

Whitney's connectivity inequalities for directed hypergraphs

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Abstract

Whitney's inequality established an important connection between vertex and edge connectivity, and the degree of a graph, which was later generalized to digraphs and undirected hypergraphs. Here we show, using the most common definitions of connectedness for directed hypergraphs, that an analogous result holds for directed hypergraphs. It relates the vertex connectivity under strong vertex elimination, edge connectivity under weak edge elimination, and a suitable degree-like parameter and it is a proper generalization of the situation in both digraphs and undirected hypergraphs. We furthermore relate the connectivity parameters of directed hypergraphs with those of its directed bipartite König representation.

Keywords: Strong and weak vertex elimination, strong connectedness, unilateral connectedness, directed hypergraph, total degree, connectivity indices.

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1 Introduction

Directed hypergraphs naturally arose as a model of dependencies e.g. in propositional logic, database theory, and model checking, see e.g. [3, 7] for reviews. Recently they also received increasing attention as models of biological [6, 9], chemical [2, 11], and transportation networks [10]. Connectivity parameters are one of the most fundamental characteristics of a network, and hence are also of directed practical relevance for applications of directed hypergraphs [6].

It is important to note that directed hypergraphs give rise to many different notions of connectedness. Here, we only consider the simplest, least restrictive, construction of hyperpath, requiring only a single vertex in the overlap of the head of one directed hyperedge and the tail of the following one. In particular in chemical reactions networks, much more restrictive notions path and reachability are also of interest, see e.g. [1, 2, 6]. The concepts of connectivity explored here remain closely related to those of bipartite graphs representation of directed hypergraphs [12] and, as we shall see, admit generalizations of well-known results for graphs and digraphs.

The connectivity in an undirected graph G is described by two parameters, the vertex connectivity index κ and the edge connectivity index κ' . They are defined as the minimum number of vertices or edges, respectively, whose removal disconnects G or gives a trivial graph. Hassler Whitney [13] showed that all undirected graphs satisfy the inequality $\kappa \leq \kappa' \leq \delta$, where δ denotes the minimal vertex degree in G . Later Geller and Harary found a generalization to digraphs [8]. In hypergraphs, the situation becomes more complicated because there are different, natural ways to delete vertices and hyperedges and thus to derive sub-hypergraphs [4]. Nevertheless, Whitney's inequalities for the connectivity parameters generalize to (undirected) hypergraphs [5].

In the present contribution we show that analogous results also hold for directed hypergraphs with respect to both strong and unilateral connectedness. In Section 2.1 we introduce the notation and give some simple preliminary results for later use. Section 3 introduces the various connectivity indices and established some universal inequalities between them. For many pairs of indices, however, we show that they are not comparable in general. The main theorem, a generalization of Whitney's inequalities, is the topic of Section 4. In the final Section 5 we explore relations between connectivities of Section 5 we explore relations between connectivities of directed hypergraphs and their bipartite digraph representation.

2 Notation and preliminaries

2.1 Directed hypergraphs

A *directed hypergraph* $H = (V, E)$ consists on a vertex set V and a set of directed hyperedges or hyperarcs $E = \{(T(e), H(e)) \mid T(e) \subseteq V \text{ and } H(e) \subseteq V\}$, where $H(e) \neq \emptyset$ and $T(e) \neq \emptyset$. The sets $T(e)$ and $H(e)$ are called the *tail* and the *head* of e , respectively. The *support* of a hyperedge $e \in E$ is the pair $\text{supp}(e) = T(e) \cup H(e)$. A directed hypergraph is called *k-uniform* if $|T(e)| = |H(e)| = k$ for all $e \in E$. Two edges $e, e' \in E$ are said to be *parallel* if $T(e) = T(e')$ and $H(e) = H(e')$. A directed hypergraph $H = (V, E)$ is called *simple* if it has neither parallel hyperarcs and nor loops, that is, edges e with $T(e) \cap H(e) = \emptyset$. A (directed) hypergraph is *trivial* if $|V| = 1$ and $E = \emptyset$, i.e., H consists of a single vertex.

We say that $u, v \in V$ are *adjacent* if there exists a hyperarc $e \in E$ such that $u \in T(e)$ and $v \in H(e)$. The *neighborhood* of a vertex v in a hypergraph (or graph) is the set of all the vertices adjacent to v not including v . The *indegree* of a vertex v , denoted as $d^-(v)$ in H , is defined as the number of hyperarcs that contain v in their head. The *outdegree* of a vertex v , denoted as $d^+(v)$ in H , is defined as the number of hyperarcs that contain v in their tail. The *minimum indegree* and *minimum outdegree* of H will be denoted by $\delta^-(H) = \min\{d^-(v)\}_{v \in V}$ and $\delta^+(H) = \min\{d^+(v)\}_{v \in V}$, respectively. The number of arcs parallel to e (including e) is the *multiplicity* of e and it is denoted as $m_H(e)$.

Every directed hypergraph $H = (V, E)$ can be represented as a bipartite digraph $G(H)$ with vertex set $V \cup E$ and directed arcs $x \rightarrow e$ iff $x \in T(e)$ and $e \rightarrow x$ iff $x \in H(e)$. The arcs of $G(H)$ are called the *bits* of the directed hypergraph. The graph $G(H)$ is called the *incidence digraph*, *Levi digraph*, or *König digraph* of H . There is a one-to-one correspondence between directed hypergraphs and bipartite graphs for which one partition (the one corresponding to the hyperarcs E) has neither sources nor sinks (since we do not allow hyperarcs with empty heads or tails.) For details we refer to [12].

2.2 Subhypergraphs

Substructures of directed hypergraphs can be constructed in two ways: In *strong* substructures the hyperedges are either retained or removed as an entity. In *weak substructures*, hyperedges can be restricted to a subset of vertices as long as their heads and tails remain non-empty. More precisely, following [5] we define:

A directed hypergraph $H' = (V', E')$ is a *weak subhypergraph* of the directed hypergraph $H = (V, E)$ if $V' \subset V$ and E' consists of edges e' with $T(e') = \{v \mid v \in T(e) \cap V'\}$ and $H(e') = \{v \mid v \in H(e) \cap V'\}$ for some $e \in E$. A directed hypergraph $H' = (V', E')$ is a *weak induced subhypergraph* of the directed hypergraph $H = (V, E)$ if $V' \subset V$ and edge set $E' = \{(T(e) \cap V', H(e) \cap V') \mid e \in E \wedge T(e) \cap V' \neq \emptyset \wedge H(e) \cap V' \neq \emptyset\}$. A directed hypergraph $H' = (V', E')$ is called a *strong subhypergraph* of the directed hypergraph $H = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *strong subhypergraph* $H' = (V', E')$ of $H = (V, E)$, is *induced* by V' if $\text{supp}(e) \subseteq V'$ and it is induced by E' if $V' = \bigcup_{e' \in E'} \text{supp}(e')$. $H' = (V', E')$ is a *spanning subhypergraph* of $H = (V, E)$ if $V' = V$.

The deletion of vertices and edges from a directed hypergraph will play a key role in this contribution. Just as the formation of subhypergraphs this can be done in two ways:

Strong vertex deletion of $v \in V$ removes v and all hyperarcs that are incident to v . Thus it creates the strong subhypergraph $H' = (V', E')$ of $H = (V, E)$ with vertex set $V' = V \setminus \{v\}$ and edge set $E' = \{e \in E \mid v \notin T(e) \cup H(e)\}$. For a subset $X \subset V$ we write $H \setminus_S X$ to denote the directed hypergraph formed by strongly deleting all the vertices of X from H .

