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e-Operators on the Extension of Lattice-valued (U, N)-Implications

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Abstract

Generalize fuzzy operators for the framework of lattices is the main objective of many recent researches. Here we present a study about how to define lattice-valued (U, N)-implications i.e. the class of implications generated from a uninorm U and a fuzzy negation N. Moreover, we discuss about its extension from a lattice to a bigger one using the e-operator.

Keywords: Uninorms, Implications, (U, N)-implications

1 Introduction

The fusion between fuzzy logic and lattice theory has been the main issue of many researches mainly because of getting new ways to interpret mathematically some particular problems [3, 4, 14, 15, 32]. For instance, lattice-valued fuzzy logic has been considered in image processing, mathematical morphology and artificial intelligent [9, 20, 21, 30].

In this framework Karaçal et al. in [16, 17] have presented a definition of uninorm and nullnorm on a bounded lattice L and developed ways to construct them and its characterization [18]. Çayli et al. in [8] have introduced a new class of lattice-valued uninorms. Also, lattice-valued fuzzy negations are considered in [22].

Another important problem that relates fuzzy and lattice theory is the construction of extension methods for fuzzy operator able to preserve their characteristics and properties [29]. In this sense, Palmeira and Bedregal have presented a series of results of extension of fuzzy operations (namely, t-norms, t-conorms and negations in [22, 23], implications in [24, 25, 26], uninorms and nullnorms in [27], among others) in the seek to uncover which properties are preserved by the extension methods via retraction [22] and via e-operators [23] created by them. In particular, for uninorms and nullnorms Palmeira and Bedregal show [27] how to extend these operators using the e-operators.

Taking that into account this paper has two main goals. The first one is presenting a formalization of notion of fuzzy implications generated from a uninorm and a fuzzy negation ((U, N)-implications) for lattices. The second one is applying the extension method via e-operator presented in [23] for (U, N)-implications.

The paper is split as follows: Section 2 presents some concepts related to lattice theory and the definitions of L-uninorms, L-negations and L-implications. Section 3 is devoted to present the extension method via e-operator. In Sections 4 the definition of lattice-valued (U, N)-implications and some results are presented and its extension is considered in Section 5. Finally, some final remarks are given in Section 6.

2 Preliminaries

In this section some important definitions and results from lattice theory are presented. For further reading about these concepts we recommend [1, 5, 6, 10, 11, 12, 19].

2.1 Lattices and Morphims

Definition 2.1 *Let L be a nonempty set. If $\wedge_L$ and $\vee_L$ are two binary operations on L, then $\langle L, \wedge_L, \vee_L\rangle$ is a lattice provided that for each $x, y, z \in L$, the following properties hold:*

1. $x \wedge_L y = y \wedge_L x$ *and* $x \vee_L y = y \vee_L x$;
2. $(x \wedge_L y) \wedge_L z = x \wedge_L (y \wedge_L z)$ *and* $(x \vee_L y) \vee_L z = x \vee_L (y \wedge_L z)$;
3. $x \wedge_L (x \vee_L y) = x$ *and* $x \vee_L (x \wedge_L y) = x$.

If in $\langle L, \wedge_L, \vee_L\rangle$ there are elements 0_L and 1_L such that, for all $x \in L$, $x \wedge_L 1_L = x$ and $x \vee_L 0_L = x$, *then $\langle L, \wedge_L, \vee_L, 0_L, 1_L \rangle$ is called a bounded lattice. A lattice L is called a complete lattice if every subset of it has a top and a bottom element.*

Remark 2.1 *A partial order can be constructed in L by considering the relation $a \le b$ if and only if $a = a \wedge_L b$ for all $a, b \in L$. In case a and b are incomparable elements of a lattice L i.e. $a \neq b$ or $a \nleq b$ or $b \nleq a$, we denote it by $a \parallel b$.*

Remark 2.2 *Notice that given a lattice L we can consider the subintervals of L as follows:*

$$[a, b] = \{x \in L \mid a \le x \le b\} \text{ for each } a, b \in L$$

Similarly, it may be defined $(a, b]$, $[a, b)$ and (a, b). Also, define the sets

$$A(e) = (0_L, e] \times [e, 1_L) \cup [e, 1_L) \times (0_L, e]$$

$$B(e) = [0_L, e] \times (e, 1_L] \cup (e, 1_L] \times [0_L, e]$$

$$C(e) = [0_L, e) \times [e, 1_L] \cup [e, 1_L] \times [0_L, e)$$

$$D(e) = [0_L, e) \times (e, 1_L] \cup (e, 1_L] \times [0_L, e)$$

and

$$I_e = \{x \in L \mid x \parallel e\}$$

Definition 2.2 *Let $(L, \wedge_L, \vee_L, 0_L, 1_L)$ and $(M, \wedge_M, \vee_M, 0_M, 1_M)$ be two bounded lattices. A mapping $f : L \longrightarrow M$ is said to be a lattice homomorphism if, for all $x, y \in L$, we have*

1. *$f(x \wedge_L y) = f(x) \wedge_M f(y)$;*
2. *$f(x \vee_L y) = f(x) \vee_M f(y)$;*
3. *$f(0_L) = 0_M$ and $f(1_L) = 1_M$.*

It is easy to verify that every lattice homomorphism preserves the lattice order.

