# Adaptive Quantum Computation, Constant Depth Quantum Circuits and Arthur-Merlin Games 

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#### Abstract

We present evidence that there exist quantum computations that can be carried out in constant depth, using 2-qubit gates, that cannot be simulated classically with high accuracy. We prove that if one can simulate these circuits classically efficiently then BQP $\subseteq$ AM.


## 1 Introduction

The idea of quantum teleportation $\mathrm{BBC}^{+} 93$, transferring a quantum state by dual usage of classical measurement data and quantum entanglement, has found a profound application in quantum computation. It has been understood by Gottesman and Chuang [GC99] that not only states, but also quantum gates, can be teleported. This observation has given rise to a new paradigm for quantum computation, which we will refer to as adaptive quantum computation.

In adaptive quantum computation, the outcomes of measurements performed throughout the course of the computation determine the quantum gates that are subsequently performed on the quantum registers ${ }^{1}$; the quantum program is "adapted" to the classical measurement data.

Nonadaptive quantum computation, which we explore in this paper, is a new computational model derived from the adaptive scheme. In this model we introduce a 'guess' bit string $g ; g$ is a prior guess of the outcome of all of the quantum measurements performed in the course of the adaptive quantum computation. This guess can be compared with the actual quantum measurement outcomes (which can now be deferred to the end of the computation, since no quantum gate operations depend on them). In the (rare) case that all these outcomes agree with the guess, the quantum computation can be called successful, and we know when it occurs. This suggests such quantum circuits could be hard to simulate classically, i.e. if we were able to consider all possible

[^0]outcomes in our classical simulation, -including the rare one that corresponds to having guessed correctly- , then our classical simulation would simulate the output of a polynomial-time quantum computer on the remaining output bits.

On the other hand, the interesting feature of nonadaptive models is that it can lead to quantum circuits of restricted depth. The adaptive model by Gottesman and Chuang (and similarly the one in [RB01) in which two-qubit gates are 'teleported' into the circuit and single-qubit gates are performed normally, maps to a nonadaptive model of constant depth (Lemma (1). That is, the resulting quantum circuit can be implemented in a constant number of time steps, which seems to make it very weak. The idea of computation by teleportation is also the basis for the Knill-Laflamme-Milburn proposal of quantum computation by linear optics KLM01 (see also the scheme by Gottesman, Kitaev and Preskill GKP01]). A nonadaptive version of the Knill-LaflammeMilburn construction also exists; however, because this model, involving a quantum circuit with passive linear optics applied to single photon states and followed by photon counting measurements, does not conform to the standard qubit model, the resulting nonadaptive circuit is not of constant depth. It could be of reduced depth, for example logarithmic depth, but this has not been proved.

The fact that nonadaptive parallelized quantum models can be of constant depth makes it likely (although not certain) that such models have no (quantum) computational power whatsoever. One may also expect that they can be simulated efficiently by a classical algorithm. However, we will provide evidence in this paper that, in fact, it may be hard to simulate these models classically. The results are the following:

1) We imagine that there is style of classical simulation that is powerful enough to follow any computational pathway, regardless of its probability of occurrence. We call this type of simulation by density computation. At the end of the paper we prove that if such a simulation of the constant depth quantum circuits is possible then $\mathrm{BPP}=\mathrm{BQP}$ and also the polynomial hierarchy collapses (Corollary (2).
2) Even if the simulation does not have this extended power, there is another well-known technique of endowing such a simulation with greater power; if an all-powerful 'Merlin' can direct the use of the simulation by 'Arthur', then Merlin can steer the simulation to the rare, successful cases. If such a simulation of constant depth quantum circuits were possible, it would mean that polynomial-time classical circuits, assisted by Merlin, would necessarily be able to simulate any polynomial-time quantum circuit. In complexity-theoretic language, this would imply the containment $\mathrm{BQP} \subseteq \mathrm{AM}$ (AM='Arthur-Merlin') (Theorem 2 2 ).

### 1.1 Previous work

Constant-depth quantum circuits, but with gates that have arbitrary fan-in, have been studied previously, see for example Ref. GHMP02. The constant depth circuits that we consider here have fan-in at most two. In the setting of classical boolean circuits, there are arguments that use a reduction to a constant-depth model by enumeration of the inputs and outputs of each gate in the circuit Hås87; these arguments are somewhat akin to the ones we use here.

## 2 Definitions

### 2.1 Basic Concepts

We use the following standard notation. All strings are over the alphabet $\Sigma=\{0,1\}, \Sigma^{*}$ is the set of all finite strings and $|x|$ denotes the length of the string $x$.

Definition 1 (Quantum Register $Q R(w)$ ). A quantum register of $w$ qubits is a tensor product of 2 dimensional 'qubit' Hilbert spaces $\mathcal{H}_{2}, \mathcal{H}=\mathcal{H}_{2}^{\otimes w}$ of total dimension $2^{w}$. Allowed states are unit-norm vectors $|\psi\rangle$ in this space. The complex inner product in this space is denoted by $\langle v \mid w\rangle=$ $\sum_{i} v_{i}^{*} w_{i}$ where $|v\rangle=\sum_{i} v_{i}|i\rangle$ and $i$ is $w$-bit string. The standard (computational) basis is formed by w-bit strings $|i\rangle=\left|i_{1}, i_{2}, i_{3}, \ldots, i_{w}\right\rangle$.