Weak vertex deletion of $v \in V$ removes v from the vertex set, and all occurrences of v from the hyperarcs of the directed hypergraph H . This creates the hypergraph $H' = (V', E')$ where $V' = V \setminus \{v\}$ and $E' = \{(T(e) \cap V', H(e) \cap V') \mid e \in E \wedge T(e) \cap V' \neq \emptyset \wedge H(e) \cap V' \neq \emptyset\}$. We use the notation $H \setminus_W v$ to denote the directed hypergraph formed by weakly deleting the vertex v from H . For any subset X of V we write $H \setminus_W X$ to denote the directed hypergraph formed by weakly deleting all the vertices of X from H .

Strong deletion of the hyperarc $e \in E$ removes e from the hypergraph and weakly deletes all the vertices incident with e . Thus it produces the weak subhypergraph $H' = (V', E')$ with $V' = V \setminus \text{supp}(e)$ and $E' = \{(T(e) \cap V', H(e) \cap V') \mid e \in E \wedge T(e) \cap V' \neq \emptyset \wedge H(e) \cap V' \neq \emptyset\}$. We write $H \setminus_S e$ to denote the hypergraph formed by strongly deleting the edge e from H . For any subset F of E , we use $H \setminus_S F$ to denote the directed hypergraph formed by strongly deleting all the hyper arcs of F from H .

Weak deletion of the hyperarc $e \in E$ simply removes the hyperarc e without affecting the rest of the hypergraph. Thus it leads to the strong subhypergraph $H' = (V, E')$ with $E' = E \setminus \{e\}$. We write $H \setminus_W e$ to denote the directed hypergraph formed by weakly deleting the hyperarc e from H . For any subset F of E , we write $H \setminus_W F$ to denote the directed hypergraph formed by weakly deleting all the hyperarcs of F from H .

It follows directly from the definition that the order in which vertices or edges are deleted has no impact on the final result. Thus the hypergraphs $H \setminus_S X$, $H \setminus_W X$, $H \setminus_S F$, and $H \setminus_W F$ are well-defined.

2.3 Connectedness

A *directed walk* in a hypergraph $H = (V, E)$ is a sequence $P_{v_0, v_k} = (v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ where $e_1, \dots, e_k \in E$ and $v_0, \dots, v_k \in V$, such that $v_{i-1} \neq v_i$, $v_{i-1} \in T(e_i)$ and $v_i \in H(e_i)$. A *directed p -path* is a walk where the vertices v_0, \dots, v_k are all distinct. A *directed cycle* is a directed walk with k distinct hyperarcs and k distinct vertices such that $v_0 = v_k$. The *length* of a directed walk, directed path, or cycle is the number of hyperarcs in the sequence; i.e., it is k in the foregoing definitions. Let $e \in E$ where $T(e) = \{u_1, \dots, u_k\}$ and $H(e) = \{v_1, \dots, v_l\}$ then the *reverse hyperarc* of e is $\bar{e} \in E$ such that $H(\bar{e}) = \{u_1, \dots, u_k\}$ and $T(\bar{e}) = \{v_1, \dots, v_l\}$.

Definition 2.1. We say that y is reachable from x in H if there is a directed p -path from x to y in H . For two hyperarcs e and e' we say that e' is reachable from e in H if there is x in $H(e)$ and $y \in T(e')$ such that y is reachable from x . Furthermore, we say v is reachable from u in $G(H)$ if there is a directed path from u to v .

There are three natural notions of connectedness in digraphs: A digraph is said to be *strongly connected* if, for every pair of vertices $x, y \in V$, x is reachable from y **and** y is reachable from x . It is said to be *unilaterally connected* if, for every pair of vertices $x, y \in V$, x is reachable from y **or** from y is reachable from x . A bipartite graph with vertex set $V_1 \cup V_2$ is *unilaterally connected on V_1* if for every pair $u, v \in V_1$, v is reachable from u or u is reachable from v . Finally, a digraph is weakly connected if its underlying graph, i.e without direction, is connected. These definitions can be generalized immediately to hypergraphs.

Definition 2.2. A directed hypergraph H is *strongly connected* if for every pair of vertices $u, v \in V$, u is reachable from v and v is reachable from u . It is *unilaterally connected* if for every two pair of vertices $u, v \in V$, v is reachable from u or u is reachable from v . It is (weakly) *connected* if the underlying hypergraph, is connected.

Corollary 2.3. For every directed hypergraph, “strongly connected” implies “unilaterally connected”, which in turn implies “weakly connected”.

Lemma 2.4. *A directed hypergraph H is strongly, unilaterally, or weakly connected if and only if its incidence (di)graph $G(H)$ is strongly connected, unilaterally connected on V , and weakly connected, respectively.*

Proof. An undirected hypergraph is connected if and only if its (undirected) incidence graph is connected, see e.g. [4, 5], hence the statement is true for weak connectedness.

To show the statement for unilateral and strong connectedness we first show that for all $x, y \in V$ there is hyperpath from x to y in H if and only if there is a path from x to y in $G(H)$. First assume that a directed hyperpath $x = v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k = y$ in H exists. Then the bits $(v_0, e_1), (e_1, v_2), \dots, (e_{k-1}, v_k)$ form a directed path for x to y in $G(H)$. Conversely, suppose such a directed path exists in $G(H)$. We note that the arcs (v_{i-1}, e_i) and (e_i, v_i) in $G(H)$ by construction are bits induced by a hyperedge e_i with $v_{i-1} \in T(e_i)$ and $v_i \in H(e_i)$. Thus the sequence $x = v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k = y$ is a directed hyperpath in H .

If $G(H)$ is strongly connected then in particular there is a directed hyperpath in H between any pair of vertices, and thus H is strongly connected. Conversely, if H is strongly connected, we know that there is a directed path between any pair x, y in V . To see that every $e \in E$ is reachable from every $x \in V$ in $G(H)$ we recall that every $T(e) \neq \emptyset$, i.e., there is $u \in T(e)$. We already know that there is a directed path from x to u in $G(H)$, which can be extended by the bit (u, e) to a directed path from x to e . Using that $H(e) \neq \emptyset$ we see that every $x \in V$ is reachable from every $e \in E$. Concatenating a directed from e to x and from x to e' we finally see that every $e' \in E$ is reachable from every $e \in E$, and thus $G(H)$ is strongly connected.

It follows immediately that H is unilaterally connected if for every $x, y \in V$ there is a directed path from x to y or from y to x in $G(H)$, i.e., if $G(H)$ is unilaterally connected on V . \square

Note that unilateral connectedness of H does not imply unilateral connectedness of $G(H)$. As a counterexample consider the directed (hyper)graph H with $V = \{u, v, w, x\}$ and hyperarcs $e_1 = (u, v)$, $e_2 = (v, w)$, $e_3 = (w, x)$, and $e_4 = (u, x)$. H is unilaterally but not strongly connected but there is no directed path from e_2 to e_4 or *vice versa* in $G(H)$.

In the following we say that $H = (V, E)$ is \mathcal{C} -connected with $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}, \mathcal{W}\}$ is strongly, unilaterally, or (weakly) connected. Correspondingly, we shall say that H is \mathcal{C} -disconnected if it is not \mathcal{C} -connected.