Definition 2.3 *A homomorphism r of a lattice L onto a lattice M is said to be a retraction if there exists a homomorphism s of M into L which satisfies $r \circ s = id_M$ [1]. A lattice M is called a retract of a lattice L if there is a retraction r, of L onto M, and s is then called a pseudo-inverse of r.*

[1] $\circ$ is the composition of functions.

2.2 (r, s)-Sublattices

Let L and M be bounded lattices and suppose there exists a retraction $r : L \to M$ with pseudo inverse $s : M \to L$. In this case M is a retract of L and we say that M is a (r, s)-sublattice of L. Notice that the pair (r, s) is not unique which means that it can exist different ways to embed M into L as a algebraic retract. So, for each pair of homomorphisms (r, s) M is a different retract of L giving us a flexible way to thing about sublattices.

Definition 2.4 *A retraction $r : L \longrightarrow M$ (with pseudo-inverse s) which satisfies $s \circ r \leqslant id_L$ is called a lower retraction. If it ($id_L \leqslant s \circ r$) we say that it is an upper retraction.*

Definition 2.5 *Let M be a (r, s)-sublattice of L. Thus, there is a retraction r_1 from L onto M and a pseudo-inverse $s : M \longrightarrow L$ such that $r_1 \circ s = id_M$. We say that*

1. *M is a lower sublattice of L if r_1 is a lower retraction;*
2. *M is an upper sublattice of L whenever r_1 is an upper retraction;*
3. *If r_1 is a lower retraction and there is an upper retraction $r_2 : L \longrightarrow M$ such that its pseudo-inverse is also s, then M is called a full sublattice of L. Notation: $M \trianglelefteq L$ over (r_1, r_2, s).*

2.3 L-uninorms

It is known from the literature that a t-norm (t-conorm) on a bounded lattice L is an operation $T : L^2 \to L$ which is commutative, associative, increasing with respect both variables and has 1_L (0_L) as the neutral element. But it is possible consider other operations on lattices that have an element $e \in L/\{0_L, 1_L\}$ as the neutral element. These operator are known as uninorms and it were first defined on a bounded lattice by Karaçal and Mesiar in [17].

Definition 2.6 *Let L be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L if for every $x, y, z \in L$,*

(U1) *$U(x, y) = U(y, x)$;*

(U2) *$U(x, U(y, z)) = U(U(x, y), z)$;*

(U3) *If $x \le_L y$ then $U(x, z) \le_L U(y, z)$;*

(U4) *There exists a neutral element $e \in L$, i.e. $U(x, e) = x$ for every $x \in L$.*

Remark 2.3 *For all uninorm U on L it holds $U(0_L, 1_L) \in \{0_L, 1_L\}$ since U is associative. In case $U(0_L, 1_L) = 0_L$ it is called a conjuntive uninorm and if $U(0_L, 1_L) = 1_L$ it is called a disjuntive uninorm.*

They also have proven the following results:

Proposition 2.1 *If U is a uninorm on L with neutral element $e \in L$ then*

1. $x \wedge y \leq U(x, y) \leq x \vee y$ *for* $(x, y) \in A(e)$;
2. $U(x, y) \leq x$ *for all* $(x, y) \in L \times [0_L, e]$;
3. $U(x, y) \leq y$ *for all* $(x, y) \in [0_L, e] \times L$;
4. $x \leq U(x, y)$ *for all* $(x, y) \in L \times [e, 1_L]$;
5. $y \leq U(x, y)$ *for all* $(x, y) \in [e, 1_L] \times L$.

Proposition 2.2 *Let L be a bounded lattice and U be an uninorm on L with neutral element $e \in L$. Then*

1. $T = U|_{[0_L,e]^2} : [0_L, e]^2 \to [0_L, e]$ *is a t-norm;*
2. $S = U|_{[e,1_L]^2} : [e, 1_L]^2 \to [e, 1_L]$ *is a t-conorm.*

Proposition 2.3 *Let L be a bounded lattice and $U : L^2 \to L$ be an idempotent uninorm with neutral element $e \in L \backslash \{0_L, 1_L\}$.*

1. $U(x, y) \leq x \wedge_L y$ *for* $(x, y) \in [0_L, e]^2$;
2. $x \vee_L y \leq U(x, y)$ *for* $(x, y) \in [e, 1_L]^2$.

2.4 Fuzzy Implications on L

Definition 2.7 *[2] A function $I : L \times L \longrightarrow L$ is a fuzzy implication on L if for each $x, y, z \in L$ the following properties hold:*

(I1) if $x \leqslant_L y$ then $I(y, z) \geqslant_L I(x, z)$;

(I2) if $y \leqslant_L z$ then $I(x, y) \leqslant_L I(x, z)$;

(I3) $I(0_L, 0_L) = 1_L$;

(I4) $I(1_L, 1_L) = 1_L$;

(I5) $I(1_L, 0_L) = 0_L$.

