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A Characterization of Uninorms by Means of a Pre-order they Induce

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Abstract

In Hliněná et al. (2014) the authors, inspired by Karaçal and Kesicioğlu (2011), introduced a pre-order induced by uninorms. This contribution is devoted to a classification of families of uninorms by means of types of pre-orders (and orders) they induce. Philosophically, the paper follows the original idea of Clifford (1954).

Keywords: Pre-order induced by uninorm, Representable uninorm, Uninorm, Uninorm with continuous underlying operations, Locally internal uninorm.

1 Introduction

In this paper we study pre-orders generated by uninorms. The main idea is based on that of Karaçal and Kesicioğlu [19], and follows the original idea of Clifford [4]. The main idea of authors is to show a relationship between families of uninorms and families of pre-orders (partial orders, in some cases) they induce (see [16]). In some sense, the pre-order (see Definion 11) follows the original idea by Clifford [4]. Another relation induced by uninorms, that is always a partial order (see Definition 12), was proposed by Erteğrul et al. [11]. Here, the main intention of authors was to get a partial order. But this relation (partial order) does not extend the relation introduced by Clifford [4].

2 Preliminaries

In this section we review some well-known types of monotone commutative monoidal operations on $[0,1]$ and provide an overview of, from the point of view of this contribution, important steps in introducing orders (and pre-orders) induced by semigroups.

2.1 Known types of monotone commutative monoidal operations on $[0,1]$

In this part we give just very brief review of well-known types of monotone commutative monoidal operations on $[0,1]$. For more details we recommend monographs [2, 20].

Definition 1 (see, e.g., [20]). *A triangular norm T (t-norm for short) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$, fulfilling the boundary condition $T(x,1) = x$, for all $x \in [0,1]$.*

Definition 2 (see, e.g., [20]). *A triangular conorm S (t-conorm for short) is a commutative, associative, monotone binary operation on the unit interval $[0,1]$, fulfilling the boundary condition $S(x,0) = x$, for all $x \in [0,1]$.*

Remark 1. *If T is a t-norm, then*

$$S(x,y) = 1 - T(1-x, 1-y)$$

is a t-conorm and vice versa. We obtain a dual pair (T,S) of a t-norm and a t-conorm.

Example 1. *Well-known examples of triangular norms and their dual t-conorms are:*

- $T_M(x,y) = \min(x,y), \quad S_M(x,y) = \max(x,y)$,
- $T_P(x,y) = x.y, \quad S_P(x,y) = x+y-x.y$,
- $T_L(x,y) = \max(x+y-1,0), \quad S_L(x,y) = \min(x+y,1)$.

Casasnovas, Mayor [3] introduced divisible t-norms.

Definition 3 ([3]). *Let L be a bounded lattice and $T : L \times L \to L$ be a t-norm. T is said to be* divisible *if the following conditions are satisfied for all $(x,y) \in L^2$*

$$(x \le y) \;\Rightarrow\; (\exists z \in L)(T(y,z) = x). \qquad (1)$$

Of course, a t-norm $T : [0,1]^2 \to [0,1]$ is divisible if and only if it is continuous.

Definition 4 (see, e.g., [2]). *Let* $*: [0,1]^2 \to [0,1]$ *be a binary commutative operation. Then*

(i) *element* c *is said to be* idempotent *if* $c * c = c$,

(ii) *element* e *is said to be* neutral *if* $e * x = x$ *for all* $x \in L$,

(iii) *element* a *is said to be* annihilator *if* $a * x = a$ *for all* $x \in L$.

Definition 5 ([26]). *A uninorm* U *is a function* $U: [0,1]^2 \to [0,1]$ *that is increasing, commutative, associative and has a neutral element* $e \in [0,1]$.

Remark 2. For any uninorm with neutral element equal to e we denote

$$A(e) = [0, e[\times]e, 1] \ \cup \ ]e, 1] \times [0, e[\,.$$

1. If $e \notin \{0,1\}$ is the neutral element of U, we say that U is a proper uninorm.
2. Every uninorm U has a distinguished element a called annihilator, for which the following holds $U(a,x) = U(0,1) = a$. A uninorm U is said to be conjunctive if $U(x,0) = 0$, and U is said to be disjunctive if $U(1,x) = 1$, for all $x \in [0,1]$.

