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On L-fuzzy partitioned automata

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Abstract

This paper is towards the study of theory of fuzzy automata with fuzzy partitions. Specifically, we study the concept of the L-fuzzy partitioned automaton corresponding to a given L-fuzzy automaton. Further, we introduce the concept of a crisp-deterministic L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept the same L-fuzzy language. Finally, the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton is introduced and a characterization of its L-fuzzy language is given.

Keywords: L-fuzzy automata; L-fuzzy languages; L-fuzzy partitions; L-fuzzy partitioned automata.

1 Introduction

Since the theory of fuzzy sets was introduced by Zadeh [43], fuzzy automata and languages have been studied as methods for bridging the gap between the precision of computer languages and vagueness. These studies were initiated by Santos [32], Wee [41], and Wee and Fu [42], and further developed by a number of researchers (cf., [18, 22, 25]). Fuzzy automata and languages with membership values in different lattice structures have attracted considerable attention from researchers in this area (cf., [1–3, 6, 10–18, 20, 21, 26–31, 33–35, 37, 38, 40]). Among these works, the work of Jin and his coworkers [14] is towards the algebraic study of fuzzy automata based on po-monoids; the work of Peeva is towards the study of minimizing the states of fuzzy automata and its application to study pattern recognition (cf., [26, 27]); the work of Kim, Kim and Cho [18] is towards the algebraic study of fuzzy automata theory; the work of Abolpour and Zahedi is towards the use of categorical concepts in the study of automata with membership values in different lattice structures (cf., [1–3]); the work of Horry and Zahedi [10] is towards the use fuzzy topologies for the study of a max-min general fuzzy automaton; the work of Das [6] is towards the fuzzy topological characterization of a fuzzy automaton; the work of Qiu is towards the algebraic and topological study of fuzzy automata theory based on residuated lattices (cf., [28–31]); the work of Li and Pedrycz [20] is towards the fuzzy automata based on lattice-ordered monoids; the work of Ćirić and his coworkers is towards the study of determinism in fuzzy automata theory (cf., [11–13]), and the work of Tiwari and his coworkers is towards the algebraic and topological study of fuzzy automata (cf., [33–35, 37, 38, 40]). In application point of view, fuzzy automata provide a useful surrounding for ambiguous computation and have shown their importance for solving meaningful problems in learning systems, pattern recognition and data base theory (cf., [4, 25, 27]).

In this paper specifically, we introduce and study

- the concept of the L-fuzzy partitioned automaton corresponding to a given L-fuzzy automaton;
- the crisp-deterministic L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language; and
- the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton.

The content of this paper is arranged as follows. Section 2 contains preliminary information about the content of the paper. In Section 3, we introduce the concept of the L-fuzzy partitioned automaton corresponding to a given L-fuzzy automaton. Further, we study the relationship among the L-fuzzy languages of the L-fuzzy partitioned automaton and L-fuzzy automaton. In Section 4, we introduce the crisp-deterministic

L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language. Finally, in section 5, the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton is introduced. Interestingly, we show that the L-fuzzy language of fuzzified L-fuzzy partitioned automaton can be obtained from the L-fuzzy language of the L-fuzzy partitioned automaton.

2 Preliminaries

In this section, we recall the concepts related to residuated lattices [5,39]; L-fuzzy relations [24,39]; L-fuzzy automata [7,23,36]; L-fuzzy languages [7,39], and L-fuzzy objects [23,24].

We begin with the following.

Definition 2.1. *An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called* **complete residuated lattice** *if it satisfies the following conditions:*

(i) $(L, \leqslant, \wedge, \vee, 0, 1)$ *is a complete lattice with the greatest element 1 and the least element 0;*

(ii) $(L, \odot, 1)$ *is a commutative monoid; and*

(iii) $x \odot y \leqslant z$ *iff* $x \leqslant y \rightarrow z$, $\forall x, y, z \in L$.

Throughout this paper, we assume L is a complete residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ and the L-fuzzy sets considered in this paper are in sense of [9], i.e., an L-fuzzy set A in a set X is a map $A : X \longrightarrow L$. For a nonempty set X, $\mathcal{F}(X)$ denotes the collection of all L-fuzzy sets in X. Also, for $x, y \in L$, $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$ and Λ denotes an indexed set.

Definition 2.2. *For L-fuzzy set A in a nonempty set X,* **core of** A*, denoted by core(A), is given as,*

$$core(A) = \{x \in X : A(x) = 1\}.$$

Further, if core(A)$\neq \phi$, then A is called **normal** L**-fuzzy set**.

Proposition 2.1. *[19, 39] Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a complete residuated lattice. Then for all $x, y, z, x_j, y_j \in L$ and $j \in \Lambda$, the following properties hold:*

(i) $x \leftrightarrow y = 1 \Leftrightarrow x = y$;

(ii) $x \leftrightarrow y \leqslant y \rightarrow x$;

(iii) $x \leftrightarrow y = y \leftrightarrow x$;

(iv) $y \leftrightarrow z \leqslant (x \odot y) \leftrightarrow (x \odot z)$; *and*

(v) $x \odot (\vee\{y_j : j \in \Lambda\}) = \vee\{x \odot y_j : j \in \Lambda\}$ *and* $(\vee\{x_j : j \in \Lambda\}) \odot y = \vee\{x_j \odot y : j \in \Lambda\}$.