Consider also the following properties of an implication I on L:

(LB) $I(0_L, y) = 1_L$, for all $y \in L$;
(RB) $I(x, 1_L) = 1_L$, for all $x \in L$;
(CC4) $I(0_L, 1_L) = 1_L$;
(NP) $I(1_L, y) = y$ for each $y \in L$ (left neutrality principle);
(EP) $I(x, I(y, z)) = I(y, I(x, z))$ for all $x, y, z \in L$ (exchange principle);
(IP) $I(x, x) = 1_L$ for each $x \in L$ (identity principle);
(OP) $I(x, y) = 1_L$ if and only if $x \leqslant_L y$ (ordering property);
(IBL) $I(x, I(x, y)) = I(x, y)$ for all $x, y, z \in L$ (iterative Boolean law);
(CP) $I(x, y) = I(N(y), N(x))$ for each $x, y \in L$ with N a fuzzy negation on L (law of contraposition);
(L-CP) $I(N(x), y) = I(N(y), x)$ (law of left contraposition);
(R-CP) $I(x, N(y)) = I(y, N(x))$ (law of right contraposition);
(P) $I(x, y) = 0_L$ if and only if $x = 1_L$ and $y = 0_L$ (Positivety).

2.5 L-Negations

Definition 2.8 *A mapping $N : L \to L$ is a negation on L or just an L-negation, if the following properties are satisfied for each $x, y \in L$:*
(N1) $N(0_L) = 1_L$ and $N(1_L) = 0_L$ and
(N2) If $x \leqslant_L y$ then $N(y) \leqslant_L N(x)$.
Moreover, the L-negation N is considered strong if it also satisfies the involution property, i.e.
(N3) $N(N(x)) = x$ for each $x \in L$.
The L-negation is strict if satisfies the property:
(N4) $N(x) <_L N(y)$ whenever $y <_L x$.
The L-negation N is frontier if satisfies the property:
(N5) $N(x) \in \{0_L, 1_L\}$ if and only if $x = 0_L$ or $x = 1_L$.

Definition 2.9 *[2] Let I be a fuzzy implication on L. If $I(1_L, \alpha) = 0_L$ for some $\alpha \in L$ then the function $N_I^\alpha : L \to L$ given by*

$$N_I^\alpha(x) = I(x, \alpha) \ \forall x \in L \tag{1}$$

is called the natural negation of I with respect to α.

3 Extension Method via e-operators

The extension problem is a very interesting issue since it can be considered in many different frameworks to discuss about how a given function may be extended preserving its properties.

We have been studied in recent years how to extend a lattice-valued fuzzy operator preserving its algebraic properties and developed two different extension method (via retraction [22] and via e-operator [23]) to extend t-norms, t-conorms, fuzzy negations and implications [24].

Definition 3.1 *[23] Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). A mapping $\odot : M \times M \longrightarrow L$ is called an e-operator on M if it is isotonic and satisfies, for each $a, b \in M$ and for each $x \in L$, the following conditions:*

$$r_1(a \odot b) = a \wedge_M b \ \ \text{and} \ \ r_2(a \odot b) = a \vee_M b \quad (2)$$

$$r_1(x) \odot r_2(x) = x \quad (3)$$

Lemma 3.1 *[23] Consider $M \trianglelefteq L$ with respect to (r_1, r_2, s) and let $\odot$ be an e-operator on M. Then, for all $a, b \in M$ and $x, y \in L$, the following properties hold:*

1. *$a \leqslant_M b$ if and only if $r_1(a \odot b) = a$ and $r_2(a \odot b) = b$;*
2. *For every $a \in M$ we have $s(a) = a \odot a$;*
3. $$r_1(x) \leqslant_M r_1(y) \text{ and } r_2(x) \leqslant_M r_2(y) \text{ iff } x \leqslant_L y; \quad (4)$$
4. *$r_1(x) = r_1(y)$ and $r_2(x) = r_2(y)$ if and only if $x = y$;*
5. *$\odot$ is commutative.*

Proposition 3.1 *[23] Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ an e-operator on M. Thus, if N is a fuzzy negation on M then*

$$N^E_\odot(x) = N(r_1(x)) \odot N(r_2(x)) \quad (5)$$

is a fuzzy negation on L. Moreover,

1. *If N is involutive then $N^E_\odot$ is also involutive.*
2. *If a is an equilibrium point of fuzzy negation N then $s(a)$ is an equilibrium point of $N^E_\odot$.*

For fuzzy implication the extension works as in the following proposition.

Proposition 3.2 *[13] Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an e-operator on M. If $I : M^2 \to M$ is a fuzzy implication then function $U^E_\odot : L^2 \longrightarrow L$ defined by*

$$I^E_\odot(x, y) = I(r_2(x), r_1(y)) \odot U(r_1(x), r_2(y)) \quad (6)$$

for all $x, y \in L$ is a fuzzy implication L.