Definition 2 (Quantum Gate). The action of a general one-qubit gate is described by an element of $S U(2)$, which is applied to the vector describing the state of that qubit. The action of a general two-qubit gate is described by an element of SU(4), which is applied to the vector describing the state of that pair of qubits. It is understood that if a one- or two-qubit gate is applied to particular qubit(s) of a register $Q R(w)$, the unitary transformation on the full $2^{w}$-dimensional state vector is obtained by a tensor product of the gate action on the specified qubit(s) with the identity operation on all other qubits.

Definition 3 (Quantum Measurement $\mathcal{M}$ in the Standard Basis). A quantum measurement in the standard basis is an operation applied to one qubit in a quantum register. If the number of qubits in the register is $w$, then the output of the measurement is a single bit $b \in \Sigma$ and a smaller quantum register, of size $w-1$, consisting of the unmeasured qubits. The bit $b$ occurs with probability $p(b)=\operatorname{Tr}(I \otimes|b\rangle\langle b||\psi\rangle\langle\psi|)$. Here $I$ is the identity operator on the $2^{w-1}$-dimensional Hilbert space of the unmeasured qubits. When the measurement outputs the bit $b$, the new state of the unmeasured qubits is $\left|\psi_{b}\right\rangle=\sum_{b^{\prime} \in \Sigma^{w-1}}\left|b^{\prime}\right\rangle\left\langle b^{\prime}, b \mid \psi\right\rangle / \sqrt{p(b)}$.

Measurements $\mathcal{M}$ on different qubits commute, so we may consider a composite standard measurement on $k$ qubits with output bit string $b$ as $k$ measurements applied to each of these qubits. Two-qubit measurements can be considered as a composite of some two-qubit quantum gate applied to the pair of qubits to be measured, immediately followed by a standard measurement $\mathcal{M}$ on the two qubits. Of particular importance is the two-qubit measurement called a Bell measurement. For this, the two qubit unitary transformation preceding $\mathcal{M}$ is of the form $U=|00\rangle\left\langle\Psi^{+}\right|+|01\rangle\left\langle\Psi^{-}\right|+|10\rangle\left\langle\Phi^{+}\right|+|11\rangle\left\langle\Phi^{-}\right|$, where the Bell states are defined as $\left|\Phi^{ \pm}\right\rangle=\frac{1}{\sqrt{2}}(|00\rangle \pm|11\rangle)$ and $\left|\Psi^{ \pm}\right\rangle=\frac{1}{\sqrt{2}}(|01\rangle \pm|10\rangle)$.

Definition 4 (Quantum Circuit $Q C_{x}\left(w, w^{\prime}, \mid\right.$ init $\left.\rangle, d\right)$ ). A quantum circuit, determined by bit string $x$, of input width $w$, depth $d$, and classical output width $w^{\prime} \leq w$ consists of a quantum register $Q R(w)$ set initially (called time-step 0) to Hilbert-space vector $\mid$ init $\rangle$. In each time-step 1 through $d-1$ the quantum register is acted upon by one qubit or two qubit gates. Time-step $d$ consists of a standard quantum measurement $\mathcal{M}$ applied to some subset of $w^{\prime} \leq w$ of the qubits; the output of the quantum circuit is a bit string of length $w^{\prime}$ and a $w-w^{\prime}$-qubit quantum state.

We will often abbreviate our notation for a quantum circuit to $Q C(\mid$ init $\rangle, d)$. Quantum circuits can be composed:

$$
\begin{equation*}
Q C_{x_{2}}\left(w-w^{\prime}, w^{\prime \prime},\left|\psi_{b}\right\rangle, d_{2}\right) * Q C_{x_{1}}\left(w, w^{\prime},|i n i t\rangle, d_{1}\right) \tag{1}
\end{equation*}
$$

denotes a new circuit in which the unmeasured qubits of circuit $Q C_{x_{1}}$, in state $\left|\psi_{b}\right\rangle$ ( $b$ is the bit string of the measurement outcome), are fed into a second quantum circuit $Q C_{x_{2}}$. So long as $Q C_{x_{2}}$ does not depend on $b$, this composed circuit is identical to some single circuit $Q C_{x}^{\prime}\left(w, w^{\prime \prime},|i n i t\rangle, d^{\prime}\right)$ with $d^{\prime} \leq d_{1}+d_{2}$. But in the following we will consider cases where $Q C_{x_{2}}$ does depend on the measurement outcomes $b$.

Definition 5 (BQP). A language $L \subseteq \Sigma^{*}$ is in BQP if $\forall x \in L M(x)=1$ with probability larger than or equal to $2 / 3$ where $M$ is a uniformly generated family of quantum circuits $Q C_{x}\left(w, w^{\prime},|0\rangle, d\right)$ with $w$ and d polynomial in $|x|$, and one of the $w^{\prime}$ output bits gives the value $M(x)$; when $x \notin L$ then $M(x)=1$ with probability smaller than or equal to $1 / 3$.

The class AM defined by Babai (Bab85] and [BM88]) is an extension of the nondeterministic class NP where we allow randomness and interaction in the verification procedure. Here is a formal definition:

Definition 6 (AM). A language $L \subseteq \Sigma^{*}$ is in $A M$ if, $\forall x \in L$ there exists a strategy for Merlin such that a polynomial time computation in $|x|$ by Arthur accepts with probability larger than or equal to $2 / 3$. If $x \notin L$ then for all strategies of Merlin, Arthur accepts with probability smaller than $1 / 3$. A 'strategy' is implemented in the following way: Arthur begins by sending a random bit string b $(|b|=\operatorname{poly}(|x|))$ to Merlin. Merlin, after performing some (arbitrarily powerful) classical computation on $b$, obtains bit string $m$ which he returns to Arthur. Arthur performs a polynomial time computation with $m$ as input, obtaining a single-bit output $M(x)$. The acceptance probability is $p(M(x)=1)$.