Definition 2.3. *An L-***fuzzy relation** *on a nonempty set X is a map $E : X \times X \longrightarrow L$. The L-fuzzy relation E is called*

(i) **reflexive** *if* $E(x, x) = 1, \forall x \in X$;

(ii) **symmetric** *if* $E(x, y) = E(y, x)$, $\forall x, y \in X$; *and*

(iii) **transitive** *if* $E(x, y) \odot E(y, z) \leqslant E(x, z)$, $\forall x, y, z \in X$.

A reflexive, symmetric, and transitive L-fuzzy relation on X is called an L**-fuzzy similarity relation** *on X.*

Now, we recall the following concepts related to the L-fuzzy automata.

Definition 2.4. *An* L**-fuzzy automaton** *is a system $\mathcal{M} = (Q, (M, *, e), T, I, F)$, where Q is a nonempty set of states, $(M, *, e)$ is a monoid inputs, $T : Q \times M \longrightarrow L^Q$ is the transition function such that $\forall p, q \in Q$ and $\forall m, n \in M$,*

$$T(p, e)(q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q, \text{ and} \end{cases}$$

$$T(p, m * n)(q) = \vee\{T(p, m)(r) \odot T(r, n)(q) : r \in Q\},$$

$I \in \mathcal{F}(Q)$ is the initial L-fuzzy state and $F \in \mathcal{F}(Q)$ is the final L-fuzzy state.

A state $q \in Q$ is called **initial** *(resp.* **final***)* **state** *of $\mathcal{M}$ if $I(q) > 0$ (resp. $F(q) > 0$). An L-fuzzy automaton whose set of states is finite is called* **finite** L**-fuzzy automaton**.

Definition 2.5. *An L-fuzzy automaton $\mathcal{M} = (Q, (M, *, e), T, I, F)$ is called*

(i) **complete** *if for all $m \in M$ and $p \in Q$ there exists q such that $T(p, m)(q) > 0$,*

(ii) **deterministic** *if there is a unique initial state q_0 with $I(q_0) > 0$ and for all $m \in M$ and $p, q, r \in Q$ if $T(p, m)(q) > 0$ and $T(p, m)(r) > 0$, then $q = r$.*

If $\mathcal{M} = (Q, (M, *, e), T, I, F)$ is a complete deterministic L-fuzzy automaton such that for all $m \in M$ and $p, q \in Q$, $T(p, m)(q) \in \{0, 1\}$ and for unique initial state q_0, $I(q_0) = 1$, then $\mathcal{M}$ is called **crisp-deterministic** L**-fuzzy automaton**. *In this case, there exists a function $\delta : Q \times M \longrightarrow Q$ such that for all $p \in Q$ and $m \in M$, $\delta(p, m) = q$ iff $T(p, m)(q) = 1$. Such crisp-deterministic L-fuzzy automaton is denoted by $(Q, (M, *, e), \delta, q_0, F)$.*

Definition 2.6. *An* L**-fuzzy language** $f_{\mathcal{M}} : M \longrightarrow L$ *is*

(i) **accepted** *by an L-fuzzy automaton $\mathcal{M} = (Q, (M, *, e), T, I, F)$ if $f_{\mathcal{M}}(m) = \vee\{I(r) \odot T(r, m)(q) \odot F(q) : r, q \in Q\}$, $\forall m \in M$; and*

(ii) **accepted** *by a crisp-deterministic L-fuzzy automaton $\mathcal{M} = (Q, (M, *, e), \delta, q_0, F)$ if $f_{\mathcal{M}}(m) = F(\delta(q_0, m))$, $\forall m \in M$.*

Definition 2.7. *Let X be a nonempty set. A system $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ of normal L-fuzzy sets in X is an* **L-fuzzy partition** *of X, if $\{core(A_\lambda) : \lambda \in \Lambda\}$ is a partition of X. A pair $(X, \mathcal{A})$ is called a* **space with an L-fuzzy partition**.

Now, we recall the following concept of L-fuzzy objects in a spaces with L-fuzzy partitions.

Definition 2.8. *Let $(L, \mathcal{L})$ be a space with an L-fuzzy partition and $\mathcal{L} = \{L_a : a \in L\}$ be an L-fuzzy partition of L such that $\forall a, b \in L$, $L_a(b) = a \leftrightarrow b$. Then an* **$L$-fuzzy object** *in a space with an L-fuzzy partition $(X, \mathcal{A})$ is a map $(A, \sigma) : (X, \mathcal{A}) \longrightarrow (L, \mathcal{L})$ such that*

(i) $A : X \longrightarrow L$ *is a map;*

(ii) $\sigma : \Lambda \longrightarrow L$ *is a map; and*

(iii) $\forall \lambda \in \Lambda$ *and* $\forall x \in X$, $A_\lambda(x) \leqslant L_{\sigma(\lambda)}(A(x)) = \sigma(\lambda) \leftrightarrow A(x)$.

