $(X_n) \sim (Y_{m_n})$, we have $|X_n \Delta Y_{m_n}| = o_{n\to\infty}(|X_n|)$, and by Proposition 4.1 (applied to (X_n) and (Y_{m_n})), $|Y_{m_n}| \sim_{n\to\infty} |X_n| \sim_{n\to\infty} |Y_n|$. Summing up, we deduce that $|X_n \Delta Y_n| = o_{n\to\infty}(|X_n|)$. □

4.2 Comparing Besicovitch submeasures

A basic tool in our set constructions will be the following elementary remark.

Remark 4.6. If (X_n) is nondecreasing and exhaustive, then for every finite set W and every $\varepsilon > 0$, there exists $n_{(X_n)}(W, \varepsilon)$ such that $\forall n \ge n_{(X_n)}(W, \varepsilon), \mathfrak{P}(W|X_n) < \varepsilon$ and $W \subset X_n$.

We deduce the following, which will be useful in our constructions.

Lemma 4.7. *Let* (X_n) *be a nondecreasing exhaustive sequence of an infinite group* G. *Let* $W = \bigcup_{i\in\mathbb{N}} W_i$ *where* $\emptyset \ne W_i \Subset G$ *for each* $i \in \mathbb{N}$, *such that, for every* $n \in \mathbb{N}$, *there are at most finitely many* i*'s such that* $W_i \cap X_n \ne \emptyset$ *(this is the case, for example, if the* W_i*'s are pairwise disjoint); in that case* $j_n = \max_{W_j \cap X_n \ne \emptyset} j$ *is well-defined for every* n. *Then:*

1. $$\mu_{(X_n)}(W) \ge \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \mathfrak{P}(W_i|X_m) \ .$$

2. *If there is a sequence* (ε_n) *converging to* 0 *such that* $\forall n \in \mathbb{N}$, $n_{(X_n)}(\bigcup_{i<j_n} W_i, \varepsilon_n) \le n$, *then:*
$$\mu_{(X_n)}(W) = \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \mathfrak{P}(W_i|X_m) \ .$$

3. *In general, there exists a nondecreasing integer sequence* $\mathbf{l}$ *such that, noting* $W_{\mathbf{l}} = \bigcup_{i\in\mathbb{N}} W_{l_i}$*:*

$$\mu_{(X_n)}(W_{\mathbf{l}}) = \lim_{i\to\infty} \max_{m\in\mathbb{N}} \mathfrak{P}\left(W_{l_i}|X_m\right) \ .$$

Proof.

1. Let $(m_i)_{i\in\mathbb{N}}$ be a sequence of integers such that $\mathfrak{P}\left(W_i|X_{m_i}\right) = \max_{m\in\mathbb{N}} \mathfrak{P}\left(W_i|X_m\right)$. We know that this sequence goes to infinity (even though it may not be nondecreasing), because only finitely many W_i's intersect each X_m, but they all intersect at least one. Hence, $\mu_{(X_n)}(W) = \limsup_{n\to\infty} \mathfrak{P}\left(W|X_n\right) \geq \limsup_{i\to\infty} \mathfrak{P}\left(W|X_{m_i}\right)$. We get the desired inequality by noting that $W_i \subset W$.

2. Point 1 already gives one inequality. For the converse:

$$\begin{aligned}
\mu_{(X_n)}(W) &= \limsup_{n\to\infty} \mathfrak{P}\left(\bigcup_{i<j_n} W_i \cup W_{j_n} \cup \bigcup_{i>j_n} W_i|X_n\right) \\
&\leq \limsup_{n\to\infty} \left(\mathfrak{P}\left(\bigcup_{i<j_n} W_i|X_n\right) + \mathfrak{P}\left(W_{j_n}|X_n\right) + \mathfrak{P}\left(\bigcup_{i>j_n} W_i|X_n\right)\right) \\
&\leq \limsup_{n\to\infty} \left(\varepsilon_n + \max_{m\in\mathbb{N}} \mathfrak{P}\left(W_{j_n}|X_m\right) + 0\right) \\
&\leq \limsup_{n\to\infty} \varepsilon_n + \limsup_{n\to\infty} \max_{m\in\mathbb{N}} \mathfrak{P}\left(W_{j_n}|X_m\right) \\
&\leq 0 + \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \mathfrak{P}\left(W_i|X_m\right) \ .
\end{aligned}$$

   The last inequality comes from the fact that the sequence (j_n) is nondecreasing (because (X_n) is nondecreasing), and not upper-bounded (because the W_i's are nonempty), so it goes to infinity.