3 Connectivity in directed hypergraphs

The degree of connectedness in an undirected hypergraph H is described by invariants describing the minimal number of vertices or edges that must be removed by either weak or strong elimination to disconnect the hypergraph or leave on a trivial hypergraph behind [5]. The situation becomes even more involved because each of these invariants or indices can be defined with respect to each of the three concepts of connectedness. We write $\kappa_{x\mathcal{C}}$ and $\kappa'_{x\mathcal{C}}$, where the prime refers to edge deletion, $x \in \{s, w\}$ indicates strong or weak vertex/edge deletion and refers to strong, unilateral, or weak connectedness. The numbers $\kappa_{x\mathcal{C}}$ and $\kappa'_{x\mathcal{C}}$, are the minimum numbers of vertices and hyperedges, respectively, such that their x -elimination leaves a hypergraph \mathcal{C} -disconnected or trivial.

Let $H = (V, E)$ be a directed hypergraph. A vertex $v \in V$ is called a *strong (weak) \mathcal{C} -cut vertex* of H if $H \setminus_s v$ ($H \setminus_w v$) is \mathcal{C} -disconnected or trivial. X is a *strong (weak) vertex*

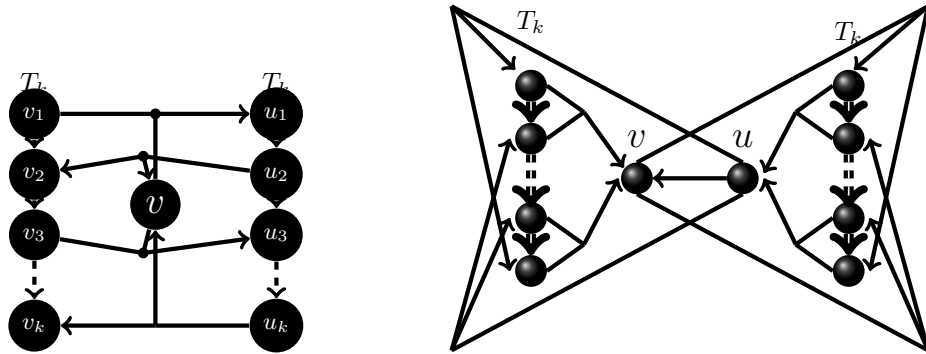


Figure 1: **Left:** T_k is the tournament of k vertices, where k is even. In this hypergraph $\kappa_{s\mathcal{S}} = 1$, $\kappa_{w\mathcal{S}} = \frac{k}{2}$, $\kappa_{s\mathcal{U}} = 1$, $\kappa_{w\mathcal{U}} = k$, the minimum strong vertex cut for $s\mathcal{S}$ and $s\mathcal{U}$ cut is $\{v\}$ and a minimum weak vertex cut for $w\mathcal{S}$ is $\{u_1, u_3, \dots, u_{k-1}\}$ and a weak vertex cut for $w\mathcal{U}$ is $\{u_1, v_2, u_3, v_4, \dots, v_k\}$. As k increases, an infinite family of hypergraphs for which this difference grows linearly is obtained. **Right:** T_k is the tournament of k vertices, where k is even. In this hypergraph $\kappa'_{s\mathcal{S}} = \kappa'_{s\mathcal{U}} = 1$ ((u, v) is a disconnecting arc), $\kappa'_{w\mathcal{S}} = \frac{k}{2}$ (all the arcs that have tail v), $\kappa'_{w\mathcal{U}} = k + 1$ (all the arcs that have tail or head u).

\mathcal{C} -cut of H if $H \setminus_s X$ ($H \setminus_w X$) is \mathcal{C} -disconnected or trivial. We adopt the convention that $\kappa_{x\mathcal{C}} = 1$ for trivial hypergraphs and $\kappa_{x\mathcal{C}} = 0$ for null hypergraphs. A subset $F \subseteq E$ is called a *strong (weak) \mathcal{C} -disconnecting set* of H if $H \setminus_s F$ ($H \setminus_w F$) is \mathcal{C} -disconnected or trivial. We set $\kappa'_{x\mathcal{C}} = 1$ for trivial hypergraphs and $\kappa'_{x\mathcal{C}} = 0$ for null hypergraphs.

The following inequalities hold for all directed hypergraphs as an immediate consequence of the definition and the implications between the connectedness classes for both $x = s$ and $x = w$.

$$\kappa_{x\mathcal{S}} \leq \kappa_{x\mathcal{U}} \leq \kappa_{x\mathcal{W}} \quad \kappa'_{x\mathcal{S}} \leq \kappa'_{x\mathcal{U}} \leq \kappa'_{x\mathcal{W}} \quad (3.1)$$

Since \mathcal{W} -connectedness coincides with the connectedness of undirected hypergraphs we focus on $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}\}$ in the following. The case of undirected hypergraphs is studied in detail in [5]. We first consider the relationships between strong and weak elimination:

Lemma 3.1. *Let $H = (V, E)$ be a directed hypergraph. Then $\kappa_{s\mathcal{C}} \leq \kappa_{w\mathcal{C}}$ for $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}\}$.*

Proof. If H is trivial or null, there is nothing to show. If H is \mathcal{C} -disconnected, then $\kappa_{x\mathcal{C}} = \kappa'_{x\mathcal{C}} = 0$, and the inequalities hold trivially. Now suppose that H is nontrivial and \mathcal{C} -connected. We note that $H \setminus_s X$ is a spanning strong subhypergraph of $H \setminus_w X$ for all $X \subseteq V$. This implies immediately that $\kappa_{s\mathcal{C}} \leq \kappa_{w\mathcal{C}}$ for $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}\}$. \square

It is worth noting that $\kappa_{w\mathcal{C}}$ is a poor upper bound for $\kappa_{s\mathcal{C}}$. Indeed, the difference between $\kappa_{w\mathcal{C}} - \kappa_{s\mathcal{C}}$ can become arbitrarily large as shown in Figure 1(left). It is important to notice that not every strong vertex cut is contained in a weak vertex cut. The situation on the left hand side of Figure 1 is an example.

Whitney's inequalities [5] and their generalization to directed graphs [8] and undirected hypergraphs [13] relate the connectivity indices with each other. In the case of directed

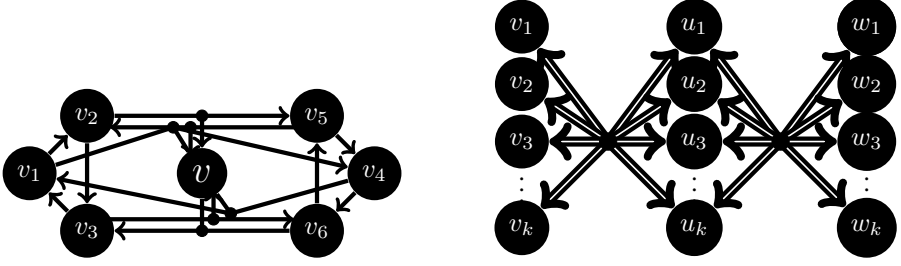


Figure 2: **Left:** In this hypergraph $\kappa_{s\mathcal{S}} = \kappa_{s\mathcal{U}} = 1$ since v is a cut vertex; $\kappa_{w\mathcal{S}} = 2$ since $\{v_2, v_4\}$ is a minimum weak vertex cut; $\kappa'_{s\mathcal{U}} = \kappa'_{s\mathcal{S}} = 2$ since $\{(v_5, v_4), (v_2, v_3)\}$ is a strong disconnecting set. **Right:** The hyperarcs in this hypergraph are $e_1 = (\{v_1, \dots, v_k\}, \{u_1, \dots, u_k\})$, \bar{e}_1 , $e_2 = (\{u_1, \dots, u_k\}, \{w_1, \dots, w_k\})$, \bar{e}_2 . We have $\kappa'_{w\mathcal{S}} = 1$ by removal of any hyperarc and $\kappa_{w\mathcal{S}} = k$ since $\{u_1, \dots, u_k\}$ is the minimal weak vertex.