$R(X, \mathcal{A})$ denotes the set of all L-fuzzy objects in $(X, \mathcal{A})$.

Now, we recall the following from [8, 24].

Let $(X, \mathcal{A})$ be a space with an L-fuzzy partition and $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ be an L-fuzzy partition of X. Then the L-fuzzy relation π on a set Λ is defined as:

$\pi(\lambda_1, \lambda_2) = (\vee\{A_{\lambda_1}(x) : x \in core(A_{\lambda_2})\}) \vee (\vee\{A_{\lambda_2}(x) : x \in core(A_{\lambda_1})\})$, $\forall \lambda_1, \lambda_2 \in \Lambda$.

It can easily verified that π is reflexive and symmetric L-fuzzy relation.

Now, consider the smallest L-fuzzy relation $\rho_{X,\mathcal{A}}$ on a set Λ with conditions:

$\rho_{X,\mathcal{A}}(\lambda_1, \lambda_2) \odot \rho_{X,\mathcal{A}}(\lambda_2, \lambda_3) \leqslant \rho_{X,\mathcal{A}}(\lambda_1, \lambda_3)$ and

$\pi(\lambda_1, \lambda_2) \leqslant \rho_{X,\mathcal{A}}(\lambda_1, \lambda_2)$, $\forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda$.

In that case, $\rho_{X,\mathcal{A}}$ is an L-fuzzy similarity relation on Λ.

Next, on the basis of $\rho_{X,\mathcal{A}}$, the L-fuzzy relation $\delta_{X,\mathcal{A}}$ on a set X is defined as:

$\delta_{X,\mathcal{A}}(x_1, x_2) = \rho_{X,\mathcal{A}}(\lambda_1, \lambda_2)$, $\forall \lambda_1, \lambda_2 \in \Lambda$, $\forall x_1 \in core(A_{\lambda_1})$, and $\forall x_2 \in core(A_{\lambda_2})$.

It can easily verified that $\delta_{X,\mathcal{A}}$ is an L-fuzzy similarity relation on X.

Proposition 2.2. *[24] For $(X, \mathcal{A})$ is a space with an L-fuzzy partition and $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ is an L-fuzzy partition of X, if $A : X \longrightarrow L$ be a map. Then the following statements are equivalent.*

(i) There exists the unique map $\sigma : \Lambda \longrightarrow L$ such that $(A, \sigma) \in R(X, \mathcal{A})$; and

(ii) For given $\lambda \in \Lambda$, $x \in core(A_\lambda)$, and $x' \in X$, $A_\lambda(x') \leqslant A(x) \leftrightarrow A(x')$ holds.

Let $(X, \mathcal{A})$ be a space with an L-fuzzy partition and $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ be an L-fuzzy partition of X. Then $R_1(X, \mathcal{A})$ is defined as,

$R_1(X, \mathcal{A}) = \{A : X \longrightarrow L : (A, \sigma)$ is an L-fuzzy object in $(X, \mathcal{A})$ for some map $\sigma : \Lambda \longrightarrow L\}$.

3 The L-fuzzy partitioned automata

In this section, we introduce and study the concept of an L-fuzzy partitioned automaton corresponding to a given L-fuzzy automaton. Further, we establish the relationship among the L-fuzzy languages of the introduced L-fuzzy partitioned automaton and L-fuzzy automaton.

We begin with the following definition of the L-fuzzy automaton with L-fuzzy partition from [23].

Definition 3.1. *A system $((X, \mathcal{A}), (M, *, e), d)$ is called an* **L-fuzzy automaton with L-fuzzy partition**, *if*

(1) X is the set of state with an L-fuzzy partition $\mathcal{A}$, where $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ is an L-fuzzy partition of X.

(2) $(M, *, e)$ is a monoid inputs; and

(3) $d : X \times M \longrightarrow R_1(X, \mathcal{A})$ is a map such that $\forall x, y \in X$, $\forall x' \in core(A_\lambda)$, and $\forall m, n \in M$,

  (i) $d(x, e)(y) = \delta_{X,\mathcal{A}}(x, y)$;

  (ii) $d(x, m * n)(y) = \vee\{d(x, m)(z) \odot d(z, n)(y) : z \in X\}$;

  (iii) $d(x, m)(y) \odot A_\lambda(x) \leqslant d(x', m)(y)$; *and*

  (iv) From the condition (iii), for each $x, x' \in core(A_\lambda)$, $d(x, m) = d(x', m)$.

In the remaining part of this section, $\mathcal{M} = (Q, (M, *, e), T, I, F)$ is an L-fuzzy automaton and $(Q, \mathcal{Q})$ is a space with an L-fuzzy partition, where $\mathcal{Q} = \{Q_\alpha : \alpha \in \Lambda\}$ is an L-fuzzy partition of Q.