3. Let us define some sequence $\mathbf{l}$ by recurrence, from any seed $l_0 \in \mathbb{N}$. Assume that l_n is defined, and write $k_n = n_{(X_n)}(\bigcup_{j\leq n} W_{l_j})$. Choose any l_{n+1} such that for every $m \geq l_{n+1}$, W_m does not intersect X_{k_n-1} (this is possible by assumption). If $j_n = \max_{W_{l_j}\cap X_n\neq\emptyset} j$, then $n_{(X_n)}(\bigcup_{j<j_n} W_{l_j}) = k_{j_n-1}$. By definition, $W_{l_{j_n}}$ does not intersect $X_{k_{j_n-1}-1}$. Since $W_{l_{j_n}}$ intersects X_n, we can deduce that $n > k_{j_n-1} - 1$. This means that (W_{l_i}) satisfies the hypothesis of Point 2.
   Replacing the lim sup by a lim can be achieved by taking, again, a subsequence. ☐

Lemma 4.8. *Let* $\varepsilon, \delta > 0$*, and* (X_n)*,* (Y_n) *be nondecreasing and exhaustive. The following are equivalent.*

1. *For every* $W \subset G$*, if* $\mu_{(Y_n)}(W) \geq \varepsilon$*, then* $\mu_{(X_n)}(W) \geq \delta$*.*

2. $\liminf_{n\in\mathbb{N}} \max_{m\in\mathbb{N}} \dfrac{\varepsilon\,|Y_n| - |Y_n \setminus X_m|}{|X_m|} \geq \delta$*.*

If m_n *realizes the maximum for each* $n \in \mathbb{N}$*, and if* $\varepsilon < 1$*, then these properties imply that*

$$\frac{\delta}{\varepsilon} \leq \liminf_{n\in\mathbb{N}} \frac{|Y_n|}{|X_{m_n}|} \leq \limsup_{n\in\mathbb{N}} \frac{|Y_n|}{|X_{m_n}|} \leq \frac{1-\delta}{1-\varepsilon} \ .$$

In particular, the properties imply that $\delta \leq \varepsilon$.

Proof.

- Let us start by proving the final inequalities. Suppose $\liminf_{n\in\mathbb{N}} \frac{\varepsilon|Y_n|-|Y_n\setminus X_{m_n}|}{|X_{m_n}|} \geq \delta$. Then on the one hand, it is clear that $\liminf_{n\in\mathbb{N}} \frac{\varepsilon|Y_n|}{|X_{m_n}|}$ is even bigger, which gives the first inequality. On the other hand, since $|Y_n \setminus X_{m_n}| \geq |Y_n| - |X_{m_n}|$, we can see that $\liminf_{n\in\mathbb{N}}(\varepsilon-1)\frac{Y_n}{X_{m_n}}+1 \geq \liminf_{n\in\mathbb{N}} \frac{\varepsilon|Y_n|-|Y_n\setminus X_{m_n}|}{|X_{m_n}|} \geq \delta$, which gives that $\limsup_{n\in\mathbb{N}} \frac{|Y_n|}{|X_{m_n}|} \leq \frac{1-\delta}{1-\varepsilon}$, provided that $\varepsilon < 1$.