hypergraphs, however, some of these quantities do not fulfill universal inequalities. We give some simple counterexamples:

$\kappa_{s\mathcal{U}} \not\leq \kappa_{w\mathcal{S}}$. In a directed cycle C_n of length $n > 3$ we have $\kappa_{s\mathcal{U}} = 2$ and $\kappa_{w\mathcal{S}} = 1$. Therefore $\kappa_{s\mathcal{U}} > \kappa_{w\mathcal{S}}$. On the other hand, the left hand side of figure 2 shows an example where $\kappa_{s\mathcal{U}} = 1$ and $\kappa_{w\mathcal{S}} = 2$, i.e., $\kappa_{s\mathcal{U}} < \kappa_{w\mathcal{S}}$.

$\kappa'_{s\mathcal{U}} \not\leq \kappa'_{w\mathcal{S}}$. In a directed cycle C_n of length $n > 4$ we have $\kappa'_{w\mathcal{S}} = 1$ and $\kappa'_{s\mathcal{U}} = 2$. Therefore $\kappa'_{s\mathcal{U}} > \kappa'_{w\mathcal{S}}$. The hypergraph on the right hand side of Figure 1 has $\kappa'_{s\mathcal{U}} = 1$ and $\kappa'_{w\mathcal{S}} = \frac{k}{2}$. For $k > 3$ we therefore have $\kappa'_{s\mathcal{U}} < \kappa'_{w\mathcal{S}}$.

$\kappa_{s\mathcal{S}} \not\leq \kappa'_{s\mathcal{U}}$. The hypergraph in Figure 2(left) satisfies $\kappa_{s\mathcal{S}} < \kappa'_{s\mathcal{U}}$. Now consider the hypergraph in Figure 1(right) with all reverse hyperarcs added. Here, $\kappa'_{s\mathcal{U}} = 1$ and $\kappa_{s\mathcal{S}} = 2$ since $\{u, v\}$ is a vertex cut. Thus $\kappa'_{s\mathcal{U}} < \kappa_{s\mathcal{S}}$.

$\kappa_{s\mathcal{U}} \not\leq \kappa'_{s\mathcal{U}}$. The hypergraph in Figure 2(left) satisfied $\kappa_{s\mathcal{U}} < \kappa'_{s\mathcal{U}}$, while the hypergraph in Figure 1(right) satisfies $\kappa_{s\mathcal{U}} = 2$ ($\{u, v\}$ is a minimal strong the vertex cut) and $\kappa'_{s\mathcal{U}} = 1$. Therefore $\kappa'_{s\mathcal{U}} < \kappa_{s\mathcal{U}}$.

$\kappa'_{s\mathcal{U}} \not\leq \kappa_{w\mathcal{S}}$. In a directed cycle C_n of length $n > 4$ we have $\kappa_{w\mathcal{S}} = 1$ and $\kappa'_{s\mathcal{U}} = 2$, i.e., $\kappa_{w\mathcal{S}} < \kappa'_{s\mathcal{U}}$. Again, we consider the hypergraph in Figure 1(right) with the reverse hyperarcs added. It satisfies $\kappa'_{s\mathcal{U}} = 1$ and $\kappa_{w\mathcal{S}} = 2$, i.e., $\kappa'_{s\mathcal{U}} < \kappa_{w\mathcal{S}}$.

$\kappa_{w\mathcal{S}} \not\leq \kappa'_{w\mathcal{S}}$. The hypergraph in Figure 2(right) satisfies $\kappa'_{w\mathcal{S}} = 1$ and $\kappa_{w\mathcal{S}} = k$, i.e., $\kappa'_{w\mathcal{S}} < \kappa_{w\mathcal{S}}$. For the hypergraph in Figure 1(right) we have $\kappa_{w\mathcal{S}} = 2$ due to the weak way the vertex cut $\{u, v\}$. Furthermore, for $k \geq 4$ we have $\kappa'_{w\mathcal{S}} = \frac{k}{2}$ and thus $\kappa_{w\mathcal{S}} < \kappa'_{w\mathcal{S}}$.

$\kappa'_{w\mathcal{U}} \not\leq \kappa_{w\mathcal{S}}$. The hypergraph in Figure 1(right) satisfies $\kappa_{w\mathcal{S}} = 2$, the set $\{u, v\}$ being a minimum vertex cut. On the other hand, we have $\kappa'_{w\mathcal{U}} = k + 1$ in the same example, hence, for $k > 2$, we have $\kappa_{w\mathcal{S}} < \kappa'_{w\mathcal{U}}$. The hypergraph in Figure 2(right) satisfies $\kappa_{w\mathcal{S}} = k$ and $\kappa'_{w\mathcal{U}} = 2$, since $\{e_1, \bar{e}_1\}$ is a minimal disconnecting set. Thus, for $k > 2$, we have $\kappa'_{w\mathcal{U}} < \kappa_{w\mathcal{S}}$.

$\kappa_{w\mathcal{S}} \not\leq \kappa'_{s\mathcal{U}}$. The hypergraph in Figure 2(right) satisfied $\kappa'_{s\mathcal{U}} = 1$ since every hyperarc contains a strong cut vertex. On the other hand we have $\kappa_{w\mathcal{S}} = k$ and this, for

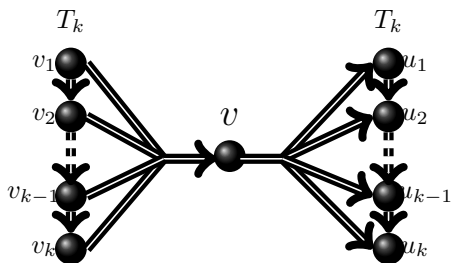


Figure 3: In this hypergraph with even k we have $\kappa_{w\mathcal{S}} = \kappa_{w\mathcal{U}} = 1$ since the vertex v is a cut vertex; $\kappa'_{s\mathcal{U}} = \kappa'_{s\mathcal{S}} = 1 + \frac{k}{2}$ since $\{(v, \{u_1, \dots, u_k\}), (v_1, v_2), (v_3, v_4), \dots, (v_{k-1}, v_k)\}$ is a strong disconnecting set.

$k > 1$, $\kappa'_{s\mathcal{U}} < \kappa_{w\mathcal{S}}$. In a directed cycle C_n with $n > 5$ we have $\kappa_{w\mathcal{S}} = 1$ and $\kappa_{s\mathcal{U}} = 2$ and therefore $\kappa_{w\mathcal{S}} < \kappa_{s\mathcal{U}}$.

$\kappa_{w\mathcal{U}} \leq \kappa'_{w\mathcal{S}}$. In a directed cycle of length $n > 3$ we have $\kappa'_{w\mathcal{S}} = 1$ and $\kappa_{w\mathcal{U}} = 2$ and this $\kappa'_{w\mathcal{S}} < \kappa_{w\mathcal{U}}$. The hypergraph in Figure 1(right) satisfied $\kappa'_{w\mathcal{S}} = \frac{k}{2}$ and $\kappa_{w\mathcal{U}} = 2$ since the set $\{u, v\}$ is a weak vertex cut. Thus, for $k > 4$, we have $\kappa_{w\mathcal{U}} < \kappa'_{w\mathcal{S}}$.