Now, we introduce the concept of the L-fuzzy partitioned automata.

Definition 3.2. *Let $\mathcal{M} = (Q, (M, *, e), T, I, F)$ be an L-fuzzy automaton. Then the* **L-fuzzy partitioned automaton** *corresponding to $\mathcal{M}$, denoted by $\mathcal{P}$, is the system $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$, where*

(i) Q is the set of state with an L-fuzzy partition $\mathcal{Q}$, where $\mathcal{Q} = \{Q_\alpha : \alpha \in \Lambda\}$ is an L-fuzzy partition of Q;

(ii) $(M, *, e)$ *is a monoid inputs; and*

(iii) $T_1(p,m)(q) = \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\}$, $\forall p, q \in Q$ *and* $\forall m \in M$, *where* $\delta_{Q,\mathcal{Q}}(q_1,q_2) = \rho_{Q,\mathcal{Q}}(\alpha_1,\alpha_2)$, $\forall \alpha_1, \alpha_2 \in \Lambda$, $\forall q_1 \in core(Q_{\alpha_1})$, *and* $\forall q_2 \in core(Q_{\alpha_2})$.

Proposition 3.1. *Let* $\mathcal{M} = (Q, (M, *, e), T, I, F)$ *be an L-fuzzy automaton. Then the L-fuzzy partitioned automaton* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *corresponding to* $\mathcal{M}$ *is an L-fuzzy automaton with L-fuzzy partition.*

Proof. (i) Let $p, s \in Q$, $s' \in core(Q_\alpha)$, and $m \in M$. Then

$$\begin{aligned} & T_1(p,m)(s) \leftrightarrow T_1(p,m)(s') \\ = & (\vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,s) : r_1, r_2 \in Q\}) \leftrightarrow (\vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,s') : r_1, r_2 \in Q\}) \\ \geqslant & (\delta_{Q,\mathcal{Q}}(p,p) \odot T(p,m)(p) \odot \delta_{Q,\mathcal{Q}}(p,s)) \leftrightarrow (\delta_{Q,\mathcal{Q}}(p,p) \odot T(p,m)(p) \odot \delta_{Q,\mathcal{Q}}(p,s')) \\ \geqslant & \delta_{Q,\mathcal{Q}}(p,s)) \leftrightarrow \delta_{Q,\mathcal{Q}}(p,s')) \\ \geqslant & Q_\alpha(s). \end{aligned}$$

Thus $Q_\alpha(s) \leqslant T_1(p,m)(s) \leftrightarrow T_1(p,m)(s')$, which implying that $T_1(p,m) \in R_1(Q,\mathcal{Q})$.

(ii) Let $p, q \in Q$. Then

$$\begin{aligned} T_1(p,e)(q) &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,e)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\} \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r) \odot \delta_{Q,\mathcal{Q}}(r,q) : r \in Q\} \\ &\leqslant \delta_{Q,\mathcal{Q}}(p,q). \end{aligned}$$

Conversly,

$$\begin{aligned} T_1(p,e)(q) &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,e)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\} \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r) \odot \delta_{Q,\mathcal{Q}}(r,q) : r \in Q\} \\ &\geqslant \delta_{Q,\mathcal{Q}}(p,p) \odot \delta_{Q,\mathcal{Q}}(p,q) \\ &= \delta_{Q,\mathcal{Q}}(p,q). \end{aligned}$$

Thus $T_1(p,e)(q) = \delta_{Q,\mathcal{Q}}(p,q)$.

(iii) Let $p, q \in Q$ and $m, n \in M$. Then

$$\begin{aligned} T_1(p,m*n)(q) &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m*n)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\} \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot \vee\{T(r_1,m)(r) \odot T(r,n)(r_2) : r \in Q\} \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\} \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r) \odot T(r,n)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r, r_1, r_2 \in Q\} \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r) \odot \delta_{Q,\mathcal{Q}}(r,r) \odot \delta_{Q,\mathcal{Q}}(r,r) \odot T(r,n)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r, r_1, r_2 \in Q\} \\ &= \vee\{T_1(p,m)(r) \odot T_1(r,n)(q) : r \in Q\}. \end{aligned}$$

Thus $T_1(p,m*n)(q) = \vee\{T_1(p,m)(r) \odot T_1(r,n)(q) : r \in Q\}$.

(iv) Let $p, s \in Q$, $s' \in core(Q_\alpha)$, and $m \in M$. Then

$$\begin{aligned} T_1(p,m)(s) \odot Q_\alpha(p) &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,s) : r_1, r_2 \in Q\} \odot Q_\alpha(p) \\ &= \vee\{\delta_{Q,\mathcal{Q}}(p,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,s) \odot Q_\alpha(p) : r_1, r_2 \in Q\} \\ &\leqslant \vee\{\delta_{Q,\mathcal{Q}}(s',r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,s) : r_1, r_2 \in Q\} \\ &= T_1(s',m)s. \end{aligned}$$

Thus $T_1(p,m)(s) \odot Q_\alpha(p) \leqslant T_1(s',m)(s)$.