### 2.2 Adaptive and nonadaptive quantum computation

We now formalize what we mean by an adaptive quantum computation model:
Definition $7\left(\mathcal{Q} \mathcal{C}_{a d}\right)$. The class of adaptive quantum computations $\mathcal{Q C}_{a d}$ consists of all composed circuits of the form

$$
Q C_{x, b_{1}, b_{2}, \ldots b_{R}}\left(\left|\psi_{b_{R}}\right\rangle, d_{R+1}\right) * \ldots * Q C_{x, b_{1}, b_{2}}\left(\left|\psi_{b_{2}}\right\rangle, d_{3}\right) * Q C_{x, b_{1}}\left(\left|\psi_{b_{1}}\right\rangle, d_{2}\right) * Q C_{x}\left(|i n i t\rangle, d_{1}\right) .
$$

This composition is adaptive in the sense that the description of the second circuit $Q C_{x, b_{1}}$ is a function of the measurement outcome of the first circuit $b_{1}$ as well as of the problem specification $x$; the description of the third circuit $Q C_{x, b_{1}, b_{2}}$ is a function of the measurement outcomes of the first two, and so on. The depth parameters $d_{1}$ through $d_{R+1}$, and the number of rounds $R+1$, should be polynomial in $|x|$, as should all the widths; also, for uniformity, the boolean functions determining the quantum circuit at every round from the previous measurement outcomes and the input $x$ should be efficiently implementable.

While $\mathcal{Q C}_{a d}$ has no greater power than the set of ordinary quantum circuits $\mathcal{Q C}$, Gottesman and Chuang's work shows that the universal operations for $\mathcal{Q C}_{a d}$ could be quite different than for $\mathcal{Q C}$. Their main theorem, stated informally, is that one-qubit gates alone are sufficient to implement all operations in $\mathcal{Q} \mathcal{C}_{a d}$, provided that $\mid$ init $\rangle$ consists of a sufficient supply of entangled states as well as qubits in the $|0\rangle$ state, and that the quantum circuits composing $\mathcal{Q C}_{a d}$ are permitted to perform Bell measurements (see Sec. (4).

Now we introduce a new model, nonadaptive quantum computation which we derive from $\mathcal{Q C}_{a d}$ and which we will use to deduce interesting constraints relating simulatability of quantum circuits and quantum complexity classes.

Definition $8\left(\mathcal{Q C} \mathcal{C a d}_{\text {nad }}\right)$. With each member $Q C_{\text {ad }} \in \mathcal{Q} \mathcal{C}_{\text {ad }}$ we associate a set of members $Q C_{n a d}(g) \in$ $\mathcal{Q C}_{\text {nad }}$, one for each distinct value of the 'guess' bit string $g$. The nonadaptive quantum computation so obtained is the composed quantum circuit

$$
\begin{equation*}
Q C_{x, g_{1}, g_{2}, \ldots g_{R}}\left(\left|\psi_{b_{R}}\right\rangle, d_{R+1}\right) * \ldots Q C_{x, g_{1}, g_{2}}\left(\left|\psi_{b_{2}}\right\rangle, d_{3}\right) * Q C_{x, g_{1}}\left(\left|\psi_{b_{1}}\right\rangle, d_{2}\right) * Q C_{x}\left(|i n i t\rangle, d_{1}\right) . \tag{2}
\end{equation*}
$$

The only difference between $Q C_{a d}$ and $Q C_{n a d}(g)$ is that the circuit's dependence on the measured values $b_{1}, b_{2}, \ldots$ in $Q C_{a d}$ is replaced by the guessed values $g_{1}, g_{2}, \ldots$ in $Q C_{n a d}(g)$. Because of this, all intermediate measurements in the nonadaptive circuit can be moved to the end, and the circuit can be viewed as a single ordinary quantum circuit, with no measurements during the computation.

## 3 Classical Simulations

We formalize the notion of classical simulatability with a certain accuracy:
Definition $9\left(S_{\epsilon}(\mathcal{Q C})\right.$ ). An efficient simulation $S_{\epsilon}(\mathcal{Q C})$ (with accuracy parameter $\epsilon$ ) of a (uniformly generated) family of quantum circuits $\mathcal{Q C}$ exists if for each $Q C\left(w, w^{\prime},|0\rangle, d\right) \in \mathcal{Q C}$ there is a classical boolean circuit with depth $d^{\prime}$ and input width $r\left(r, d^{\prime}=\operatorname{poly}(w, d)\right)$, and output width $w^{\prime}$, such that $\forall b,\left|N(b) / 2^{r}-p_{Q C}(b)\right| \leq \epsilon p_{Q C}(b)$. Here, $p_{Q C}(b)$ is the probability that the measured state of the output quantum register is $b$ for a particular quantum circuit $Q C ; N(b)$ is the number of settings of the input register of the classical circuit that simulates $Q C$, for which the classical circuit outputs $b$. The classical circuits are uniformly generated from the description of the QC-circuits.