$\kappa_{w\mathcal{U}} \leq \kappa'_{w\mathcal{U}}$. The hypergraph in Figure 2(right) satisfies $\kappa'_{w\mathcal{U}} = 2$ and $\kappa_{w\mathcal{U}} = k$. Thus, for $k > 2$ we have $\kappa'_{w\mathcal{U}} < \kappa_{w\mathcal{U}}$. The hypergraph in Figure 1(right) satisfied $\kappa_{w\mathcal{U}} = 2$ and $\kappa'_{w\mathcal{U}} = k + 1$. This, for $k > 2$ we have $\kappa_{w\mathcal{U}} < \kappa'_{w\mathcal{U}}$.

$\kappa_{s\mathcal{U}} \leq \kappa'_{w\mathcal{S}}$. In a directed cycle C_n of length $n > 3$ we have $\kappa'_{w\mathcal{S}} = 1$ and $\kappa_{s\mathcal{U}} = 2$, i.e., $\kappa'_{w\mathcal{S}} < \kappa_{s\mathcal{U}}$. The hypergraph in Figure 1(right) satisfies $\kappa_{s\mathcal{U}} = 2$ and $\kappa'_{w\mathcal{S}} = \frac{k}{2}$. Thus, for $k > 4$ we have $\kappa_{s\mathcal{U}} < \kappa'_{w\mathcal{S}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa_{s\mathcal{U}}$. The hypergraph in Figure 1(right) satisfied $\kappa'_{s\mathcal{S}} = 1$ and $\kappa_{s\mathcal{U}} = k + 1$, i.e., $\kappa'_{s\mathcal{S}} < \kappa_{s\mathcal{U}}$. For the hypergraph in Figure 2(left) we have $\kappa_{s\mathcal{U}} = 1$ and $\kappa'_{s\mathcal{S}} = 2$ and thus $\kappa_{s\mathcal{U}} < \kappa'_{s\mathcal{S}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa_{s\mathcal{S}}$. The hypergraph in Figure 1(right) satisfies $\kappa'_{s\mathcal{S}} = 1$ and $\kappa_{s\mathcal{S}} = 2$, and thus $\kappa'_{s\mathcal{S}} < \kappa_{s\mathcal{S}}$. For the hypergraph in Figure 2(left) we have $\kappa_{s\mathcal{S}} = 1$ and $\kappa'_{s\mathcal{S}} = 2$, and therefore $\kappa_{s\mathcal{S}} < \kappa'_{s\mathcal{S}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa_{w\mathcal{S}}$. The hypergraph in Figure 3 satisfies that $\kappa'_{s\mathcal{S}} = 1 + \frac{k}{2}$ and $\kappa_{w\mathcal{S}} = 1$, hence for $k > 2$ we have $\kappa'_{s\mathcal{S}} > \kappa_{w\mathcal{S}}$. The hypergraph in Figure 2(right) satisfies $\kappa'_{s\mathcal{S}} = 1$ and $\kappa_{w\mathcal{S}} = k$, thus for $k > 1$ we have $\kappa'_{s\mathcal{S}} < \kappa_{w\mathcal{S}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa_{w\mathcal{U}}$. The hypergraph in Figure 3 satisfies $\kappa'_{s\mathcal{S}} = 1 + \frac{k}{2}$ and $\kappa_{w\mathcal{U}} = 1$ thus for $k > 2$ we have $\kappa'_{s\mathcal{S}} > \kappa_{w\mathcal{U}}$. The hypergraph in Figure 2(right) satisfies $\kappa'_{s\mathcal{S}} = 1$ and $\kappa_{w\mathcal{U}} = k$, hence for $k > 1$ we have $\kappa'_{s\mathcal{S}} < \kappa_{w\mathcal{U}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa'_{w\mathcal{S}}$. The hypergraph in Figure 3 satisfies $\kappa'_{s\mathcal{S}} = 1 + \frac{k}{2}$ and $\kappa'_{w\mathcal{S}} = 1$, hence for $k > 2$ we have $\kappa'_{s\mathcal{S}} > \kappa'_{w\mathcal{S}}$. The hypergraph in Figure 1(left) satisfies $\kappa'_{s\mathcal{S}} = 1$ and $\kappa'_{w\mathcal{S}} = \frac{k}{2}$, hence for $k > 1$ we have $\kappa'_{s\mathcal{S}} < \kappa'_{w\mathcal{S}}$.

$\kappa'_{s\mathcal{S}} \leq \kappa'_{w\mathcal{U}}$. The hypergraph in Figure 3 satisfies $\kappa'_{s\mathcal{S}} = 1 + \frac{k}{2}$ and $\kappa'_{w\mathcal{U}} = 1$, hence for $k > 2$ we have $\kappa'_{s\mathcal{S}} > \kappa'_{w\mathcal{U}}$. The hypergraph in Figure 1(left) satisfies $\kappa'_{s\mathcal{S}} = 1$ and $\kappa'_{w\mathcal{U}} = \frac{k}{2}$, hence for $k > 1$ we have $\kappa'_{s\mathcal{S}} < \kappa'_{w\mathcal{U}}$.

4 Whitney's theorem for directed hypergraphs

Let $H = (V, E)$ a directed hypergraph (or digraph) and $v \in V$. The *total degree of the vertex v* is $d^t(v) = d^+(v) + d^-(v)$. Denote by δ_{id} , δ_{od} , and $\delta_{\mathcal{W}}$ the minimum of d^- , d^+ and d^t over all $v \in V$, respectively. Furthermore we introduce

$$\delta_{id}^{\mathcal{W}} = \min_{v \in V} \{d^-(v) + \delta_{id}(H \setminus_w v)\} \quad \text{and} \quad \delta_{od}^{\mathcal{W}} = \min_{v \in V} \{d^+(v) + \delta_{od}(H \setminus_w v)\}.$$

With this notation we define $\delta_{\mathcal{S}} = \min\{\delta_{id}, \delta_{od}\}$ and $\delta_{\mathcal{W}} = \min\{\delta_{id}^{\mathcal{W}}, \delta_{od}^{\mathcal{W}}\}$. These parameters are direct generalizations of the corresponding quantities for directed hypergraphs, see e.g. [8].

The next theorem is a generalization of Whitney's inequalities for directed hypergraphs. The proof follows ideas from [8] for the analogous result for digraphs.

Theorem 4.1. *Let $H = (V, E)$ a directed hypergraph. Then $\kappa_{s\mathcal{W}}(H) \leq \kappa'_{w\mathcal{W}}(H) \leq \delta_{\mathcal{W}}(H)$ and $\kappa_{s\mathcal{S}}(H) \leq \kappa'_{w\mathcal{S}}(H) \leq \delta_{\mathcal{S}}(H)$.*

Proof. If H is trivial or null, the statements of the theorem are obviously valid.