(v) From the condition (iv), $T_1(s,m) = T_1(s',m)$, $\forall s, s' \in core(Q_\alpha)$ and $\forall m \in M$.

Hence $\mathcal{P} = ((Q,\mathcal{Q}), (M, *, e), T_1, I, F)$ is an L-fuzzy automaton with L-fuzzy partition. ☐

The following is to establish the relationship between L-fuzzy languages of the introduced L-fuzzy partitioned automaton and L-fuzzy automaton.

Proposition 3.2. *Let* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton* $\mathcal{M} = (Q, (M, *, e), T, I, F)$. *Then* $f_\mathcal{M} \subseteq f_\mathcal{P}$.

Proof. Let $m \in M$. Then

$$\begin{aligned} f_\mathcal{P}(m) &= \vee\{I(r) \odot T_1(r,m)(q) \odot F(q) : r, q \in Q\} \\ &= \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r_1) \odot T(r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) : r, r_1, r_2, q \in Q\} \\ &\geqslant \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r) \odot T(r,m)(q) \odot \delta_{Q,\mathcal{Q}}(q,q) \odot F(q) : r, q \in Q\} \\ &= \vee\{I(r) \odot T(r,m)(q) \odot F(r)\} \\ &= f_\mathcal{M}(m). \end{aligned}$$

Thus $f_\mathcal{M} \subseteq f_\mathcal{P}$. ☐

Proposition 3.3. *Let* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton* $\mathcal{M} = (Q, (M, *, e), T, I, F)$, *where* $\mathcal{Q} = \{Q_\alpha : \alpha \in \Lambda\}$ *such that for all* $\alpha \in \Lambda$, *there exists unique* $q \in Q$ *with* $Q_\alpha(q) = 1$ *and* 0, *otherwise. Then* $f_\mathcal{P} = f_\mathcal{M}$.

Proof. Let $m \in M$. Then

$$\begin{aligned} f_{\mathcal{P}}(m) &= \vee\{I(r) \odot T_1(r,m)(q) \odot F(q) : r, q \in Q\} \\ &= \vee\{I(r) \odot (\vee\{\delta_{Q,\mathcal{Q}}(r,r_1) \odot T(r_1,m)(r_2) \\ &\quad \odot \delta_{Q,\mathcal{Q}}(r_2,q) : r_1, r_2 \in Q\}) \odot F(q) : r, q \\ &\quad \in Q\} \\ &= \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r_1) \odot T(r_1,m)(r_2) \odot \\ &\quad \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) : r, r_1, r_2, q \in Q\} \\ &= \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r) \odot T(r,m)(q) \odot \delta_{Q,\mathcal{Q}} \\ &\quad (q,q) \odot F(q) : r \in Q\} \\ &= \vee\{I(r) \odot T(r,m)(q) \odot F(q)\} \\ &= f_{\mathcal{M}}(m). \end{aligned}$$

Thus $f_{\mathcal{P}} = f_{\mathcal{M}}$. □

Proposition 3.4. *If the L-fuzzy partitioned automaton $\mathcal{P} = ((Q,\mathcal{Q}),(M,*,e),T_1,I,F)$ corresponding to given L-fuzzy automaton $\mathcal{M} = (Q,(M,*,e),T,I,F)$ is crisp-deterministic L-fuzzy automaton, then $\mathcal{Q} = \{Q_\alpha : \alpha \in \Lambda\}$ such that for all $\alpha \in \Lambda$, there exists unique $q \in Q$ with $Q_\alpha(q) = 1$ and 0, otherwise.*

Proof. Follows from Proposition 3.1. □

4 Determinization of L-fuzzy partitioned automata

In this section, we introduce the crisp-deterministic L-fuzzy automaton corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language.

Definition 4.1. *Let $\mathcal{P} = ((Q,\mathcal{Q}),(M,*,e),T_1,I,F)$ be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton $\mathcal{M} = (Q,(M,*,e),T,I,F)$. Then the* **crisp-deterministic L-fuzzy automaton** *corresponding to $\mathcal{P}$, denoted by $\mathcal{F}$, is the system $\mathcal{F} = (\mathcal{F}(Q),(M,*,e),T_{\mathcal{F}},I_{\mathcal{F}},F_{\mathcal{F}})$, where*

(i) *$\mathcal{F}(Q) = \{\mu : \mu : Q \longrightarrow L\}$ is the set of states;*

(ii) *$(M,*,e)$ is a monoid inputs;*

(iii) *$T_{\mathcal{F}} : \mathcal{F}(Q) \times M \longrightarrow \mathcal{F}(Q)$ is a transition function such that $\forall A \in \mathcal{F}(Q)$, $\forall q \in Q$, and $\forall m \in M$, $T_{\mathcal{F}}(A,m)(q) = \vee\{A(r) \odot T(r,m)(q) : r \in Q\}$;*