A stronger type of classical simulation is one where one can explicitly calculate the (conditional) probability of a certain set of outcomes, and then sample this probability distribution. Here is our definition:

Definition $10\left(S^{C}(\mathcal{Q C})\right)$. An efficient density computation of a (uniformly generated) family of quantum circuits $\mathcal{Q C}$ is one that proceeds as follows: we first divide up the full quantum measurement at the end of $Q C \in \mathcal{Q C}$ into separate measurements $\mathcal{M}_{1}, \mathcal{M}_{2}, \ldots$ on disjoint sets of qubits that contain a constant number of qubits. Let $b_{i}$ denote the set of potential outcomes of measurement $M_{i}$, and let $\mathrm{b}_{\mathrm{i}}$ denote an outcome. An efficient density computation exists if there exist polynomialtime (in the width and depth of $Q C\left(w, w^{\prime},|0\rangle, d\right) \in \mathcal{Q C}$ ) uniformly generated classical procedures for evaluating the conditional probabilities

$$
\begin{equation*}
p\left(b_{i} \mid \mathrm{b}_{\mathrm{j}_{1}} \ldots \mathrm{~b}_{\mathrm{j}_{k}}\right) \tag{3}
\end{equation*}
$$

Here the set of indices $j_{1}, \ldots, j_{k}$ can be any subset (including the empty set) of the set of measurements $\mathcal{M}=\left\{\mathcal{M}_{1}, \mathcal{M}_{2}, \ldots\right\}$ and $i \neq j_{1} \ldots j_{k}$ labels any other measurement.

For an example of a density computation, see the simulation algorithm in Ref. Val02. This definition leads obviously to

Proposition 1. If an efficient density computation exists, it provides the means for performing an efficient simulation with $\epsilon=0$.

Proof. We proceed as follows. We pick a first measurement $\mathcal{M}_{1}$ (it does not matter which one since they all commute) and calculate $p\left(b_{1}\right)$, i.e. we calculate a constant number $c_{1}$ of probabilities, where $c_{1}$ is the number of outcomes of $\mathcal{M}_{1}$. We flip coins according to the probability distribution $p\left(b_{1}\right)$ and fix the outcome $\mathbf{b}_{1}$. Then we pick a next measurement $\mathcal{M}_{2}$ and we calculate $p\left(b_{2} \mid \mathbf{b}_{1}\right)$, again a constant number of probabilities. We flip coins, fix the outcome and proceed to the next measurement $\mathcal{M}_{3}$ etc. There are no more than $w$ measurements (for a quantum circuit $Q C$ of width $w$ ) and thus in total we calculate at $\operatorname{most} w \max _{i} c_{i}$ conditional probabilities.

Remark: One may also consider density computations with accuracy parameter $\delta$ in which the conditional probabilities can be calculated as $\left|p_{\text {sim }}(. \mid)-.p(. \mid).\right| \leq \delta p(. \mid$.$) . If \delta=2^{\frac{1}{w} \log (1+\epsilon)}-1$ where $w$ is the width of the quantum circuit (for large width $w, \delta \rightarrow \frac{1}{w} \ln (1+\epsilon)$ ), then this will provide a way to do an simulation with accuracy parameter $\epsilon$.

Definitions 9 and 10 relate to regular quantum circuits $\mathcal{Q C}$ but can be extended in a straightforward way to adaptive quantum circuits $\mathcal{Q C}_{a d}$ by restricting ourselves in the density computation Definition 10 to estimating conditional probabilities where (intermediate) measurement $\mathcal{M}_{i}$ occurs after the (intermediate) measurements labeled by $j_{1}, \ldots, j_{k}$.

In the 'density computation' setting, there is a close connection between the simulation of adaptive and nonadaptive circuits:

Theorem 1. If there exists an efficient density computation of $Q C_{n a d}(g) \in \mathcal{Q C}_{\text {nad }}$ for all $g$, then there is a simulation with accuracy parameter $(\epsilon=0)$ for $Q C_{a d} \in \mathcal{Q C}$ ad .

Proof. The simulation of $Q C_{a d}$ will be a direct adaptation of the algorithm for the density computation of $Q C_{n a d}(g)$ for some $g$. We consider the composed quantum circuit in Definition 77 for $Q C_{a d}$. The first member of this composition, $Q C_{x}\left(|i n i t\rangle, d_{1}\right)$, is by itself an example of a nonadaptive circuit and can be simulated efficiently by the hypothesis. We flip coins biased according to the outcome probabilities of $\mathcal{M}_{1}$ (the measurement following $Q C_{x}$ ) and we fix the outcome, say $b_{1}=\mathrm{b}_{1}$. This fixes the second circuit in the composition $Q C_{x, \mathrm{~b}_{1}}$ and any further choices of gates and measurements depending on $b_{1}$. Now we consider two stages of the quantum circuit, $Q C_{x, \mathbf{b}_{1}}\left(\left|\psi_{\mathbf{b}_{1}}\right\rangle, d_{2}\right) * Q C_{x}\left(|i n i t\rangle, d_{1}\right)$. Since the second circuit is now fixed, it is no longer adaptive and we may consider the pair as a single nonadaptive quantum circuit with guess bit $g=\mathrm{b}_{1}$; the measurements $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ (the measurement following $Q C_{x, b 1}$ ) may be moved to the end. Since we have already fixed the outcome of measurement $\mathcal{M}_{1}$, we are interested in sampling from the probability distribution of outcomes of $\mathcal{M}_{2}$ given $b_{1}=\mathrm{b}_{1}$. By hypothesis, our classical algorithm can do this by, for example, letting us calculate the conditional probability distributions $p\left(b_{2}\left(\mathcal{M}_{2}\right) \mid \mathrm{b}_{1}\right)$. By computing the constant number of quantities $p\left(b_{2}\left(\mathcal{M}_{2}\right) \mid \mathbf{b}_{1}\right)$ and flipping coins we can implement measurement $\mathcal{M}_{2}$. We fix its outcome, say $b_{2}=b_{2}$ and proceed as before to the third measurement by moving it to the end and using the density computation of some nonadaptive circuit with guess bits $g=\mathrm{b}_{1} \mathrm{~b}_{2}$, etc. Note that if $\mathcal{M}_{1}, \mathcal{M}_{2}$, etc., do not have a constant number of outcomes, the circuit can always be broken up into a larger number of stages, where the number of outcomes for each of these sub-stages is constant.