2⇒1

$$\begin{aligned}
\mu_{(X_n)}(W) &= \limsup_{m\to\infty} \mathfrak{P}\left(W|X_m\right) \\
&\geq \limsup_{n\to\infty} \mathfrak{P}\left(W|X_{m_n}\right) \\
&\geq \limsup_{n\to\infty} \frac{|W\cap Y_n\cap X_{m_n}|}{|X_{m_n}|} \\
&= \limsup_{n\to\infty} \frac{|W\cap Y_n| - |W\cap Y_n\setminus X_{m_n}|}{|X_{m_n}|} \\
&\geq \limsup_{n\to\infty} \frac{|W\cap Y_n| - |Y_n\setminus X_{m_n}|}{|X_{m_n}|} \\
&= \limsup_{n\to\infty} \left(\frac{\varepsilon\,|Y_n| - |Y_n\setminus X_{m_n}|}{|X_{m_n}|} + \frac{|W\cap Y_n| - \varepsilon\,|Y_n|}{|Y_n|}\frac{|Y_n|}{|X_{m_n}|}\right) \\
&\geq \liminf_{n\to\infty} \frac{\varepsilon\,|Y_n| - |Y_n\setminus X_{m_n}|}{|X_{m_n}|} + \left(\limsup_{n\to\infty}\frac{|W\cap Y_n|}{|Y_n|} - \varepsilon\right)\liminf_{n\in\mathbb{N}}\frac{|Y_n|}{|X_{m_n}|} \\
&\geq \delta + 0\frac{\delta}{\varepsilon} \text{ by the two premises and the first inequalities.}
\end{aligned}$$

1⇒2 Assume that $\liminf_{i\to\infty} \frac{\varepsilon|Y_i|-|Y_i\setminus X_{k_i}|}{|X_{k_i}|} < \delta$. Let us build a set W that contradicts Point 1. For each $n\in\mathbb{N}$, there exists $k_n = \min\left\{k|\,|Y_n\setminus X_k| \leq \varepsilon\,|Y_n|\right\}$, because for large k, $Y_n\setminus X_k = \emptyset$ (because (X_k) is exhaustive and Y_n is finite). By noting that $(Y_n\cap X_{k_n})\setminus X_{k_n-1} = (Y_n\setminus X_{k_n-1})\setminus(Y_n\setminus X_{k_n})$ (by convention X_{-1} is empty), we can write that $|(Y_n\cap X_{k_n})\setminus X_{k_n-1}| = |Y_n\setminus X_{k_n-1}| - |Y_n\setminus X_{k_n}|$, which is bigger than $\varepsilon\,|Y_n| - |Y_n\setminus X_{k_n}|$, by minimality of k_n. Hence $(Y_n\cap X_{k_n})\setminus X_{k_n-1}$ admits a subset Z_n of cardinality $|Z_n| = \lfloor\varepsilon\,|Y_n|\rfloor - |Y_n\setminus X_{k_n}|$. Define $W_n = (Y_n\setminus X_{k_n})\bigsqcup Z_n$. Note that $W_n\subset Y_n$, and that $\varepsilon - \frac{1}{|Y_n|} < \mathfrak{P}\left(W_n|Y_n\right) \leq \varepsilon$. The W_i satisfy the hypotheses of Lemma 4.7, so that Point 3 gives an integer sequence $\mathbf{l}$, with $\mu_{(X_n)}(W_{\mathbf{l}}) = \lim_{i\to\infty}\max_{m\in\mathbb{N}}\mathfrak{P}\left(W_{l_i}|X_m\right)$. By construction, we have:

$$\begin{aligned}
\mathfrak{P}\left(W_i|X_m\right) &= \mathfrak{P}\left(Y_i\setminus X_{k_i}|X_m\right) + \mathfrak{P}\left(Z_i|X_m\right) \\
&= \frac{|Y_i\cap X_m\setminus X_{k_i}| + |Z_i\cap X_m|}{|X_m|}\ .
\end{aligned}$$

If $m < k_i$, then $X_m\subseteq X_{k_i}$, and $Z_i\cap X_m\subseteq Z_i\cap X_{k_i-1} = \emptyset$, so that this quantity is 0. On the contrary, if $m\geq k_i$, then $Z_i\subseteq X_{k_i}\subseteq X_m$, and $Y_i\cap X_m\setminus X_{k_i} = (Y_i\setminus X_{k_i})\setminus(Y_i\setminus X_m)$, so that:

$$\begin{aligned}
\mathfrak{P}\left(W_i|X_m\right) &= \frac{|Y_i \cap X_m \setminus X_{k_i}| + |Z_i \cap X_m|}{|X_m|} \\
&= \frac{|Y_i \cap X_m \setminus X_{k_i}| + |Z_i|}{|X_m|} \\
&= \frac{|Y_i \setminus X_{k_i}| - |Y_i \setminus X_m| + \lfloor \varepsilon \, |Y_i| \rfloor - |Y_i \setminus X_{k_i}|}{|X_m|} \\
&\leq \max_{m \in \mathbb{N}} \frac{|\lfloor \varepsilon \, |Y_i| \rfloor| - |Y_i \setminus X_m|}{|X_m|} \\
&< \delta \text{ by hypothesis.}
\end{aligned}$$