Let H be a \mathcal{W} -connected hypergraph and let $u, v \in V$ such that $d^-(u) = \delta_{id}(H)$ and $d^-(v) = \delta_{id}(H \setminus_w u)$. Weakly eliminate the hyperarcs such that their heads contain u and v ; in this way, there is no (u, v) -directed path and there is no (v, u) -directed path on H . So $\kappa'_{w\mathcal{W}}(H) \leq \delta_{in}^{\mathcal{W}}$. Applying the same dual argument we conclude that $\kappa'_{w\mathcal{W}}(H) \leq \delta_{od}^{\mathcal{W}}$ and so $\kappa'_{w\mathcal{W}}(H) \leq \delta_{\mathcal{W}}$.

On the other hand, if $\kappa'_{w\mathcal{W}}(H) = 1$, there is $e \in E$, such that $H \setminus_w e$ is not \mathcal{W} -connected. If we eliminate in a strong way the vertex $v \in T(e) \cup H(e)$ then $H \setminus_s v$ is not \mathcal{W} -connected, so $\kappa_{s\mathcal{W}}(H) = 1$ and the result is valid. Let $\kappa'_{w\mathcal{W}}(H) > 1$, for proving that $\kappa_{s\mathcal{W}}(H) \leq \kappa'_{w\mathcal{W}}(H)$ let weakly eliminate set of hyperarcs F , the cardinality of F is $\kappa'_{w\mathcal{W}}(H) - 1$ such that the directed hypergraph $H' = (V', E') = H \setminus_w F$ has $\kappa'_{w\mathcal{W}}(H') = 1$. Let $e \in (E')$ a hyperarc such that $H' \setminus_w e$ is not unilaterally connected. Now we strongly eliminate the set of vertices $X \in V$ such that each vertex on X is in exactly one hyperarc of F (there are enough vertices due to $|e| > 1$ for all $e \in E$), we denote this directed hypergraph $H'' = (V'', E'')$. If $e \notin E''$ then H'' is not unilaterally connected, so $\kappa_{s\mathcal{W}}(H) < \kappa'_{w\mathcal{W}}(H)$. If $e \in E''$ then $\kappa'_{w\mathcal{W}}(H'') = 1$ and $\kappa_{s\mathcal{W}}(H'') = \kappa'_{w\mathcal{W}}(H'') = 1$ so $\kappa_{s\mathcal{W}}(H) = \kappa'_{w\mathcal{W}}(H)$.

Let H be \mathcal{S} -connected hypergraph and let $v \in V$ such that $d^-(v) = \delta_{id}(H)$. Weakly eliminate the hyperarcs such that their heads contain v ; in this way, there is no (u, v) -directed path on H for all $u \in V$. So $\kappa'_{w\mathcal{S}}(H) \leq \delta_{in}$. Applying the same dual argument we conclude that $\kappa'_{w\mathcal{S}}(H) \leq \delta_{od}$ and so $\kappa'_{w\mathcal{S}}(H) \leq \delta_{\mathcal{S}}$.

The proof that $\kappa_{s\mathcal{S}}(H) \leq \kappa'_{w\mathcal{S}}(H)$ parallels the proof of the inequality $\kappa_{s\mathcal{W}}(H) \leq \kappa'_{w\mathcal{W}}(H)$. \square

Note that Theorem 4.1 reduces the corresponding statement for digraphs whenever all $e \in E$ hyperarcs satisfy $|T(e)| = |H(e)| = 1$.

Corollary 4.2. *Let $H = (V, E)$ a directed hypergraph. Then $\kappa_{s\mathcal{S}} \leq \kappa_{w\mathcal{W}}$ and $\kappa_{s\mathcal{S}} \leq \kappa'_{w\mathcal{W}}$.*

Proof. The first inequality follows from the note at the beginning of the previous section and Lemma 3.1. The second inequality follows from Theorem 4.1 and Equation 3.1. \square

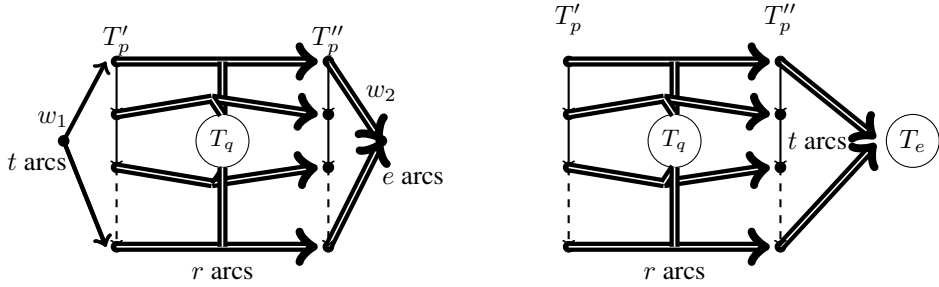


Figure 4: Left: Construction (A); Right: Construction (B)

We next show that the parameters $\kappa_{s\mathcal{C}}(H)$, $\kappa'_{s\mathcal{C}}(H)$, $\kappa_{w\mathcal{C}}(H)$ and $\kappa'_{w\mathcal{C}}(H)$ are independent for strongly or unilaterally connected directed hypergraphs satisfying $\kappa_{s\mathcal{C}}(H) \leq \min\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\}$.

Theorem 4.3. *For every choice of natural numbers a, b, c, d , and e with $a \leq \min\{c, d\}$, $b \leq c$ and $\max\{b, c, d\} \leq e$ and connectedness classes $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}, \mathcal{W}\}$ there exists a directed hypergraph such that $\kappa_{s\mathcal{C}}(H) = a$, $\kappa'_{s\mathcal{C}}(H) = b$, $\kappa_{w\mathcal{C}}(H) = c$, $\kappa'_{w\mathcal{C}}(H) = d$ and $\delta_{\mathcal{C}}(H) = e$.*

Proof. We explicitly construct hypergraphs with the desired properties.

Consider unilateral connectedness, i.e., $\mathcal{C} = \mathcal{U}$. We construct a hypergraph A from by five disjoint components: Two tournaments T'_p and T''_p on p vertices, a tournament T_q , and two single vertices w_1 and w_2 ; each arc in the tournaments T'_p and T''_p has multiplicity $e - p + 1$ and each arc in T_q has multiplicity $e - q + 1$. We then insert r hyperarcs consisting of one vertex from T'_p and T_q in its tail and a vertex from T''_p in its head involved, with all reverse hyperarcs added and with one of these hyperarcs with multiplicity t . Finally, we insert t arcs from w_1 to T'_p and e arcs from w_2 to T''_p with all reverse arcs added. The arcs are inserted in such a way that the vertices of the tournaments are covered as uniformly as possible. The construction is illustrated in Figure 4(left).

Let $a = q, b = r, c = p, d = t$. The minimum strong vertex cut in A is $V(T_q)$, the minimum strong disconnecting set is the r different hyperarcs, a minimum weak vertex cut is the set $V(T'_p)$ and the minimum weak disconnecting set is the t arcs (recall that one of the r hyperarcs has multiplicity t); then $q = \kappa_{s\mathcal{U}}$, $r = \kappa'_{s\mathcal{U}}$, $p = \kappa_{w\mathcal{U}}$ and $t = \kappa'_{w\mathcal{U}}$. Finally $\delta_{\mathcal{U}} = e$ because $d^-(w_1) = 0$ and $d^-(w_2) = e$, since all the vertices in the tournaments have in-degree and out-degree at least e , because of the multiplicities of the arcs inside the tournaments.