(iv) *$I_{\mathcal{F}} \in \mathcal{F}(Q)$ is an initial state such that $\forall q \in Q$, $I_{\mathcal{F}}(q) = \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,q) : r \in Q\}$; and*

(v) *$F_{\mathcal{F}} : \mathcal{F}(Q) \longrightarrow L$ is a final L-fuzzy state such that $\forall A \in \mathcal{F}(Q)$, $F_{\mathcal{F}}(A) = \vee\{A(r) \odot \delta_{Q,\mathcal{Q}}(r,q) \odot F(q) : r, q \in Q\}$.*

Proposition 4.1. *Let $\mathcal{P} = ((Q,\mathcal{Q}),(M,*,e),T_1,I,F)$ be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton $\mathcal{M} = (Q,(M,*,e),T,I,F)$ and $\mathcal{F} = (\mathcal{F}(Q),(M,*,e),T_{\mathcal{F}},I_{\mathcal{F}},F_{\mathcal{F}})$ be a crisp-deterministic L-fuzzy automaton corresponding to $\mathcal{P}$. Then $f_{\mathcal{F}} = f_{\mathcal{P}}$.*

Proof. Let $m \in M$. Then

$$\begin{aligned} f_{\mathcal{F}}(m) &= F_{\mathcal{F}}(T_{\mathcal{F}}(I_{\mathcal{F}},m)) \\ &= \vee\{T_{\mathcal{F}}(I_{\mathcal{F}},m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) \\ &\quad : r_2, q \in Q\} \\ &= \vee\{(\vee\{I_{\mathcal{F}}(r_1) \odot T(r_1,m)(r_2) : r_1 \in Q\}) \\ &\quad \odot \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) : r_2, q \in Q\} \\ &= \vee\{(\vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r_1) : r \in Q\}) \odot T \\ &\quad (r_1,m)(r_2) \odot \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) : r_1, r_2, \\ &\quad q \in Q\} \\ &= \vee\{I(r) \odot \delta_{Q,\mathcal{Q}}(r,r_1) \odot T(r_1,m)(r_2) \odot \\ &\quad \delta_{Q,\mathcal{Q}}(r_2,q) \odot F(q) : r, r_1, r_2, q \in Q\} \\ &= \vee\{I(r) \odot T_1(r,m)(q) \odot F(q) : r, q \in Q\} \\ &= f_{\mathcal{P}}(m). \end{aligned}$$

Thus $f_{\mathcal{F}} = f_{\mathcal{P}}$. □

5 The fuzzified L-fuzzy partitioned automata

In this section, we introduce and study the notion of the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton. Further, we study the L-fuzzy language of such fuzzified L-fuzzy partitioned automaton in terms of L-fuzzy language of the L-fuzzy partitioned automaton.

Definition 5.1. *Let $\mathcal{P} = ((Q,\mathcal{Q}),(M,*,e),T_1,I,F)$ be the L-fuzzy partitioned automaton corresponding to L-fuzzy automaton $\mathcal{M} = (Q,(M,*,e),T,I,F)$. The* **fuzzified L-fuzzy partitioned automaton** *corresponding to $\mathcal{P}$, denoted by $\mathcal{W}$, is the system $\mathcal{W} = ((Q,\mathcal{Q}),(\mathcal{F}(M),\otimes,1_e),T_2,I,F)$, where*

(i) *$(\mathcal{F}(M),\otimes,1_e)$ is a monoid inputs, where $\mathcal{F}(M) = \{A : A : M \longrightarrow L\}$ and $1_e \in \mathcal{F}(M)$ such that $\forall x \in M$,*

$$1_e(x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{if } x \neq e, \text{ and} \end{cases}$$

(ii) *$T_2(p,A)(q) = \vee\{A(m) \odot T_1(p,m)(q) : m \in M\}$, $\forall p, q \in Q$ and $\forall A \in \mathcal{F}(m)$.*

Definition 5.2. *An* **L-fuzzy language** *$f_{\mathcal{W}} : \mathcal{F}(M) \longrightarrow L$ is* **accepted** *by a fuzzified L-fuzzy partitioned automaton $\mathcal{W} = ((Q,\mathcal{Q}),(\mathcal{F}(M),\otimes,1_e),T_2,I,F)$ if $f_{\mathcal{W}}(A) = \vee\{I(r) \odot T_2(r,A)(q) \odot F(q) : r, q \in Q\}$, $\forall A \in \mathcal{F}(M)$.*

Proposition 5.1. *Let* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *be the* L*-fuzzy partitioned automaton corresponding to* L*-fuzzy automaton* $\mathcal{M} = (Q, (M, *, e), T, I, F)$*. Then the fuzzified* L*-fuzzy partitioned automaton* $\mathcal{W} = ((Q, \mathcal{Q}), (\mathcal{F}(M), \otimes, 1_e), T_2, I, F)$ *corresponding to* $\mathcal{P}$ *is an* L*-fuzzy automaton with* L*-fuzzy partition.*