Taking the limit, we get that $\mu_{(X_n)}(W_{\mathrm{I}}) < \delta$.
On the other hand, applying now Point 1 of Lemma 4.7 to sequence (Y_n):

$$\mu_{(Y_n)}(W_{\mathrm{I}}) \geq \lim_{i \in \mathbb{N}} \max_{m \in \mathbb{N}} \mathfrak{P}\left(W_{l_i}|Y_m\right) \geq \mathfrak{P}\left(W_{l_i}|Y_{l_i}\right) = \varepsilon \,. \quad \square$$

The previous lemma now allows to characterize the main properties of interest for comparing two Besicovitch submeasures.

Proposition 4.9. *Let (X_n) and (Y_n) be nondecreasing and exhaustive.*

1. *$\mu_{(Y_n)}$ is λ-Lipschitz with respect to $\mu_{(X_n)}$, where $\lambda > 0$, if and only if*

$$\forall \varepsilon > 0, \liminf_{n \to \infty} \max_{m \in \mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon} |Y_n \setminus X_m|}{|X_m|} \geq \frac{1}{\lambda} \;.$$

2. *$\mu_{(Y_n)}$ is absolutely continuous with respect to $\mu_{(X_n)}$ if and only if it is Lipschitz.*
3. *$\mu_{(Y_n)} \leq \mu_{(X_n)}$ if and only if $(Y_n) \preceq (X_n)$.*
4. *$\mu_{(Y_n)} = \mu_{(X_n)}$ if and only if $(Y_n) \sim (X_n)$.*

One can even see from the proof that $(Y_n) \preceq (X_n)$ if and only if there exists $\varepsilon \in]0,1[$ such that $\forall W \subset G, \mu_{(X_n)}(W) < \varepsilon \implies \mu_{(Y_n)}(W) < \varepsilon$.

Proof.

1. Just note that the λ-Lipschitz property of $\mu_{(Y_n)}$ is equivalent to the properties in Lemma 4.8, for every δ and $\varepsilon = \lambda\delta$, and hence to:

$$\liminf_{n \in \mathbb{N}} \max_{m \in \mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon} |Y_n \setminus X_m|}{|X_m|} \geq \frac{1}{\lambda} \;.$$

2. From Lemma 4.8, $\mu_{(Y_n)}$ is absolutely continuous with respect to $\mu_{(X_n)}$ if and only if

$$\forall \varepsilon > 0, \liminf_{n \to \infty} \max_{m \in \mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon} |Y_n \setminus X_m|}{|X_m|} > 0 \;.$$

From Point 1, this is equivalent to the existence of some λ such that $\mu_{(Y_n)}$ is λ-Lipschitz with respect to $\mu_{(X_n)}$.

3. Consider a sequence (m_n) which witnesses that $(Y_n) \preceq (X_n)$: $\lim_{n\to\infty} \frac{|Y_n \Delta X_{m_n}|}{|Y_n|} = 0$. Then

$$\lim_{n\in\mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon}|Y_n \setminus X_{m_n}|}{|X_{m_n}|} = \lim_{n\in\mathbb{N}} \frac{|Y_n|}{|X_{m_n}|}\left(1 - \frac{1}{\varepsilon}\lim_{n\in\mathbb{N}} \frac{|Y_n \setminus X_{m_n}|}{|Y_n|}\right)$$
$$= 1 \ .$$

We can conclude by Point 1.
Conversely, suppose that

$$\liminf_{n\in\mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon}|Y_n \setminus X_{m_n}|}{|X_{m_n}|} \geq 1 \ .$$

By the last inequalities in Lemma 4.8, we know that $\lim_{n\in\mathbb{N}} \frac{|Y_n|}{|X_{m_n}|} = 1$. Moreover,

$$\lim_{n\to\infty} \frac{|Y_n \setminus X_{m_n}|}{|X_{m_n}|} \leq \lim_{n\to\infty} \frac{\varepsilon|Y_n|}{|X_{m_n}|} - \varepsilon \liminf_{n\in\mathbb{N}} \frac{|Y_n| - \frac{1}{\varepsilon}|Y_n \setminus X_{m_n}|}{|X_{m_n}|} \qquad = \varepsilon - \varepsilon = 0 \ .$$

By Point 3 of Proposition 4.1, we obtain that $(Y_n) \preceq (X_n)$.