In our recent researches we have applied the about method for many lattice-valued fuzzy operators and proved that the method is efficient in preserving some properties. For applications, we have extended functions related to image processing and mathematical morphology such as retricted equivalence functions [7, 28].

For lattice-valued uninorms we have got the following results (see [27]).

Proposition 3.3 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an e-operator on M. Thus, given an uninorm U on M, the function $U^E_\odot : L^2 \longrightarrow L$ defined by*

$$U^E_\odot(x, y) = U(r_1(x), r_1(y)) \odot U(r_2(x), r_2(y)) \quad (7)$$

is an uninorm on L with respect to the neutral element $e' = s(e)$.

Theorem 3.1 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ be an e-operator on M and $e' = s(e)$. Then $U^E_\odot$ defined by Equation (7) is such that*

1. *$U^E_\odot$ is conjuntive whereas U is conjuntive;*
2. *$U^E_\odot$ is disjuntive whereas U is disjuntive;*
3. *If U is idempotent then $U^E_\odot$ is idempotent;*
4. *$x \wedge_L y \leq U^E_\odot(x, y) \leq x \vee_L y$ for $(x, y) \in A(e')$;*
5. *$U^E_\odot(x, y) \leq x$ for all $(x, y) \in L \times [0_L, e']$;*
6. *$U^E_\odot(x, y) \leq y$ for all $(x, y) \in [0_L, e'] \times L$;*
7. *$x \leq U^E_\odot(x, y)$ for all $(x, y) \in L \times [e', 1_L]$;*
8. *$y \leq U^E_\odot(x, y)$ for all $(x, y) \in [e', 1_L] \times L$.*

Proposition 3.4 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an e-operator on M. Thus, if U is an uninorm on M with respect to neutral element $e \in M$ then the Identity*

$$T^E_\odot = (U|_{[0_M, e]^2})^E_\odot = U^E_\odot|_{[0_L, e']^2} = T_{U^E_\odot} \quad (8)$$

holds[2].

Proposition 3.5 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s) and $\odot$ be an e-operator on M. If $U : M^2 \to M$ is an idempotent uninorm with neutral element $e \in L \backslash \{0_L, 1_L\}$ then*

1. *$U^E_\odot(x, y) \leq x \wedge_L y$ for $(x, y) \in [0_L, e']^2$;*
2. *$x \vee_L y \leq U^E_\odot(x, y)$ for $(x, y) \in [e', 1_L]^2$.*

[2] Similarly for $S^E_\odot = (U|_{[e, 1_M]^2})^E_\odot = U^E_\odot|_{[e', 1_L]^2} = S_{U^E_\odot}$

4 (U,N)-implications on lattices

The (S,N)-implications are those fuzzy implications generated from a t-conorm S and a negation N and a generalization of them can be obtained naturally by means changing the t-conorm S for a uninorm U (namely (U,N)-implications) as defined by Baczynski and Jayaram in [2]. Here we present an extension of the class of (U,N)-implications for the framework of lattices as follows.

Let L be a bounded lattice, $N : L \to L$ be a fuzzy negation and $U : L^2 \to L$ be a uninorm with neutral element $e \in L$. Then function $I_{(U,N)} : L^2 \to L$ given by

$$I_{(U,N)}(x,y) = U(N(x),y),\ \forall x,y \in L \qquad (9)$$

satisfies (I1), (I2) and (I5). Indeed, notice first that $I_{(U,N)}(1_L,0_L) = U(N(1_L),0_L) = U(0_L,0_L) = 0_L$ which proves that (I5) holds.

Now, let $x \leq_L y$ and $z \in L$. Since $N(y) \leq_L N(x)$ whereas $x \leq_L y$ then

$$I_{(U,N)}(x,z) = U(N(x),z) \geq_L U(N(y),z) = I_{(U,N)}(y,z)$$

and

$$I_{(U,N)}(z,x) = U(N(z),x) \leq_L U(N(z),y) = I_{(U,N)}(z,y)$$

i.e. properties (I1) and (I2) hold for $I_{(U,N)}$.

In addition, supposing U is disjunctive it can be proved that $I_{(U,N)}$ is a fuzzy implication on L as follows:

$$I_{(U,N)}(0_L,0_L) = U(1_L,0_L) = U(0_L,1_L) = 1_L$$

$$I_{(U,N)}(1_L,1_L) = U(N(1_L),1_L) = U(0_L,1_L) = 1_L$$

In this way, (U,N)-implications can be defined.