Proof. (i) Let $p, s \in Q$, $s' \in core(Q_\alpha)$, and $A \in \mathcal{F}(M)$. Then

$$\begin{aligned} & T_2(p, A)(s) \leftrightarrow T_2(p, A)(s') \\ = \;& \vee\{A(m_1) \odot T_1(p, m_1)(s) : m_1 \in M\} \leftrightarrow \vee\{A \\ & (m_2) \odot T_1(p, m_2)(s') : m_2 \in M\} \\ \geqslant \;& (A(m) \odot T_1(p, m)(s)) \leftrightarrow (A(m) \odot T_1(p, m)(s')) \\ \geqslant \;& T_1(p, m)(s) \leftrightarrow T_1(p, m)(s') \\ \geqslant \;& Q_\alpha(s). \end{aligned}$$

Thus $Q_\alpha(s) \leqslant T_2(p, m)(s) \leftrightarrow T_2(p, m)(s')$, which implying that $T_2(p, m) \in R_1(Q, \mathcal{Q})$.

(ii) Let $p, q \in Q$. Then

$$\begin{aligned} T_2(p, 1_e)(q) &= \vee\{1_e(m) \odot T_1(p, m)(q) : m \in M\} \\ &= T_1(p, e)(q) \\ &= \delta_{Q,\mathcal{Q}}(p, q). \end{aligned}$$

Thus $T_2(p, 1_e)(q) = \delta_{Q,\mathcal{Q}}(p, q)$.

(iii) Let $p, q \in Q$ and $A_1, A_2 \in \mathcal{F}(M)$. Then

$$\begin{aligned} T_2(p, A_1 \otimes A_2)(q) &= \vee\{(A_1 \otimes A_2)m \odot T_1(p, m)(q) \\ &\quad : m \in M\} \\ &= \vee\{A_1(m_1) \odot A_2(m_2) \odot T_1(p, \\ &\quad m)(q) : m = m_1 * m_2\} \\ &= \vee\{A_1(m_1) \odot A_2(m_2) \odot T_1(p, \\ &\quad m_1 * m_2)(q) : m_1, m_2 \in M\} \\ &= \vee\{A_1(m_1) \odot A_2(m_2) \odot T_1(p, \\ &\quad m_1)(r) \odot T_1(r, m_1)(q) : r \in Q \\ &\quad , m_1, m_2 \in M\} \\ &= \vee\{T_2(p, A_1)(r) \odot T_2(r, A_2)(q) \\ &\quad : r \in Q\}. \end{aligned}$$

Thus $T_2(p, A_1 \otimes A_2)(q) = \vee\{T_2(p, A_1)(r) \odot T_2(r, A_2)(q) : r \in Q\}$.

(iv) Let $p, s \in Q$, $s' \in core(Q_\alpha)$, and $A \in \mathcal{F}(M)$. Then

$$\begin{aligned} T_2(p, A)(s) \odot Q_\alpha(p) &= \vee\{A(m) \odot T_1(p, m)(s) : m \\ &\quad \in M\} \odot Q_\alpha(p) \\ &= \vee\{A(m) \odot T_1(p, m)(s) \odot Q_\alpha \\ &\quad (p) : m \in M\} \\ &\leqslant \vee\{A(m) \odot T_1(s', m)(s) : m \\ &\quad \in M\} \\ &= T_2(s', A)(s). \end{aligned}$$

Thus $T_2(p, A)(s) \odot Q_\alpha(p) \leqslant T_2(s', A)(s)$.

(v) From the condition (iv), $T_2(s, A) = T_2(s', A)$, $\forall s, s' \in core(Q_\alpha)$ and $A \in \mathcal{F}(M)$.

Hence $\mathcal{W} = ((Q, \mathcal{Q}), (\mathcal{F}(M), \otimes, 1_e), T_2, I, F)$ is an L-fuzzy automaton with L-fuzzy partition. □

Proposition 5.2. *Let* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *be the* L*-fuzzy partitioned automaton corresponding to* L*-fuzzy automaton* $\mathcal{M} = (Q, (M, *, e), T, I, F)$*,* $\mathcal{W} = ((Q, \mathcal{Q}), (\mathcal{F}(M), \otimes, 1_e), T_2, I, F)$ *be the fuzzified* L*-fuzzy partitioned automaton corresponding to* $\mathcal{P}$*, and* $W = A_1 \otimes ... \otimes A_n$*,* $\forall A_1, ..., A_n \in \mathcal{F}(M)$*. Then* $T_2(p, W)(q) = \vee\{T_1(p, m_1 \odot ... \odot m_n)(q) \odot A_1(m_1) \odot ... \odot A_n(m_n) : m_1, ..., m_n \in M\}$*,* $\forall p, q \in Q$*.*