4. This is direct from the definitions and the Point 3. □

The following is direct from Proposition 4.9 and Remark 2.3.

Corollary 4.10. *Let (X_n) and (Y_n) be nondecreasing and exhaustive. Then $(Y_n) \preceq (X_n)$ (resp. $(Y_n) \sim (X_n)$) if and only if the identity map from space A^G endowed with $d_{(X_n)}$ onto space A^G endowed with $d_{(Y_n)}$ is 1-Lipschitz (resp. an isometry).*

Here are particular classes of sequences, where the proposition can be applied.

Corollary 4.11. *Let (X_n) and (Y_n) be nondecreasing and exhaustive.*

1. *If there exist a real number $\lambda > 0$ and a sequence (m_n) such that $\liminf_{n\to\infty} \mathfrak{P}(X_n|Y_{m_n}) \geq \frac{1}{\lambda}$ and $X_n \subset Y_{m_n}$, then $\mu_{(X_n)}$ is λ-Lipschitz with respect to $\mu_{(Y_n)}$.*
2. *If for cofinitely many $n \in \mathbb{N}$, $Y_n \subset X_{n+1}$ and $\liminf_{n\to\infty} \mathfrak{P}(X_n|X_{n+1}) \geq \lambda$, then $\mu_{(X_n)}$ is λ-Lipschitz with respect to $\mu_{(Y_n)}$.*
3. *On the other hand, if $|X_n| \sim_{n\to\infty} |Y_n|$ but (X_n) and (Y_n) are not (synchronously) Følner-equivalent , and $n_{(Y_m)}(X_n, \varepsilon_n) = n+1$ for some real sequence (ε_n) converging to 0, then $\mu_{(X_n)}$ is not absolutely continuous with respect to $\mu_{(Y_n)}$.*

Proof.

1. For every $\varepsilon > 0$,

$$\liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_m|}{|Y_m|} \geq \liminf_{n\to\infty} \frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_{m_n}|}{|Y_{m_n}|} = \liminf_{n\to\infty} \frac{|X_n|}{|Y_{m_n}|} \geq \frac{1}{\lambda} \ .$$

2. Apply Point 1 with $m_n = \min\{m \in \mathbb{N} | X_n \subset Y_m\}$; the hypothesis is that m_n is ultimately $n+1$.

3. Suppose $|X_n| \sim_{n\to\infty} |Y_n|$ and (X_n) and (Y_n) are not synchronously Følner-equivalent. By Proposition 4.5, $(X_n) \not\preceq (Y_n)$, that is, $\varepsilon = \limsup_{n\to\infty} \frac{|X_n \setminus Y_n|}{|Y_n|} > 0$. We can write $\liminf_{n\to\infty} \frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_n|}{|Y_n|} = 0$.
By the second assumption, for every $m > n$, $X_n \setminus Y_m = \emptyset$ and $\frac{|X_n|}{|Y_m|} \leq \varepsilon_n$. We get:
$$\max_{m\in\mathbb{N}} \frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_m|}{|Y_m|} \leq \max\left(\frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_n|}{|Y_n|}, \varepsilon_n\right).$$
Putting things together, $\liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{|X_n| - \frac{1}{\varepsilon}|X_n \setminus Y_m|}{|Y_m|}$ is 0. We conclude by Point 2 of Proposition 4.9. □

Corollary 4.12. *Let (X_n) and (Y_n) be nondecreasing and exhaustive. Assume that $|X_n| \sim_{n\to\infty} |Y_n|$. Then the following are equivalent.*

1. *(X_n) and (Y_n) are synchronously Følner-equivalent.*
2. *$\mu_{(Y_{l_n})} = \mu_{(X_{l_n})}$, for every increasing sequence $(l_n) \in \mathbb{N}^{\mathbb{N}}$.*
3. *$\mu_{(Y_{l_n})}$ is absolutely continuous with respect to $\mu_{(X_{l_n})}$, for every increasing sequence (l_n).*

Proof.