A special case occurs when the probability distribution over the output bits $b_{1}, \ldots, b_{k}$ of a circuit $Q C_{a d}$ does not depend on the outcomes of the intermediate measurements, in other words $Q C_{a d}$ represents the same logical circuit on the qubits independent of the outcome of these intermediate measurements. An example is the Gottesman-Chuang construction considered in the next section. These kind of circuits are the ones of interest since we want to implement a fixed circuit and not a
mixture of circuits．We denote this class of circuits with an additional label＇fix＇，i．e． $\mathcal{Q C}_{a d, f i x}$ ．In that case we also have

Theorem 2．If there exists an efficient density computation of $Q C_{n a d, f i x}(g) \in \mathcal{Q C}_{\text {nad，fix }}$ for all $g$ ， then there is an efficient density computation for $Q C_{a d, f i x} \in \mathcal{Q C}_{a d, \text { fix }}$ ．

Proof．Consider $Q C_{a d, f i x}$ with a certain set of intermediate measurement outcomes $g^{\prime}$ ．We know that the probability distribution of outcomes of $Q C_{a d, f i x} p\left(b_{1} \ldots b_{k} \mid g^{\prime}\right)$ is identical to the probability distribution of outcomes of $Q C_{n a d, f i x}\left(g^{\prime}\right)$ ．Since also $p\left(b_{1} \ldots b_{k} \mid g^{\prime}\right)=p\left(b_{1} \ldots b_{k}\right)$ the efficient density computation of $Q C_{n a d, f i x}\left(g^{\prime}\right)$ can be directly used to do an efficient density computation of $Q C_{a d, f i x}$ ．

Our second theorem has interesting consequences when applied to a subset of $\mathcal{Q} \mathcal{C}_{a d, f i x}$ circuits that give universal quantum computation：

Corollary 1．Suppose the set $\left\{Q C_{\text {ad，fix }}^{U}\right\} \subset \mathcal{Q C}_{\text {ad，fix }}$ contains a universal set of quantum circuits， sufficient to implement all polynomial－time quantum computations（where each $Q C_{\text {ad，fix }}^{U}$ is such that the logical circuit on the qubits does not depend on the outcomes of the intermediate measurements）． If there is an efficient density computation of the corresponding nonadaptive set $\left\{Q C_{\text {nad，fix }}^{U}(g)\right\}$ for all $g$ ，then for the polynomial hierarchy PH we have $\mathrm{PH}=\mathrm{BPP}=\mathrm{BQP}$（thus the polynomial hierarchy would collapse to $\Sigma_{2}^{P}$ since $\mathrm{BPP} \in \Sigma_{2}^{P}$（Lau83］）．

Proof．If the simulations of the nonadaptive circuits are possible，then by Theorem 1 all the adaptive simulations are also possible．The density computation of Theorem 2 does more than simply providing a simulation；it provides a means of calculating the outcome probability of any polynomial depth quantum circuit．It has been shown［FGHP99］that determining the acceptance probability of a quantum computation which we would be able to do if we could calculate all joint probabilities， is equivalent to the complexity class coC $=\mathrm{P}$ ．Therefore we would have coC $=\mathrm{P} \subseteq \mathrm{BPP}$ ．On the other hand，it is known that $\mathrm{PH} \subseteq \mathrm{BPP}^{\mathrm{coC}=P}$ and thus $\mathrm{PH} \subseteq \mathrm{BPP}^{\mathrm{BPP}}=\mathrm{BPP}$ ．

This corollary has more explicit consequences if we consider its application to the Gottesman－ Chuang adaptive computation model，which we now examine in more detail．

## 4 Constant－Depth Quantum Circuits

The Gottesman－Chuang construction for quantum computation starts from the well－known fact that there exist universal quantum gate sets containing only a single two－qubit gate，the controlled－NOT， along with certain one－qubit gates．They obtain adaptive circuits from such a standard quantum circuit by a one－for－one replacement of each CNOT in the circuit by the teleportation protocol outlined in Fig． $\begin{aligned} & \text {（a）．The entangled four－qubit state }\left|\Psi_{C}\right\rangle \text { can be created＂offline＂at the beginning }\end{aligned}$ of the computation by the procedure shown in Fig．（b）．