Proposition 4.1 *Let L be a bounded lattice, $N : L \to L$ be a fuzzy negation and $U : L^2 \to L$ be a disjunctive uninorm with neutral element $e \in L$. Then the function $I_{(U,N)} : L^2 \to L$ given by Eq. (9) is a fuzzy implication.*

Proposition 4.2 *Let $I_{(U,N)}$ be a (U,N)-fuzzy implication on L. Then*

1. *$I_{(U,N)}$ satisfies (CC4) and (EP);*
2. *$N^e_{I_{(U,N)}} = N$ and $I_{(U,N)}$ satisfies R-CP(N);*
3. *if N is strong then $I_{(U,N)}$ satisfies L-CP(N).*

Proof: Let be $x,y,z \in L$. Then

1. (CC4) is straightforward. Moreover,
$$\begin{aligned} I_{(U,N)}(x, I_{(U,N)}(y,z)) &= U(N(x),U(N(y),z)) \\ &= U(N(x),U(z,N(y))) \\ &= U(U(N(x),z),N(y)) \\ &= U(N(y),U(N(x),z)) \\ &= I_{(U,N)}(y, I_{(U,N)}(x,z)) \end{aligned}$$
2. (R-CP)
$$\begin{aligned} I_{(U,N)}(x,N(y)) &= U(N(x),N(y)) \\ &= U(N(y),N(x)) \\ &= U(y,N(x)) \end{aligned}$$
Also, notice that
$$N^e_{I_{(U,N)}}(x) = I_{(U,N)}(x,e) = U(N(x),e) = N(x)$$
3. (L-CP) Suppose that N is strong. Hence
$$\begin{aligned} I_{(U,N)}(N(x),y) &= U(N(N(x)),y) \\ &= U(x,y) \\ &= U(y,x) \\ &= U(N(N(y)),x) \\ &= I_{(U,N)}(N(y),x) \end{aligned}$$

□

Theorem 4.1 *Let $I : L^2 \to L$ be a fuzzy implication and $N : L \to L$ be a fuzzy negation. Define function $U_I : L^2 \to L$ as*

$$U_I(x,y) = I(N(x),y)\ \forall x,y \in L \qquad (10)$$

If I satisfies (L-CP) and (EP) and N is such that $N^\alpha_I \circ N = Id_L$ for an arbitrary but fixed $\alpha \in L/\{0_L,1_L\}$ then U_I as in Eq. (10) is an uninorm with neutral element α.

Proof: Let be $x,y,z \in L$. Thus

(U1) Since I satisfies (L-CP) then
$$U_I(x,y) = I(N(x),y) = I(N(y),x) = U_I(y,x)$$
i.e. U_I is commutative;

(U2) By (L-CP) and (EP) it follows that
$$\begin{aligned} U_I(x,U_I(y,z)) &\overset{(10)}{=} I(N(x),I(N(y),z)) \\ &\overset{(L-CP)}{=} I(N(x),I(N(z),y)) \\ &\overset{(EP)}{=} U_I(N(z),U_I(N(x),y)) \\ &= U_I(U_I(x,y),z) \end{aligned}$$

(U3) Suppose that $x \leqslant_L y$ and hence $N(y) \leqslant_L N(x)$. Thus
$$U_I(x,z) = I(N(x),z) \overset{(I1)}{\leqslant} I(N(y),z) = U_I(y,z)$$

(U4) Finally, let's prove that α is an neutral element of U_I. Indeed, since $N^\alpha_I \circ N = Id_L$ it follows
$$U_I(\alpha,x) = U_I(x,\alpha) = I(N(x),\alpha) = N^\alpha_I(N(x)) = x$$

□

5 Extension of (U,N)-implications

Let L be a bounded lattice, $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ an e-operator on M. Let's prove that for every conjunctive uninorm $U : M^2 \to M$ and fuzzy negation $N : M \to M$, function

$$I_{U^E_\odot, N^E_\odot}(x,y) = U^E_\odot(N^E_\odot(x), y)\ \forall x,y \in L \tag{11}$$

is a fuzzy implication L whereas $U^E_\odot$ and $N^E_\odot$ are extensions via e-operator $\odot$ of U and N respectively. Indeed, first notice that for all $x, y \in L$ we have that $r_1(x) \leqslant_M r_2(x)$ hence $N(r_2(x)) \leqslant_M N(r_1(x))$, $r_1(N(r_1(x)) \odot N(r_2(x))) = N(r_2(x))$ and $r_2(N(r_1(x)) \odot N(r_2(x))) = N(r_1(x))$. Thus

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(x,y) &= U^E_\odot(N^E_\odot(x), y) \\
&= U^E_\odot(N(r_1(x)) \odot N(r_2(x)), y) \\
&= U(r_1(N(r_1(x)) \odot N(r_2(x))), r_1(y)) \\
&\quad \odot U(r_2(N(r_1(x)) \odot N(r_2(x))), r_2(y)) \\
&= U(N(r_2(x)), r_1(y)) \\
&\quad \odot U(N(r_1(x)), r_2(y))
\end{aligned}$$

Taking this into account and supposing that U is conjunctive (i.e. $U(0_M, 1_M) = U(1_M, 0_M) = 1_M$), we have that

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(0_L, 0_L) &= U(N(r_2(0_L)), r_1(0_L)) \odot \\
&\quad U(N(r_1(0_L)), r_2(0_L)) \\
&= U(1_M, 0_M) \odot U(1_M, 0_M) \\
&= s(U(1_M, 0_M)) = s(1_M) = 1_L
\end{aligned}$$

Similary, one can prove that $I_{U^E_\odot, N^E_\odot}(1_L, 1_L) = 1_L$. Also,

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(1_L, 0_L) &= U(N(r_2(1_L)), r_1(0_L)) \odot \\
&\quad U(N(r_1(1_L)), r_2(0_L)) \\
&= U(0_M, 0_M) \odot U(0_M, 0_M) \\
&= s(U(0_M, 0_M)) = s(0_M) = 0_L
\end{aligned}$$

Therefore, (I3), (I4) and (I5) hold for $I_{U^E_\odot, N^E_\odot}$.