$1 \Longrightarrow 2$ By Remark 4.3, synchronous Følner equivalence is transmitted to all subsequences (provided that one takes the same subsequence for (X_n) and for (Y_n)). We conclude thanks to Proposition 4.9.

$2 \Longrightarrow 3$ This is obvious.

$\not 1 \Longrightarrow \not 3$ If (X_n) and (Y_n) are not synchronously Følner-equivalent, then there exists an infinite set $I \subset \mathbb{N}$ and a real number $\alpha > 0$ such that $\forall n \in I$, $\frac{|X_n \Delta Y_n|}{|X_n|} \geq \alpha$. This implies that for every increasing sequence $(l_n) \in I^{\mathbb{N}}$, (X_{l_n}) and (Y_{l_n}) are not synchronously Følner-equivalent. We can take an increasing sequence $(l_n) \in I^{\mathbb{N}}$ such that $n_{(Y_m)}(X_{l_n}, \varepsilon_{l_n}) = l_{n+1}$, for some real sequence (ε_n) converging to 0. Then (X_{l_n}) and (Y_{l_n}) satisfy the assumptions for Point 3 of Corollary 4.11. □

4.3 Shift

If G is a group and $(X_n) \sim (gX_n)$, then we say that (X_n) is *(left) g-Følner*. Since $|X_n| = |gX_n|$, Proposition 4.5 says that it is enough to require $(X_n) \preceq (gX_n)$, and that in this case, (X_n) and (gX_n) are synchronously Følner-equivalent.

A *(left) Følner sequence* for a countable group G is a g-Følner sequence for every $g \in G$. A countable group is *amenable* if and only if it admits a Følner sequence: cf. [CSC10, Chapter 4], in particular for many alternative definitions.

A group G is *finitely generated* (briefly, f.g.) if $E \Subset G$ exists such that for every $g \in G$ there exists $e_1, \ldots, e_n \in E \cup E^{-1}$ such that $e_1 \cdots e_n = g$. Remarkably (cf. [Pet, Lemma 5.3]) if a f.g. group is amenable, then it has a nondecreasing exhaustive Følner sequence. In addition, if the size of the balls grows polynomially with the radius, then they form a Følner sequence, so Point 3 of Corollary 4.13 generalizes [HM17, Cor 4.1.4].

The following is a rephrasing of Corollary 4.10.

Corollary 4.13. *Let G be a countable group and let (X_n) be a nondecreasing exhaustive sequence.*

1. (X_n) *is* g*-Følner if and only if* $\mu_{(X_n)} = \mu_{(g^{-1}X_n)}$ *if and only if the shift by* g *is an isometry.*

2. (X_n) *is Følner if and only if every shift is an isometry.*

3. *If* G *is finitely generated, then* G *is amenable if and only if there exists a nondecreasing sequence* (X_n) *of finite subsets of* G *such that every shift is an isometry.*

Note that one implication of Point 3 was already stated in [Cap09, Theorem 3.5], but the proof contains a confusion between left and right Følner. The full equivalence generalizes [HM17, Cor 4.1.4] (since balls are Følner in polynomial-growth groups).

Corollary 4.14. *Let* G *be a finitely generated group.*

1. *If* (X_n) *is the sequence of balls with respect to some generating set of cardinality* α*, then every shift is* α*-Lipschitz.*

2. *If* $g \in G$*, a nondecreasing exhaustive sequence is* g*-Følner if and only if all of its subsequences yield a Besicovitch pseudodistance for which the shift by* g *is continuous.*

3. G *is amenable if and only if it admits a nondecreasing exhaustive sequence of finite subsets of which all subsequences yield a Besicovitch distance for which every shift is continuous.*

The first point generalizes [HM17, Prop 4.1.3]. Note that it still applies in nonamenable groups, but the shifts are no longer isometries, and there is a subsequence of balls with respect to which the Besicovitch pseudodistance makes them non-continuous.