We formally define the class of circuits $\mathcal{G C}{ }_{a d} \subset \mathcal{Q C}_{a d}$ obtained in this fashion：
Definition $11\left(\mathcal{G C}_{a d}\right)$ ．A quantum circuit $G \in \mathcal{G C}_{a d} \subset \mathcal{Q C}_{a d}$ when 1）All the quantum circuits $Q C_{b_{1}, b_{2}, \ldots .}$ composing $G$ contain only one－qubit gates；2）The initial state $\mid$ init $\rangle$ consists of some qubits in state $|0\rangle$ and others in the entangled four qubit state $\left|\Psi_{C}\right\rangle$ ．The qubits of $\Psi_{C}$ are labeled ＂1＂to＂4＂，see Fig． $\mathbf{1}(b)$ ；3）Each intermediate measurement consists of two Bell measurements． The first of these Bell measurements（see Fig．$⿴ 囗 十\left(\right.$（a））uses qubit＂ 1 ＂of a＇new＇$\Psi_{C}$ that has not been


Figure 1: (a) The Gottesman-Chuang implementation of the CNOT gate by teleportation. In addition to the two qubit inputs $q_{1}$ and $q_{2}$, the teleportation circuit has four additional ancilla qubit inputs ("1", " 2 ", " 3 ", " 4 ") preset to the entangled state $\left|\Psi_{C}\right\rangle$. Two Bell measurements (B.M.) are performed as indicated, resulting in two bit-pairs $\mathbf{b}_{1}$ and $\mathbf{b}_{2}$ as output. These bit pairs determine the parameters of two one-qubit quantum gates $U$ and $V$. (b) Construction of the entangled state $\left|\Psi_{C}\right\rangle . H$ is the one-qubit Hadamard gate, specified by the $2 \times 2$ matrix $\binom{11}{1-1} / \sqrt{2}$. The dotted box (which can be completed in one time step) causes the creation of the entangled state $\Phi^{+}$between ancilla bits " 1 " and " 2 " and between " 3 " and " 4 ". The final CNOT constitutes the "offline" application of the two-qubit gate as mentioned in the text.
acted upon previously, and some other arbitrary qubit in the circuit. The second measurement uses qubit " 4 " of the same $\Psi_{C}$ and some other arbitrary qubit.

The nonadaptive circuits corresponding to $\mathcal{G C}_{a d}$ are of constant depth:
Lemma 1. Consider any circuit $G \in \mathcal{G C}_{a d}$, and consider the corresponding set of nonadaptive circuits $G(g)$ which together make up the set $\mathcal{G C}_{\text {nad }}$. Any circuit in $\mathcal{G C}_{\text {nad }}$ has at most depth four, using one and two-qubit gates and starting with the state $|00 \ldots 0\rangle$.

Proof. The essential idea behind the depth-reduction is that in the nonadaptive model the Bell measurements can be done at the end and all further gates in the quantum circuits are one-qubit gates given a preparation of the states $\Psi_{C}$ (which can also be done in a few time-steps). The nonadaptive circuit has the following form. In the first time step $(I)$ one creates a set of entangled states $\Phi^{+}$by performing a set of 2-qubit gates all acting in parallel. At the same time, we perform some single-qubit gates on the other qubits, the 'data qubits'. The second time step (II) is built out of two steps. We do CNOT gates on halves of the entangled states; the target and control bits are each half of a different entangled state (so as to create $\Psi_{C}$ ). Then we do some single-qubit gates on the qubits " 2 " and " 3 " (see Fig. [(b)) that went through these CNOT gates. These two steps can be merged into one by permitting arbitrary 2 -qubit gates on halves of the entangled states. Then
we rotate some qubits from the Bell basis to the standard basis (III) and in the final time step we perform single-qubit measurements in the computational basis (IV).

The restricted form of the $\mathcal{G} \mathcal{C}_{n a d}$ circuits leads to a conclusion about the simulatability of constant-depth quantum circuits:

Corollary 2. If there is an efficient density computation of all quantum circuits $Q C(|0\rangle, d=4)$, then $\mathrm{PH} \subseteq \mathrm{BPP}$ and $\mathrm{BQP} \subseteq \mathrm{BPP}$ and thus $\mathrm{PH}=\mathrm{BPP}=\mathrm{BQP}$.

Proof. Immediate, by application of Corollary $\mathbb{1}$ and Lemma $\mathbb{1}$
This corollary indicates that a classical simulation, at least of the density computation type, of constant depth quantum circuits, is quite unlikely to be possible. This may be considered a surprising result, since it is clear that for a quantum circuit of constant depth $d$ and width $n$, any measurement on $\log n$ qubits can be efficiently reproduced by a classical density computation, that is, the family of quantum circuits $\{Q C(n, \log n,|0\rangle, d)\}$ is simulatable for constant $d$. In the past history of a single qubit there is only a total constant number $\left(2^{d}\right)$ of qubits with which it has interacted and therefore the output state of $\log n$ qubits can be determined by following the evolution of at most $2^{d} \log n$ input qubits. We can thus simulate the measurements on $\log n$ qubits by following the $n^{2^{d}}$ amplitudes during the computation and tracing over outputs that are no longer in the past history cone of the output qubits of interest.

We now note that the depth of four, singled out by the Gottesman-Chuang construction, is the smallest depth for which Corollary 2 holds:

Proposition 2. An efficient density computation exists of all quantum circuits of depth three, $Q C(|0\rangle, d=3)$.