Proof. We prove the result by induction on length W, denoted by $|W|$. Let $|W| = n, n \geqslant 0$. Then for $n = 0$, the result is obvious from the definition of T_2. Suppose that the result is true for W of length n, then we have to show that the result holds for length $n+1$. Now, let $W = A_1 \otimes ... \otimes A_n \otimes A_{n+1}, \forall A_1, ..., A_n, A_{n+1} \in \mathcal{F}(M)$. Then

$$\begin{aligned} T_2(p, W)(q) &= \vee\{T_2(p, A_1 \otimes ... \otimes A_n)(r) \odot T_2(r, \\ &\quad A_{n+1})(q) : r \in Q\} \\ &= \vee\{(\vee\{T_1(p, m_1 \odot ... \odot m_n)(r) \odot A_1 \\ &\quad (m_1) \odot ... \odot A_n(m_n) : m_1, ..., m_n \\ &\quad \in M\}) \odot (\vee\{T_1(r, m_{n+1})(q) \odot A \\ &\quad (m_{n+1}) : m_{n+1} \in M\})q : r \in Q\} \\ &= \vee\{T_1(p, m_1 \odot ... \odot m_n)(r) \odot A_1(m_1 \\ &\quad) \odot ... \odot A_n(m_n) \odot T_1(r, m_{n+1})(q) \\ &\quad \odot A_{n+1}(m_{n+1}) : r \in Q, m_1, ...m_n, \\ &\quad m_{n+1} \in M\} \\ &= \vee\{T_1(p, m_1 \odot ... \odot m_n \odot m_{n+1})(q) \\ &\quad \odot A_1(m_1) \odot ... \odot A_n(m_n) \odot A_{n+1} \\ &\quad (m_{n+1}) : m_1, ...m_n, m_{n+1} \in M\}. \end{aligned}$$

□

The following is towards the L-fuzzy language of the introduced fuzzified L-fuzzy partitioned automaton in terms of the L-fuzzy language of the L-fuzzy partitioned automaton.

Proposition 5.3. *Let* $\mathcal{P} = ((Q, \mathcal{Q}), (M, *, e), T_1, I, F)$ *be the* L*-fuzzy partitioned automaton corresponding to* L*-fuzzy automaton* $\mathcal{M} = (Q, (M, *, e), T, I, F)$*,* $\mathcal{W} = ((Q, \mathcal{Q}), (\mathcal{F}(M), \otimes, 1_e), T_2, I, F)$ *be the fuzzified* L*-fuzzy partitioned automaton corresponding to* $\mathcal{P}$*, and* $W = A_1 \otimes ... \otimes A_n$*,* $\forall A_1, ..., A_n \in \mathcal{F}(M)$*. Then* $f_{\mathcal{W}}(W) = \vee\{f_{\mathcal{P}}(m_1 \odot ... \odot m_n) \odot A_1(m_1) \odot ... \odot A_n(m_n) : m_1, ..., m_n \in M\}$*.*

Proof. Let $W = A_1 \otimes ... \otimes A_n$, $\forall A_1, ..., A_n \in \mathcal{F}(M)$. Then

$$\begin{aligned} f_{\mathcal{W}}(W) &= \vee\{I(r) \odot T_2(r, W)(q) \odot F(q) : r, q \in Q\} \\ &= \vee\{I(r) \odot (\vee\{T_1(r, m_1 \odot ... \odot m_n)(q) \odot \\ &\quad A_1(m_1) \odot ... \odot A_n(m_n) : m_1, ..., m_n \\ &\quad \in M\}) \odot F(q) : r, q \in Q\} \\ &= \vee\{I(r) \odot T_1(r, m_1 \odot ... \odot m_n)(q) \odot A_1 \\ &\quad (m_1) \odot ... \odot A_n(m_n) \odot F(q) : r, q \in Q, \\ &\quad m_1, ..., m_n \in M\} \\ &= \vee\{f_{\mathcal{P}}(m_1 \odot ... \odot m_n) \odot A_1(m_1) \odot ... \odot \\ &\quad A_n(m_n) : m_1, ..., m_n \in M\}. \end{aligned}$$

Thus $f_{\mathcal{W}}(W) = \vee\{f_{\mathcal{P}}(m_1 \odot ... \odot m_n) \odot A_1(m_1) \odot ... \odot A_n (m_n) : m_1, ..., m_n \in M\}$. □

6 Conclusion

In this paper, we have introduced and studied the concept of the L-fuzzy partitioned automaton corresponding to a given L-fuzzy automaton, whose set of states is a space with an L-fuzzy partition of the set of states of such given automaton. Also, we have obtained the relationship among the L-fuzzy languages of the L-fuzzy partitioned automaton and L-fuzzy automaton. Further, the crisp-deterministic L-fuzzy automaton is introduced corresponding to the L-fuzzy partitioned automaton such that both accept same L-fuzzy language. Finally, the concept of the L-fuzzy sets have been used to introduce the fuzzified L-fuzzy partitioned automaton corresponding to a given L-fuzzy partitioned automaton. Interestingly, it is shown here that the L-fuzzy language of fuzzified L-fuzzy partitioned automaton can be obtained from the L-fuzzy language of the L-fuzzy partitioned automaton.

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