Next we consider strongly connectedness, $\mathcal{C} = \mathcal{S}$. We construct a hypergraph B from by five disjoint components: Two tournaments T'_p and T''_p on p vertices, a tournament T_q , and a tournament T_e ; each arc in the tournaments T'_p and T''_p has multiplicity $e - p + 1$ and each arc in T_q has multiplicity $e - q + 1$. We then insert r hyperarcs consisting of one vertex from T'_p and T_q in its tail and a vertex from T''_p in its head involved, with all reverse hyperarcs added and with all of these hyperarcs with multiplicity t . The arcs are inserted in such a way that the vertices of the tournaments are covered as uniformly as possible. Finally, we insert t hyperarcs consisting of at least one vertex from T_e in its tail and a

vertex from T_p'' in its head involved, all vertices in T_e are in a head of these hyperarcs, all the reverse hyperarcs are added. The construction is illustrated in Figure 4(right).

Let $a = q$, $b = r$, $c = p$, and $d = t$. The minimum strong vertex cut in B is $V(T_q)$, the minimum strong disconnecting set is the r different hyperarcs, the minimum weak vertex cut is the set $V(T_p'')$ and a minimum weak disconnecting set is the t arcs from T_e to T_p'' (don't forget that each of the r hyperarcs has multiplicity t); then $q = \kappa_{s\mathcal{S}}$, $r = \kappa'_{s\mathcal{S}}$, $p = \kappa_{w\mathcal{S}}$ and $t = \kappa'_{w\mathcal{W}}$. Finally $\delta_{\mathcal{S}} = e$ because all the vertices in T_e have indegree and outdegree e (all the vertices in the other tournaments have indegree and outdegree at least e , because of the multiplicities of the arcs inside the tournaments).

Next, we consider weak connectedness, $\mathcal{C} = \mathcal{W}$. We construct a hypergraph C from by five disjoint components: Two complete graphs K_p' and K_p'' on p vertices, a complete graph K_q , and a complete graph K_e ; each edge in the complete graphs K_p' and K_p'' has multiplicity $e - p + 1$ and each edge in K_q has multiplicity $e - q + 1$. We then insert r edges consisting of one vertex from K_p' , K_q and K_p'' , with all of these edges with multiplicity t . The edges are inserted in such a way that the vertices of the complete graphs are covered as uniformly as possible.

Finally, we insert t edges consisting of at least one vertex from T_e and one vertex from T_p'' , all vertices in T_e are incident with these edges. The explanation of why this hypergraph has the desired parameters is analogous to the strong case. \square

5 König digraph of a directed hypergraph

The connectivity invariants in digraphs are defined in the same way as in directed hypergraphs, the difference is that the weak or strong elimination of vertices or arcs is not relevant so we only have to consider a single connectivity index for each connectedness class, which we denote by $\kappa_{\mathcal{C}}$ and $\kappa'_{\mathcal{C}}$ with $\mathcal{C} \in \{\mathcal{S}, \mathcal{U}\}$.

Lemma 5.1. *Let $H = (V, E)$ be a directed hypergraph and let $G(H) = (V \cup E, A)$ be its König digraph. Then*

$$\kappa_{s\mathcal{C}}(H) \leq \kappa_{\mathcal{C}}(G(H)) \leq \min\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\} \quad (5.1)$$

holds for $\mathcal{C} \in \{U, S\}$.

Proof. Let $S \subseteq V \cup E$ be a vertex cut in $G(H)$ with $|S| = \kappa_{\mathcal{C}}(G(H))$.

Case 1: $S \subseteq V$, then $S = \{v_1, \dots, v_k\}$ is a weak vertex cut in H and so it is a strong vertex cut in H , so $\kappa_{s\mathcal{C}}(H) \leq \kappa_{\mathcal{C}}(G(H))$.

Case 2: Suppose $S \subseteq E$. We step-wisely construct a vertex cut S' as follows by interacting over the hyperedges in S . In each step we add to S' a single vertex $v_i \in e_i$ that is not contained in $\bigcup_{j < i} \text{supp}(e_j)$. By construction we have $|S'| \leq |S|$. Since $H \setminus_s S'$ is not strongly or unilaterally connected we have $\kappa_{s\mathcal{C}}(H) \leq \kappa_{\mathcal{C}}(G(H))$.

Case 3: Suppose $S \cap V \neq \emptyset$ and $S \cap E \neq \emptyset$. We write $S = \{v_1, \dots, v_l, e_{l+1}, \dots, e_k\}$ where $v_i \in V$ for $i \in \{1, \dots, l\}$ and $e_i \in E$ for $i \in \{l+1, \dots, k\}$. Let $S' = \{v_1, \dots, v_l\}$. We iterate over the $e_i \in S \cap E$ and, in each step, we add to S' , if it exists, a vertex $v_i \in e_i$ satisfying $v_i \notin S \cap V$ and $v_i \notin \bigcup_{j < i} \text{supp}(e_j)$. This yields a strong vertex cut containing $S \cap V$ and at most one vertex from each $e_i \in S \cap E$, thus $|S'| \leq |S|$. So $H \setminus_s S'$ is not strongly or unilaterally connected and $\kappa_{s\mathcal{C}}(H) \leq \kappa_{\mathcal{C}}(G(H))$.

Considering the other inequality, let $S \subseteq V$ be a minimal weak vertex cut in H with $|S| = \kappa_{w\mathcal{C}}(H)$. Since $H \setminus_w S$ is not strongly or unilaterally connected, then by Lemma 2.4,

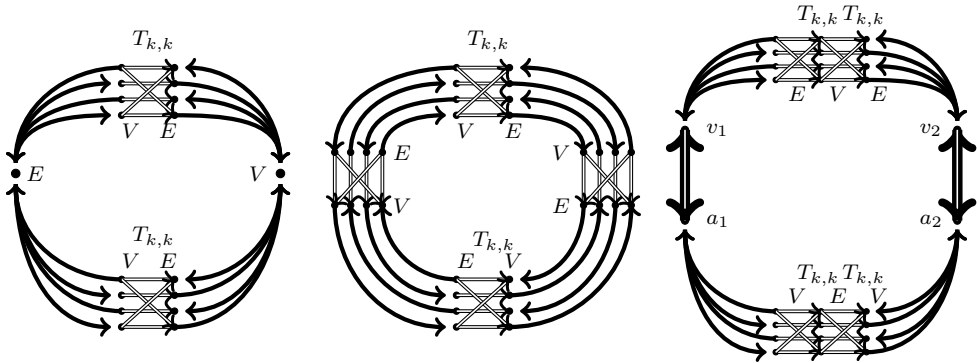


Figure 5: **Left:** König graph $G(H)$ of a hypergraph with $\kappa'_{w\mathcal{U}}(H) = \kappa_{w\mathcal{U}}(H) = k + 1$ and $\kappa'_{w\mathcal{S}}(H) = \kappa_{w\mathcal{S}}(H) = \frac{k}{2} + 1$. **Middle:** König graph $G(H)$ of a hypergraph with $\kappa_{s\mathcal{C}}(H) = \kappa'_{s\mathcal{C}} = 1$, $\kappa_{\mathcal{U}}(G(H)) = k + 1$ and $\kappa_{\mathcal{S}}(G(H)) = k$. **Right:** König graph $G(H)$ of a hypergraph with $\kappa_{w\mathcal{C}}(H) = \kappa'_{w\mathcal{C}}(H) = 2$, $\kappa'_{\mathcal{U}}(G(H)) = 2k$ and $\kappa'_{\mathcal{S}}(G(H)) = k + \frac{k}{2}$.