Proof. We give the simulation. After the first time step the quantum state of the circuit consists of a set of 2-qubit entangled states and possibly some 1-qubit states and thus the amplitudes of this state can be efficiently represented classically. We may consider the second computing step and the final measurement step as one single step in which a set of final measurements are performed in arbitrary 2 -qubit bases. We pick a first measurement. It is simple to calculate the probabilities for the various outcomes since they depend on the state of no more than four qubits. We flip coins according to the outcome probabilities and fix the outcome. We replace the 4 -qubit state by the post-measurement projected state consisting now of 2 qubits. We pick the next measurement and proceed similarly etc. If only a subset of these measured bits are required as output, the rest are simply discarded.

It is clear that adding a fourth layer of computation will break the method of proof for the proposition. The problem is that each final measurement in the Bell basis leaves an entangled state with more qubits which may not have an efficient classical representation. The reason that a constant-depth quantum computation could be hard to simulate classically may be precisely this.

### 4.1 General Simulations

As we argued above, the constant-depth $\mathcal{G C}_{\text {nad }}$ model is able to perform quantum computation - the probability of success is exponentially small, but there is a flag that indicates when the successful outcome is achieved. This means that if there were a classical algorithm of any sort that simulates
the constant-depth $\mathcal{G C} \mathcal{C a d}_{\text {nad }}$ model, then there would exist an efficient classical probabilistic algorithm that could simulate a polynomial-time quantum computation with exponentially small probability, but with a success flag. Even though such a simulation would not be useful, it would nevertheless have some interesting consequences for computational complexity classes. In particular, we will relate the class BQP to the classical nondeterministic complexity class AM (for Arthur-Merlin). We consider the circuits $G(g=0) \in \mathcal{G C}_{\text {nad }}$ which have guessed outcome corresponding to $U=I$ and $V=I$ after every Bell measurement GC99. The following theorem is proved for these circuits with $g=0$, but holds equally well for other values of $g$ :

Theorem 3. If an efficient simulation with accuracy parameter $\epsilon<1 / 3$ exists for the family of circuits $\{G(g=0)\} \subset \mathcal{G C}$ nad , then $\mathrm{BQP} \subseteq \mathrm{AM}$.

The idea behind the proof is the following. The classical probabilistic simulation of $G(g=0) \in$ $\mathcal{G C}{ }_{\text {nad }}$ uses a certain number of random coins. For some set of values of these coins the simulation outputs (1) the Bell measurement outcomes corresponding to the guess string $g=0$ such that we know that the simulated circuit performs a successful simulation of some quantum computation $M$ and (2) the bit value 1 as the outcome of this quantum computation $M$. Thus the size of this set of coin values depends on whether $M$ outputs 1 with large probability or small probability, corresponding to the decision problem that $M$ solves. The estimation of the approximate size of a set is a problem that is known to be in AM.

Here are the details of the proof:
Proof. Let $L \in \mathrm{BQP}$ and let $\left\{G_{x}\right\} \in \mathcal{G C}_{a d}$ be the uniformly generated family of quantum circuits, of the Gottesman-Chuang type, that output a bit $M(x)$ that decides $L$; thus if $x \in L, M(x)=1$ with probability at least $2 / 3$, while if $x \notin L, M(x)=1$ with probability less than $1 / 3$. Consider the corresponding nonadaptive version of these $G C$ circuits $G_{x}(g=0)$ where $g$ is the guess bit string $(|g|=k)$ for all the Bell measurements. We will be interested in two of the outputs of the circuit $G_{x}(g=0)$ : The bit string $y$ giving all the Bell measurement outcomes, and the decision bit $M^{\prime}(x)$ where $M^{\prime}$ is a function that coincides with $M$ when $y=g=0$, and does not, in general, coincide with $M$ when $y \neq 0$. Since all Bell measurement outcomes are equally likely (see Appendix A), that is, $p(y)=1 / 2^{k}$ for all $y$, then if $x \in L, p(y=g, M(x)=1) \geq(2 / 3) \times\left(1 / 2^{k}\right)$.

The classical probabilistic simulation $S_{\epsilon}$ of the circuit $G_{x}(g=0)$ takes as input a set of $n$ random bits $r(n=\operatorname{pol} y(|x|))$, the description of the circuit $G_{x}(g=0)$ (note that it is necessary that $n \geq|g|=k$ ), and the input $x$. The machine $S_{\epsilon}$ outputs $\left(y, M^{\prime}(x)\right)$ in poly $(n)$ time; thus, we can express the input-output relation of the simulation as $\left(r, G_{x}(g=0), x\right) \xrightarrow{S_{\epsilon}}\left(y, M^{\prime}(x)\right)$. Since $S_{\epsilon}$ is a good simulation of $G_{x}$, there exist a set of values for the random bits $r$ such that the Bell outcomes agree with the guesses, $\left(r, G_{x}(g=0), x\right) \xrightarrow{S_{\epsilon}}(y=0, M(x))$. In fact, according to the accuracy parameter $\epsilon$ condition of Definition 0 we have that, for this simulation, $p(y=0, M(x)=$ $1)_{\text {sim }} \geq(1-\epsilon)(2 / 3) \times 2^{-k}$. Since there are $2^{n}$ strings $r$, the total size of this set $\mathbf{S}$ of random coin settings for which $y=0$ and $M(x)=1$ is at least $\mathrm{BIG}=(1-\epsilon)(2 / 3) \times 2^{n-k}$; if $x \notin L$, then the size of the set $\mathbf{S}$ for which $y=0$ and $M(x)=1$ is guaranteed to be less than or equal to SMALL $=(1+\epsilon)(1 / 3) \times 2^{n-k}$.