We remark (cf. [dlH00, VII.34]) that the sequence of balls is Følner if and only if the group has polynomial growth, and has a Følner subsequence if and only if it has subexponential growth.

Proof.

1. If E is the generating set and E_n the corresponding radius-n ball, then $E_0 = \{e\}$ where e is the identity of G and $E_{n+1} = (E \cup E^{-1}) \cdot E_n$, so $|E_n| \leq (2|E| + 1)^n$. We can apply Point 2 of Corollary 4.11.

2. This comes from Corollary 4.12.

3. This comes from Point 2 and the characterization of amenability through Følner sequences. $\square$

There are nondecreasing non-Følner sequences for which the shift is Lipschitz (but not an isometry) in $\mathbb{Z}^d$. Here's an example: $X_n = (\llbracket -n, n \rrbracket \cup 2\,\rrbracket -n, n\llbracket)^d$. Indeed, for every n, $1 + X_n \subset X_{2n}$ and $\frac{|X_{2n}|}{|X_n|} = \frac{(8n-1)^d}{(4n-1)^d}$, which converges to 2^d when n goes to infinity. We conclude by Point 1 of Corollary 4.11, with $m_n = 2n$ and $\alpha = 2^d$. But the shift is not an isometry because the sequence is not Følner: $\mu((2\mathbb{Z})^d) = 2^d/3^d > \mu((2\mathbb{Z}+1)^d) = 1/3^d$.

4.4 Propagations and right Følner sequences

Let G be a group and let $g \in G$. A sequence (X_n) of finite subsets of G is *right* g*-Følner* if $(X_n) \sim (X_n g)$; equivalently, if (X_n^{-1}) is left g^{-1}-Følner. A *right Følner sequence* is then a sequence which is right g-Følner for every $g \in G$.

Let now A be an alphabet. The *propagation* in direction $g \in G$ is the function $\pi^g : A^G \to A^G$ defined by $\pi^g(x)(i) = x(ig^{-1})$ for every $x \in A^G$ and $i \in G$. With this definition, the value of $\pi^g(x)$ at point ig equals the value of x at point i: that is, the information moves in direction g. Points 1, 2 and 3 of Corollary 4.13 can then be *dualized* to right Følner sequences and propagations:

Corollary 4.15. *Let G be a group.*

1. *A nondecreasing exhaustive sequence (X_n) is right g-Følner if and only if $\mu_{(X_n)} = \mu_{(X_n g^{-1})}$ if and only if the propagation in direction g is an isometry.*

2. *A nondecreasing exhaustive sequence is right Følner if and only if every propagation is an isometry.*

3. *A finitely generated group is amenable if and only if there exists a nondecreasing exhaustive sequence (X_n) of finite subsets of G such that every propagation is an isometry.*

Proof. The first equivalence of Point 1 is immediate. For the other one, given $x \in A^G$, let $\overline{x}(i) = x(i^{-1})$ for every $x \in A^G$ and $i \in G$: then for every $x, y \in A^G$ and $g \in G$ it is $\overline{\overline{x}} = x$, $\Delta(\overline{x}, \overline{y}) = (\Delta(x, y))^{-1}$ and $\overline{\pi^g(x)} = \sigma^g(\overline{x})$, thus also $d_{(X_n)}(x, y) = d_{(X_n^{-1})}(\overline{x}, \overline{y})$ and $d_{(X_n)}(\pi^g(x), \pi^g(y)) = d_{(X_n^{-1})}(\sigma^g(\overline{x}), \sigma^g(\overline{y}))$, so that the propagation in direction g is an isometry for $d_{(X_n)}$ if and only if the shift by g is an isometry for $d_{X_n^{-1}}$. Points 2 and 3 follow easily. □