Moreover, for all $x, y, z \in L$ such that $x \leqslant_L y$. In this case, for $i \in \{1, 2\}$ we have that $r_i(x) \leqslant_M r_i(y)$ and hence $N(r_i(y)) \leqslant_M N(r_i(x))$. Thus

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(x,z) &= U(N(r_2(x)), r_1(z)) \\
&\quad \odot U(N(r_1(x)), r_2(z)) \\
&\geqslant_L U(N(r_2(y)), r_1(z)) \\
&\quad \odot U(N(r_1(y)), r_2(z)) \\
&= I_{U^E_\odot, N^E_\odot}(y,z)
\end{aligned}$$

and

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(z,x) &= U(N(r_2(z)), r_1(x)) \\
&\quad \odot U(N(r_1(z)), r_2(x)) \\
&\leqslant_L U(N(r_2(z)), r_1(y)) \\
&\quad \odot U(N(r_1(z)), r_2(y)) \\
&= I_{U^E_\odot, N^E_\odot}(z,y)
\end{aligned}$$

which proves that $I_{U^E_\odot, N^E_\odot}$ satisfies (I1) and (I2).

Theorem 5.1 *Let L be a bounded lattice, $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ an e-operator on M. For every conjunctive uninorm $U : M^2 \to M$ and fuzzy negation $N : M \to M$, function defined by Eq. (11) is a (U,N)-implication on L.*

Proposition 5.1 *Let L be a bounded lattice, $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ an e-operator on M. Given an uninorm $U : M^2 \to M$ and a fuzzy negation $N : M \to M$ it holds that*

$$I_{U^E_\odot, N^E_\odot} = (I_{U,N})^E_\odot \tag{12}$$

.

Proof: For all $x, y, z \in L$ it follows that

$$\begin{aligned}
I_{U^E_\odot, N^E_\odot}(x,y) &= U^E_\odot(N^E_\odot(x), y) \\
&= U^E_\odot(N(r_1(x)) \odot N(r_2(x)), y) \\
&= U(r_1(N(r_1(x)) \odot N(r_2(x))), r_1(y)) \\
&\quad \odot U(r_2(N(r_1(x)) \odot N(r_2(x))), r_2(y)) \\
&= U(N(r_2(x)), r_1(y)) \odot U(N(r_1(x)), r_2(y)) \\
&= I_{U,N}(r_2(x), r_1(y)) \odot I_{U,N}(r_1(x), r_2(y)) \\
&= (I_{U,N})^E_\odot(x,y)
\end{aligned}$$

□

Proposition 5.2 *Let L be a bounded lattice, $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ an e-operator on M, $U : M^2 \to M$ be an uninorm with neutral element e and $N : M \to M$ be a fuzzy negation. If $I_{U,N}$ is a (U,N)-fuzzy implication on M then*

1. *$I_{U^E_\odot, N^E_\odot}$ satisfies (CC4) and (EP) whereas $I_{U,N}$ satifies that properties;*
2. *$N^{e'}_{I_{(U^E_\odot, N^E_\odot)}} = N^E_\odot$ (where $e' = s(e)$) and $I_{(U^E_\odot, N^E_\odot)}$ satisfies R-CP(N);*
3. *if $N^E_\odot$ is strong then $I_{(U^E_\odot, N^E_\odot)}$ satisfies L-CP(N).*

Proof: Let us consider $x, y, z \in L$.
1. The proof for (CC4) is similar to (I3) in the Theorem 5.1. Moreover, we have

$$\begin{aligned}
I_{(U^E_\odot, N^E_\odot)}(x, I_{(U^E_\odot, N^E_\odot)}(y,z)) &= U^E_\odot(N^E_\odot(x), U^E_\odot(N^E_\odot(y), z)) \\
&= U(N(r_2(x)), U(N(r_2(y), r_1(z))) \\
&\quad \odot U(N(r_1(x)), U(N(r_1(y), r_2(z))) \\
&= U(N(r_2(y)), U(N(r_2(x), r_1(z))) \\
&\quad \odot U(N(r_1(y)), U(N(r_1(x), r_2(z))) \\
&= I_{(U^E_\odot, N^E_\odot)}(y, I_{(U^E_\odot, N^E_\odot)}(x,z))
\end{aligned}$$

2. For any $x \in L$ it follows that

$$N^{e'}_{I_{(U^E_\odot, N^E_\odot)}}(x) = I_{(U^E_\odot, N^E_\odot)}(x, e') = U^E_\odot(N^E_\odot(x), s(e)) =$$

$$= U(N(r_2(x)), e) \odot U(N(r_1(x), e)) =$$

$$= N(r_2(x)) \odot N(r_1(x)) = N^E_\odot(x)$$

Similarly to the prove of item (2) of Proposition 4.2 one can verify that R-(CP) holds for $I_{(U^E_\odot, N^E_\odot)}$;