Lemma 1 ([12]). *Let* U *be a uninorm with the neutral element* e*. Then, for* $(x,y) \in [0,1]^2$ *the following holds*

(i) $T(x,y) = \frac{U(ex,ey)}{e}$ *is a t-norm,*

(ii) $S(x,y) = \frac{U((1-e)x+e,(1-e)y+e)-e}{1-e}$ *is a t-conorm.*

For all $(x,y) \in A(e)$ *we have*

$$\min(x,y) \le U(x,y) \le \max(x,y).$$

Definition 6. *Let* U *be a uninorm. We say that* U *is* internal *if* $U(x,y) \in \{x,y\}$ *for all* $(x,y) \in [0,1]^2$.
A uninorm U *is* locally internal on a set $G \subset [0,1]^2$ *if* $U(x,y) \in \{x,y\}$ *for all* $(x,y) \in G$.

Remark 3. (a) Particularly, a uninorm U is *locally internal on the boundary* if $U(x,0) \in \{x,0\}$ and $U(x,1) \in \{x,1\}$ holds for all $x \in [0,1]$. Some examples of uninorms which are not locally internal on the boundary can be found, e.g., in [14, 15, 16], see also Fig. 1.

(b) An important family of uninorms is that of internal ones. From results by Drewniak and Drygaś [6] follows that the family of all internal uninorms is identical with that of idempotent uninorms. Some further study of locally internal uninorms can be found, e.g., in [8] and in literature referenced therein.

From results in [6, 21, 25] we have the following.

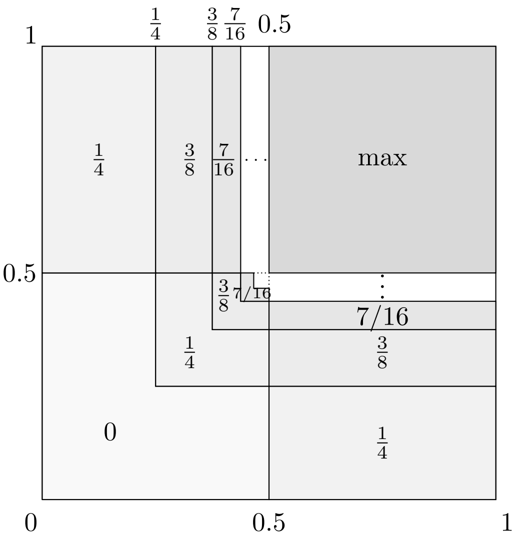

Figure 1: Example of a uninorm not locally internal on the boundary (the L-shaped areas of constantness being right-side closed)

Lemma 2. *Let* U *be a uninorm.* U *is idempotent if and only if it is* U *internal.*

Proposition 1 (E.g., [10]). *Let* $f: [-\infty, \infty] \to [0,1]$ *be an increasing bijection. Then*

$$U(x,y) = f^{-1}\big(f(x) + f(y)\big) \tag{2}$$

is a uninorm that is continuous everywhere except at points $(0,1)$ *and* $(1,0)$*, and is strictly increasing on* $]0,1[^2$*.* U *is conjunctive if we adopt the convention* $-\infty + \infty = -\infty$*, and* U *is disjunctive adopting the convention* $-\infty + \infty = \infty$.

Definition 7 (E.g., [10]). *The uninorm* U *fulfilling formula* (2) *for an increasing bijection* $f: [-\infty, \infty] \to [0,1]$ *adopting either of the conventions,* $-\infty + \infty = -\infty$ *or* $-\infty + \infty = \infty$*, is said to be a* representable uninorm.

Remark 4. Representable uninorms, under the name *aggregative operators* were studied already by Dombi [5].

Another important class of uninorms is that of continuous ones on $]0,1[^2$. These uninorms were characterized by Hu and Li [17], and further studied by Drygaś [7]. From results in [17] we have the following characterization.

Proposition 2. *A uninorm* U *with neutral element* $e \in]0,1[$ *is continuous on on* $]0,1[^2$ *if and only if one of the following conditions is satisfied:*

(i) U *is representable,*

(ii) *there exists* $0 < a < e$*, a continuous t-norm* T *a representable uninorm* U_r *and an increasing bijection* $\varphi: [a,1] \to [0,1]$ *such that*

$U(x,y) = \varphi^{-1}\big(U_r(\varphi(x),\varphi(y))\big)$ *for* $(x,y) \in [a,1]^2$,
$U(x,y) = aT(\frac{x}{a},\frac{y}{a})$ *for* $(x,y) \in [0,a]^2$,
$U(x,y) = \min\{x,y\}$ *for* $(x,y) \in [0,a[\,\cap\,]a,1[\ \cup\ ]a,1[\,\cap\,[0,a[$,
and U is locally internal on the boundary,