4.5 Block maps

A *block map* on a group G with *source alphabet* A, *target alphabet* B, *neighborhood* $N = \{j_1, \ldots, j_k\}$ and *local rule* $\phi : A^k \to B$ is a function $F : A^G \to B^G$ defined as the synchronous application of ϕ at the "N-shaped neighborhood" of each point of the group: that is, for every $x \in A^G$ and $i \in G$, $F(x)(i) = \phi(x(ij_1), \ldots, x(ij_n))$. By the *Curtis-Lyndon-Hedlund theorem* [Hed69] (see also [CSC10, Chapter 1]), block maps are all and only those functions from A^G to B^G which are continuous in the prodiscrete topology and commute with all the shifts. Every propagation is a block map, but the shift by $g \in G$ is a block map if and only if g is *central* in G, that is, $gh = hg$ for every $h \in G$; in this case, $\sigma^g = \pi^{g^{-1}}$. Note that the local rule ϕ itself can be identified with a block map Φ with source alphabet A^k, target alphabet B, and neighborhood $N = \{e\}$, where e is the identity element of G. Such a function is surely 1-Lipschitz, but not necessarily an isometry: for example, ϕ could be constant.

Block maps can be defined equivalently as follows. For $f_1, \ldots, f_k : A^G \to A^G$ define the *product* $f = f_1 \times \ldots \times f_k : A^G \to (A^k)^G$ by $f(x)(i) = (f_1(x)(i), \ldots, f_k(x)(i))$ for every $x \in A^G$ and $i \in G$. Then a block map F with source alphabet A, target alphabet B, neighborhood $N = \{j_1, \ldots, j_k\}$ and local rule ϕ has the form $F = \Phi \circ (\pi^{j_1^{-1}} \times \ldots \times \pi^{j_k^{-1}})$, where Φ is as in the previous paragraph.

Lemma 4.16. *Suppose $f_1, \ldots, f_k : A^G \to A^G$ are such that f_q is α_q-Lipschitz with respect to $d_{(X_n)}$. Then $f = f_1 \times \ldots \times f_k : A^G \to (A^k)^G$ is $(\sum_{q=1}^k \alpha_q)$-Lipschitz with respect to $d_{(X_n)}$. In particular, if $\alpha_q = \alpha$ for every $q \in [1 : k]$, then f is $k\alpha$-Lipschitz, and if each f_q is an isometry, then f is k-Lipschitz.*

Proof. For every $x, y \in A^G$ and $i \in G$, we have $f(x)(i) \neq f(y)(i)$ if and only if $f_q(x)(i) \neq f_q(y)(i)$ for at least one $q \in [1 : k]$, that is, $\Delta(f(x), f(y)) = \bigcup_{q=1}^k \Delta(f_q(x), f_q(y))$. Consequently, for every $n \geq 1$, we have $|\Delta(f(x), f(y)) \cap X_n| \leq \sum_{q=1}^k |\Delta(f_i(x), f_i(y)) \cap X_n|$, and the thesis follows easily. □

The composition of an α-Lipschitz function with a β-Lipschitz function is an $\alpha\beta$-Lipschitz function. As every propagation is a block map, Lemma 4.16 and Point 2 of Corollary 4.15 allow us to dualize Points 2 and 3 of Corollary 4.14 and recover (cf. [Cap09, Theorem 3.7] and [Cap11, Theorem 18]) the following characterization.

Corollary 4.17.

1. *A nondecreasing exhaustive sequence of finite subsets of a group G is right Følner if and only if all of its subsequences yield a Besicovitch pseudodistance for which every block map is continuous.*

2. *A finitely generated group is amenable if and only if it admits a nondecreasing exhaustive sequence (X_n) of finite subsets such that, for every $k \geq 1$ and every increasing $(l_n) \in \mathbb{N}^{\mathbb{N}}$, every block map with neighborhood size k is k-Lipschitz with respect to $d_{(X_{l_n})}$.*

5 Conclusions

We have presented a way to compare Besicovitch submeasures (in terms of absolute continuity, Lipschitz continuity, equality) thanks to the sequences of finite sets which describe them. In a shift space (with respect to a countable group) endowed with the Besicovitch topology, we have derived conditions on the defining sequence for the shift maps to be continuous, Lipschitz or isometric. As part of this, we gave another characterization of amenable groups.

Possible future work could involve extension to configuration spaces on possibly uncountable groups. This would require the use of the more general notions of *directed set* and of *net*, and although the definition of Besicovitch pseudodistance and submeasure would be immediate to extend, the techniques used to prove the main lemmas could need a major revision.

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