3. Analogous to the proof of item (3) of Proposition 4.2.

□

Theorem 5.2 *Let L be a bounded lattice, $M \trianglelefteq L$ with respect to (r_1, r_2, s), $\odot$ an e-operator on M, $I : M^2 \to M$ be a fuzzy implication and $N : M \to M$ be a fuzzy negation. Define function $U_{I^E_\odot} : L^2 \to L$ as*

$$U_{I^E_\odot}(x, y) = I^E_\odot(N^E_\odot(x), y) \text{ for all } x, y \in L \quad (13)$$

If I satisfies (L-CP) and (EP) and N is such that $N^{e'}_{I_{(U^E_\odot, N^E_\odot)}} \circ N^E_\odot = I_d$ where $e' = s(e)$ then $U_{I^E_\odot}$ is an uninorm on L with neutral element e'.

Proof: Let be $x, y, z \in L$. Thus
(U1) Since I satisfies (L-CP) then

$$\begin{aligned}
U_{I^E_\odot}(x, y) &= I^E_\odot(N^E_\odot(x), y) \\
&= I(r_2(N^E_\odot(x)), r_1(y)) \odot I(r_1(N^E_\odot(x)), r_2(y)) \\
&= I(N(r_1(x)), r_1(y)) \odot I(N(r_2(x)), r_2(y)) \\
&= I(N(r_1(y)), r_1(x)) \odot I(N(r_2(y)), r_2(x)) \\
&= I(r_2(N^E_\odot(y)), r_1(x)) \odot I(r_1(N^E_\odot(y)), r_2(x)) \\
&= I^E_\odot(N^E_\odot(y), x) \\
&= U_{I^E_\odot}(y, x)
\end{aligned}$$

i.e. $U_{I^E_\odot}$ is commutative;

(U2) By (L-CP) and (EP) it follows that

$$\begin{aligned}
U_{I^E_\odot}(x, U_{I^E_\odot}(y, z)) &\overset{(13)}{=} I^E_\odot(N^E_\odot(x), I^E_\odot(N^E_\odot(y), z)) \\
&= I(N(r_1(x)), I(N(r_1(y)), r_1(z))) \\
&\quad \odot I(N(r_2(x)), I(N(r_2(y)), r_2(z))) \\
&\overset{(L-CP)}{=} I(N(r_1(x)), I(N(r_1(z)), r_1(y))) \\
&\quad \odot I(N(r_2(x)), I(N(r_2(z)), r_2(y))) \\
&\overset{(EP)}{=} I(N(r_1(z)), I(N(r_1(x)), r_1(y))) \\
&\quad \odot I(N(r_2(z)), I(N(r_2(x)), r_2(y))) \\
&= U_{I^E_\odot}(U_{I^E_\odot}(x, y), z)
\end{aligned}$$

(U3) If $x \leqslant_L y$ then $N^E_\odot(y) \leqslant_L N^E_\odot(x)$. Thus

$$U_I(x, z) = I^E_\odot(N^E_\odot(x), z) \overset{(I1)}{\leqslant} I^E_\odot(N^E_\odot(y), z) = U_{I^E_\odot}(y, z)$$

(U4) Finally, let's prove that α is an neutral element of U_I. Indeed, since $N^\alpha_I \circ N = Id_L$ it follows

$$\begin{aligned}
U_{I^E_\odot}(e', x) &= U_{I^E_\odot}(x, e') \\
&= I^E_\odot(N^E_\odot(x), s(e)) \\
&= I(N(r_1(x)), e) \odot I(N(r_2(x)), e) \\
&= N^e_I(N(r_1(x))) \odot N^e_I(N(r_2(x))) \\
&= r_1(x) \odot r_2(x) = x
\end{aligned}$$

□

6 Final Remarks

In this paper we introduced the concept of implications in the context of lattices, besides characterizing the construction of uninorms from L-implications and L-negation. Further, we presented a way to extend (U, N)-implications and studied some of the properties that are preserved by the extension method via e-operators. Results once again proved the efficiency and robustness of the method, as we have seen in [23, 25, 26, 27, 28].

As future works, we wish to continue the study of the application of the extension method via e-operators to other functions in the context of the lattice fuzzy theory and also to propose a new and more flexible extension method.

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References

[1] P. Akella. Structure of n-uninorms. *Fuzzy Sets and Systems*, 158:1631–1651, 2007.

[2] M. Baczyński and B. Jayaram. Fuzzy Implications. *Studies in Fuzziness and Soft Computing, Volume 231*, Springer-Verlag Berlin Heidelberg, 2008.

[3] B. C. Bedregal, H. S. Santos and R. Callejas-Bedregal. T-norms on bounded lattices: t-norm morphisms and operator. *IEEE International Conference on Fuzzy Systems*, 22–28, 2006.