(iii) *or there exists $e < b < 1$ a continuous t-conorm S and a representable uninorm U_r and an increasing bijection $\varphi : [0,b] \to [0,1]$ such that*
$U(x,y) = \varphi^{-1}\big(U_r(\varphi(x),\varphi(y))\big)$ *for* $(x,y) \in [0,b]^2$,
$U(x,y) = b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b})$ *for* $(x,y) \in [b,1]^2$,
$U(x,y) = \max\{x,y\}$ *for* $(x,y) \in\ ]b,1]\cap[0.b[\ \cup\ [0,b[\,\cap\,]b,1]$,
and U is locally internal on the boundary.

2.2 An overview of pre-orders induced by a semigroup

The study of orders (pre-orders) induced by a semigroup operation had started by Clifford [4]. Later, Hartwig [13] and independently also Nambooripad [23], defined a partial order on regular semigroups. Their definition is the following.

Definition 8 ([13, 23]). *Let $(S,\oplus)$ be a semigroup and E_S the set of its idempotent elements. Then*

$$a \leq_\oplus b \ \Leftrightarrow\ (\exists e,f \in E_S)(a = b\oplus e = f \oplus b).$$

If the relation $\leq_\oplus$ is a partial order on S, it is called natural.

Definition 8 was generalized by Mitch [22].

Definition 9 ([22]). *Let $(S,\oplus)$ be an arbitrary semigroup. By $\leq_\oplus$ we denote the following relation*

$$a \leq_\oplus b \quad\Leftrightarrow\quad a = b\oplus z_1 = z_2 \oplus b,\ a\oplus z_1 = a$$

for some $z_1, z_2 \in E_{S^1}$, where

$$S^1 = \begin{cases} S & \text{if } S \text{ has a neutral element,}\\ S\cup\{e\} & \text{otherwise, where } e \text{ plays}\\ & \quad\text{the role of the neutral element,}\end{cases}$$

and E_{S^1} is the set of all idempotents of S^1.

Lemma 3 ([22]). *Let $(S,\oplus)$ be an arbitrary semigroup. The relation $\leq_\oplus$ is reflexive and antisymmetric on S.*

Proposition 3 ([22]). *Let $(S,\oplus)$ be an arbitrary semigroup. The relation*

$$a \lesssim_\oplus b \ \Leftrightarrow\ a = x\oplus b = b\oplus y \tag{3}$$

for some $x,y \in S^1$, is a partial order on S.

From now on, we restrict our attention to commutative semigroups. Lemma 3 and Proposition 3 immediately imply the following.

Lemma 4. *Let $(S,\oplus)$ be a commutative semigroup. By $a_{\lesssim_\oplus}$ we denote the set*

$$a_{\lesssim_\oplus} = \{z \in S; z \lesssim_\oplus a\},$$

where $a \in S$. Then for all $a,b \in S$ it holds that $a \lesssim_\oplus b$ if and only if $a_{\lesssim_\oplus} \subseteq b_{\lesssim_\oplus}$.

Directly by Definition 9 we get

Proposition 4. *Let $(S,\oplus)$ be a commutative semigroup. Then the set $a_{\lesssim_\oplus}$ is an ideal in $(S,\oplus)$.*

Lemma 5. *Let $(S,\oplus)$ be a commutative semigroup. Let I_S be an ideal of $(S,\oplus)$. Then $(I_S,\oplus_{I_S})$ is a subsemigrup of $(S,\oplus)$, where $\oplus_{I_S} = \oplus \restriction I_S^2$.*

Karaçal and Kesicioğlu [19] defined a partial order on bounded lattices L by means of t-norms.

Definition 10 ([19]). *Let L be a bounded lattice and $T : L\times L \to L$ a t-norm. We write $x \preceq_T y$ for arbitrary $x,y \in L$, if there exists $z \in L$ such that $x = T(y,z)$.*

Proposition 5 ([19]). *Let L be a bounded lattice and $T : L\times L \to L$ a t-norm. Then the relation $\preceq_T$ is a partial order on L.*

Remark 5. For arbitrary t-norm T, the partial order $\preceq_T$ from Definition 10 extends the partial order $\lesssim_T$ from Definition 9 in the following sense:
let L be arbitrary bounded lattice and T a commutative semigroup-operation on L with a neutral element such that $(L,\preceq_T)$ is a partially ordered set. Then

$$a \lesssim_T b \quad\Rightarrow\quad a \preceq_T b$$

for all $a,b \in L$.