Thus, if the simulation $S_{\epsilon}$ exists, then the problem of deciding whether input $x$ is in a BQP language L or not is equivalent to determining whether this set $\mathbf{S}$ of $n$-bit strings is larger than or equal to size BIG or smaller than or equal to SMALL, where membership in the set is easy to determine (in polynomial time) by running the simulation $S_{\epsilon}$. If $\epsilon<1 / 3$, then we are guaranteed
that BIG $>$ SMALL. This problem of determining "approximate set size" is known to be solvable as a two-round AM game (see Lemma 1 in [GS86]). In Appendix B we explain how the game proceeds.

How unlikely is the containment $\mathrm{BQP} \subseteq \mathrm{AM}$ ? Nothing is definitely known, but the consensus is that it is rather unlikely.

## 5 Conclusion

The nonadaptive Gottesman-Chuang circuit is a very curious resource. According to the evidence given by this paper, its multiple-bit output is hard to generate classically. Still, it is an open question whether, for example, a class such as BPP ${ }^{\mathcal{G C}}$ nad would have additional power over BPP (see for example the recent results in Ref. [FGHZ03]).

From an experimental physics point of view it is clear that it would be extremely interesting to find a problem in $\mathrm{BPP}^{\mathcal{G} \mathcal{C}_{n a d}}$ which is not known to be in BPP; a constant-depth quantum circuit should be easier to build than a universal quantum computer.

Our Theorem 1 holds for any adaptive quantum computation model with its corresponding nonadaptive version, including, for example, the Knill-Laflamme-Milburn (KLM) scheme. So, our results can be viewed as evidence that a nonadaptive KLM scheme, i.e. mere linear optics on Fock states followed by photon counting measurements, may perform some interesting non-classical computation.

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## A Appendix: Bell outcomes

In the Gottesman-Chuang circuit (adaptive or nonadaptive), each Bell measurement is performed on uncorrelated qubits, one of which is a qubit " 1 " or qubit " 4 " of the state $\Psi_{C}$. Thus, the density matrix of the two-qubit system has the form

$$
\begin{equation*}
\rho_{A B}=\frac{1}{2} I \otimes \rho_{B} \tag{4}
\end{equation*}
$$

for some choice of the AB labels. The probability that the Bell measurement outcome is, say, $\Phi^{+}$, is by standard quantum mechanical rules $\left\langle\Phi^{+}\right| \rho_{A B}\left|\Phi^{+}\right\rangle=$

$$
\begin{equation*}
\frac{1}{4}\left(\langle 00| I \otimes \rho_{B}|00\rangle+\langle 00| I \otimes \rho_{B}|11\rangle+\langle 11| I \otimes \rho_{B}|00\rangle+\langle 11| I \otimes \rho_{B}|11\rangle\right)=\frac{1}{4} \operatorname{Tr} \rho_{B}=\frac{1}{4}, \tag{5}
\end{equation*}
$$

and similarly for the other three Bell states. Thus the probability of outputting any bit pair is uniform ( $p=1 / 4$ ) as claimed.

## B Appendix: Approximate Set Size

We begin with a guaranteed separation of set sizes, BIG/SMALL $\geq \frac{2-2 \epsilon}{1+\epsilon}$. The first step is to amplify this ratio to a larger number, say BIG/SMALL $=d / 8$ (we follow here the notation of the lemma in GS86] , by considering $u=\log (d / 8) / \log \left(\frac{2-2 \epsilon}{1+\epsilon}\right)$ runs of the simulation; that is, we consider the new set $\mathbf{S}^{\prime}=\underbrace{\mathbf{S} \times \mathbf{S} \times \ldots \times \mathbf{S}}_{u}$. Given that $\mathbf{S}^{\prime}$ is a subset of all $u \times n$-bit strings, and $p=\lfloor 8 \log$ BIG $\rfloor$ (i.e. $p \sim u(n-k)$ ), the game proceeds as follows:

- Arthur, the verifier, picks at random $l=p+1$ hash functions $h_{1}, \ldots, h_{l}, h: \Sigma^{u n} \rightarrow \Sigma^{p}$ and $l^{2}$ random bit strings $Z=\left\{z_{1}, \ldots, z_{l^{2}}\right\}, z_{i} \in \Sigma^{p}$; all these are sent to Merlin, the prover. The hash may be of the Carter-Wegman type CW79, so each hash function is specified by a $u n \times p$ random Boolean matrix.
- Merlin attempts to respond with $t \in \mathbf{S}^{\prime}$ such that, for some $i, h_{i}(t) \in Z$.

With a suitably chosen amplification factor, the game can succeed with almost certainty (but not absolute certainty); that is, if $x \in L$, according to [GS86] (see also GS89], the probability that Merlin can supply a proof if $\left|\mathbf{S}^{\prime}\right| \geq$ BIG is at least as large as $1-2^{-l / 8}$, while if $\left|\mathbf{S}^{\prime}\right| \leq$ SMALL the probability that Merlin can give a proof is no greater than $l^{3} / d$. Thus, we can make the failure probability in both directions exponentially small in $|x|$ (an exponentially large $d$ only requires a polynomially large number of repetitions $u$ ).


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    ${ }^{1}$ Bernstein and Vazirani BV97 proved that any model of quantum computation with measurements during the computation is no more powerful than a model with measurements at the end of the computation. Thus the rationale behind adaptive quantum computation is merely that the set of quantum gates that is needed to implement adaptive quantum computation may be smaller and therefore simpler to implement physically.