$G_H \setminus S$ is not strongly or unilaterally connected on $V \setminus S$, so $\kappa_{\mathcal{C}}(G(H)) \leq \kappa_{w\mathcal{C}}(H)$. Let $F \subseteq E$ be a minimal weak disconnecting set in H with $|S| = \kappa_{w\mathcal{C}}(H)$, as $H \setminus_w F$ is not strongly or unilaterally connected, then by Lemma 2.4, $G_H \setminus F$ is not unilaterally connected on V , so $\kappa_{\mathcal{C}}(G(H)) \leq \kappa'_{w\mathcal{C}}(H)$. Then $\kappa_{\mathcal{C}}(G(H)) \leq \min\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\}$. \square

In practice, however, $\min\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\}$ is not a particularly good upper bound for $\kappa_{\mathcal{C}}(G(H))$ for either strong or unilateral connectedness. We show that the difference, in fact, can become arbitrarily large. Similarly, the difference $\kappa_{\mathcal{C}}(G(H)) - \kappa_{s\mathcal{C}}(H)$ can also become arbitrarily large for both strong and unilateral connectedness.

The graph in the left panel of Figure 5 is the König digraph $G(H)$ of a directed hypergraph $H = (V, E)$ with $\kappa_{\mathcal{U}}(G(H)) = \kappa_{\mathcal{S}}(G(H)) = 2$ (as seen by removing the vertices that are not in any $T_{k,k}$ subgraph). On the other hand, removing only vertices in V , we need to eliminate $k + 1$ vertices for $G(H)$ to destroy unilateral connectedness, namely k vertices in one of the $T_{k,k}$ in the V set partition, and one vertex V that is not in these complete digraphs. We need to remove $\frac{k}{2} + 1$ vertices for $G(H)$ not being strongly connected, $\frac{k}{2}$ vertices in any of the $T_{k,k}$ in the V set (the ones that are in- or out-neighbors of a vertex in E that is not in any $T_{k,k}$), and a vertex V that is not in any $T_{k,k}$ digraph. Therefore $\kappa_{w\mathcal{U}}(H) = k + 1$ and $\kappa_{w\mathcal{S}}(H) = \frac{k}{2} + 1$. The same argument is correct for eliminating vertices only in E so $\kappa'_{w\mathcal{U}}(H) = k + 1$ and $\kappa'_{w\mathcal{S}}(H) = \frac{k}{2} + 1$. As k increases, an infinite family of hypergraphs for which the difference $\min\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\} - \kappa_{\mathcal{C}}(G(H))$ grows linearly is obtained.

The middle panel of Figure 5 shows the König graph $G(H)$ of a directed hypergraph $H = (V, E)$ with $\kappa_{\mathcal{U}}(G(H)) = k + 1$ (removing the k vertices in of one partition set in $T_{k,k}$ and one in the same partition set in the adjacent complete digraph) and $\kappa_{\mathcal{S}}(G(H)) = k$ (removing the k vertices in of one partition set in $T_{k,k}$ subgraph and then any neighbor of any vertex in the other partition set of the same subgraph). In the hypergraph, on the other hand, it suffices to strongly eliminate any vertex in $G(H)$ to destroy strong connectedness, since this amount to removing a vertex together with its neighborhood from $G(H)$. Thus $\kappa_{s\mathcal{C}}(H) = 1$. Therefore, $\kappa_{\mathcal{C}}(G(H)) - \kappa_{s\mathcal{C}}(H) = k$. As k increases, we obtain an infinite

family of hypergraphs for which this difference grows linearly.

Lemma 5.2. *Let $H = (V, E)$ be a directed hypergraph and $G(H) = (V \cup E, A)$ its König digraph. Then for $\mathcal{C} \in \{\mathcal{U}, \mathcal{S}\}$ holds*

$$\max\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\} \leq \kappa'_{\mathcal{C}}(G(H)) \leq \delta_{\mathcal{C}}(G(H)). \quad (5.2)$$

Proof. Let $S \subseteq A$ be a minimal disconnecting set in $G(H)$ with $|S| = \kappa'_{\mathcal{C}}(G(H)) = k$ and let $S' = \{v \in V \mid v \in \text{supp}(e) \wedge e \in S\}$. Note that $|S'| \leq |S|$. $G(H) \setminus S'$ is not strong or unilaterally connected (and in particular not unilaterally connected on V). Therefore S' is a weak disconnecting set in H by Lemma 2.4, and thus we have $\kappa_{w\mathcal{C}}(H) \leq |S'| \leq |S| = \kappa'_{\mathcal{C}}(G(H))$. An analogous argument using $F = \{e \in E \mid e \in \text{supp}(e') \wedge e' \in S\}$ yields $\kappa'_{w\mathcal{C}}(H) \leq |S| = \kappa'_{\mathcal{C}}(G(H))$. The remaining inequalities are the Whitney inequalities for digraphs [8]. \square

Recall that in general we have $\delta_{\mathcal{C}}(H) \leq \delta_{\mathcal{C}}(G(H))$. The inequality is strict for some hypergraphs. This is the case even if $\max\{\kappa'_{w\mathcal{C}}(H), \kappa_{w\mathcal{C}}(H)\} = \kappa_{w\mathcal{C}}(H)$. For example in Figure 2 (right) we have $\kappa_{w\mathcal{C}}(H) = k$ and $\delta_{\mathcal{C}}(H) = 1$ for both unilateral and strong connectedness assuming $\max\{\kappa'_{w\mathcal{C}}(H), \kappa_{w\mathcal{C}}(H)\} = \kappa_{w\mathcal{C}}(H)$.

We note, finally, that the difference $\kappa'_{\mathcal{C}}(G(H)) - \max\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\}$ can be arbitrarily large. The right panel of Figure 5 shows the König digraph $G(H)$ of a directed hypergraph $H = (V, E)$ with even k is even and the arcs $(v_1, a_1), (v_2, a_2)$ are k parallel arcs of each direction between these vertices. We $\kappa'_{\mathcal{U}}(G(H)) = 2k$ (due to removal of the $2k$ arcs incident with v_1 or v_2 or a_1 or a_2) and $\kappa'_{\mathcal{S}}(G(H)) = k + \frac{k}{2}$ (due to removal of the k arcs with tail (or head) any of the vertices v_1, v_2, a_1, a_2). In the hypergraph H , $\{v_1, v_2\}$ is a weak vertex cut, hence $\kappa_{w\mathcal{C}}(H) = 2$. The same is true for removing $\{a_1, a_2\}$, thus $\kappa'_{w\mathcal{C}}(H) = 2$. Thus $\kappa'_{\mathcal{C}}(G(H)) - \max\{\kappa_{w\mathcal{C}}(H), \kappa'_{w\mathcal{C}}(H)\} \geq k + \frac{k}{2} - 2$. We therefore obtain an infinite family of hypergraphs for which this difference grows linearly with k .

6 Concluding remarks

We have seen that some of the connectivity invariants of directed hypergraphs are “ill-behaved” in the sense that they are not bounded by any other connectivity invariant. This is in particular the case for $\kappa'_{s\mathcal{C}}$. It is an interesting open question, therefore, whether there are interesting structural constraints on the directed hypergraph for which $\kappa'_{s\mathcal{C}}$ is bounded by some of the other connectivity parameters. A class of hypergraphs that is relevant in this context are those whose minimal cut sets are covered by collections of hyperedges that form a disconnecting set. It remains a question for future research whether such connectivity properties are related to classes of hypergraphs that have already received attention in the literature.

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