[4] B. C. Bedregal and A. Takahashi. The best interval representations of t-norms and automorphisms. *Fuzzy Sets and Systems*, 157:3220-3230, 2006.

[5] G. Birkhoff. Lattice Theory. *American Mathematical Society. Providence. RI*, 1973.

[6] S. Burris and H. P. Sankappanavar. A Course in Universal Algebra. *The Millennium Edition, New York* 2005.

[7] B. Bedregal, H. Bustince, E.S. Palmeira, G.P. Dimuro and J. Fernández. Generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions. Int. J. Approx. Reasoning 90: 1–16, 2017.

[8] G.D. Çayli, F. Karaçal and R. Mesiar. On a new class of uninorms on bounded lattices. *Information Sciences*, 367-368: 221–231, 2016.

[9] T. Chaira and A.K. Ray. Region extraction using fuzzy similarity measures. *J. Fuzzy Math.*, 11(3): 601–607, 2003.

[10] G. Chen and T. T. Pham. Fuzzy Sets, Fuzzy Logic and Fuzzy Control Systems. *CRC Press, Boca Raton*, 2001.

[11] B. A. Davey and H. A. Priestley. Introduction to Lattices and Order. *2nd ed. Cambridge University Press. Cambridge*, 2002.

[12] P. Hajek. Metamathematics of Fuzzy Logic. *Kluwer Academic Publishers, Dordrecht*, 1998.

[13] Y.L. Han and F.G. Shi. A new way to extend fuzzy implications. *Iranian Journal of Fuzzy Systems*, 15(3):79–97, 2018.

[14] A.K. Hans-Peter and L.B. Shapiro. On simultaneous extension of continuous partial functions. *Proceedings of America Mathematical Society*, 125(6): 1853–1859, 1997.

[15] K. Horiuchi and H. Murakami. Extension of the concept of mappings using fuzzy sets. *Fuzzy Sets and Systems*, 56(1): 79–88, 1993.

[16] F. Karaçal, M.A Ince and R. Mesiar. Nullnorms on bounded lattices. *Information Sciences*, 325: 227–236, 2015.

[17] F. Karaçal and R. Mesiar. Uninorms on bounded lattices. *Fuzzy Sets and Systems*, 261: 33–43, 2015.

[18] F. Karaçal, U. Ertuğrul and R. Mesiar. Characterization of uninorms on bounded lattices. *Fuzzy Sets and Systems*, 308: 54–71, 2017.

[19] E. P. Klement, R. Mesiar and E. Pap. Triangular Norms. *Kluwer Academic Publishers, Dordrecht*, 2000.

[20] K. Murota and A. Shioura. Extension of M-convexity and L-convexity of polyedral convex functions. *Advances in Applied Mathematics*, 25(4): 352–427, 2000.

[21] M. Nachtegael, P. Sussner, T. Mélange, and E.E. Kerre. On the role of complete lattices in mathematical morphology: From tool to uncertianty model. *Information Sciences*, 181:1971–1988, 2011.

[22] E.S. Palmeira and B.C. Bedregal. Extension of fuzzy logic operators defined on bounded lattices via retractions. *Computer & Mathematics with Applications*, 63: 1026–1038, 2012.

[23] E.S. Palmeira, B. Bedregal and R. Mesiar. A new way to extend t-norms, t-conorms and negations. *Fuzzy Sets and Systems*. 240: 1–21, 2014.

[24] E.S. Palmeira, B. Bedregal, J. Fernandez and A. Jurio. On extension of lattice-valued implications via retractions. *Fuzzy Sets and Systems*. 240:66–85 , 2014.

[25] E.S. Palmeira, B. Bedregal and J.A.G. dos Santos. Some results on extension of lattice-valued QL-implications. J. Braz. Comp. Soc. 22(1): 4:1-4:9, 2016.

[26] E.S. Palmeira and B. Bedregal. Some results on extension of lattice-Valued XOR, XOR-implications and E-implications. IPMU (2): 809-820, 2016.

[27] E.S. Palmeira and B. Bedregal. Using e-operators to extend lattice-valued uninorms and nullnorms. 2017 IEEE International Conference on Fuzzy Systems - FUZZ-IEEE, 1–6, 2017.

[28] E.S. Palmeira, B. Bedregal, H. Bustince, D. Paternain, L. De Miguel. Application of two different methods for extending lattice-valued restricted equivalence functions used for constructing similarity measures on L-fuzzy sets. Information Sciences 441: 95–112, 2018.

[29] S. Saminger-Platz, E. P. Klement and R. Mesiar. On extensions of triangular norms on bounded lattices. *Indagationes Mathematicae*, 19(1): 135–150, 2008.

[30] W. Siler and J.J. Buckley. Fuzzy expert systems and fuzzy reasoning. *Indag. Mathem., N. S.*, 19(1):135–150, 2008.

[31] Y. Su, W. Zong and H. Liu. Migrativity property of uninorms. *Fuzzy Sets and Systems*, 287: 213–226, 2016.

[32] M. Takano. Strong completeness of lattice-valued logic. *Arch. Math. Logic*, 41:497–505, 2002.