Remark 6. Concerning a correspondence between properties of binary aggregation function $A : L^2 \to L$ and relation $\preceq_A$ (changing a t-norm T for A in Definition 10), the following can be said:

- if A has a neutral element, or A is idempotent, then $\preceq_A$ is reflexive,
- if A is associative, then $\preceq_A$ is transitive,
- the anti-symmetry of $\preceq_A$ fails if there exist elements $x \neq z$ and y_1, y_2 such that $z = A(x,y_1)$ and $x = A(z,y_2)$. Hence, if one of the following

  $$x \preceq_A z \quad\Rightarrow\quad x \leq_L z,$$
  $$x \preceq_A z \quad\Rightarrow\quad z \leq_L x$$

  holds then $\preceq_A$ is anti-symmetric.

Hliněná et al. [16] introduced the following relation $\preceq_U$.

Definition 11 ([16])**.** *Let $U : [0,1]^2 \to [0,1]$ be an arbitrary uninorm. By $\preceq_U$ we denote the following relation*

$$x \preceq_U y \text{ if there exists } \ell \in [0,1] \text{ such that } U(y,\ell) = x.$$

Immediately by Definition 11 we get the next lemma.

Lemma 6. *Let U be an arbitrary uninorm. Then $\preceq_U$ is transitive and reflexive. If a and e are the annihilator and the neutral elements of U, respectively, then*

$$a \preceq_U x \preceq_U e$$

holds for all $x \in [0,1]$.

Remark 7. In Definition 11 we have used the same notation $\preceq_U$ for the pre-order defined from a uninorm U, as in Definition 10 for the corresponding partial order $\preceq_T$ defined from a t-norm T. These two relations really coincide if $U = T$, i.e., the notation should not cause any problems.

The pre-order $\preceq_U$ extends the partial order $\lesssim_U$ from Definition 9 in the following sense.

Proposition 6. *Let U be an arbitrary uninorm. Then*

$$x \lesssim_U y \quad \Rightarrow \quad x \preceq_U y \tag{4}$$

for all $(x,y) \in [0,1]^2$.

A different type of partial order induced by uninorms has been defined by Ertuğrul et al. [11].

Definition 12 ([11])**.** *Let U be a uninorm and $e \in]0,1[$ its neutral element. For $(x,y) \in [0,1]^2$ denote $x \trianglelefteq_U y$ if one of the following properties is satisfied:*

1. *there exists $\ell \in [0,e]$ such that $x = U(y,\ell)$ and $(x,y) \in [0,e]^2$,*
2. *there exists $\ell \in [e,1]$ such that $y = U(x,\ell)$ and $(x,y) \in [e,1]^2$,*
3. *$0 \le x \le e \le y \le 1$.*

Proposition 7 ([11])**.** *For an arbitrary uninorm U, the relation $\trianglelefteq_U$ from Definition 12 is a partial order.*

Example 2. Consider the following uninorm U

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,\frac{1}{2}]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Then $\trianglelefteq_U$ coincides with the usual order of $[0,1]$, while $x \lesssim_U y$ if one of the following possibilities is satisfied

- $y \le x$ for $x > 0.5$,
- $x \le y$ for $(x,y) \in [0, 0.5[^2$,
- $y = 0.5$.

Remark 8. Let U be a uninorm. To compare the relation $\preceq_U$ from Definition 11 with $\trianglelefteq_U$ from Definition 12, the following should be remarked.

(i) The relation $\preceq_U$, given in Definition 11 is a pre-order, but not necessarily a partial order. Unlike this, the relation $\trianglelefteq_U$ defined by Definition 12, is always a partial order.

(ii) As illustrated by Example 2, the partial order $\trianglelefteq_U$ does not necessarily extends the partial order $\lesssim_U$ on the semigroup $([0,1], U)$, i.e.,

$$x \lesssim_U y \not\Rightarrow x \trianglelefteq_U y.$$

As shown by Proposition 6, the pre-order $\preceq_U$ always extends the partial order $\lesssim_U$ on $([0,1], U)$, see formula (4).

Further in the text, we will consider only the pre-order $\preceq_U$ to distinguish several families of uninorms.

Definition 13. *Let U be an arbitrary uninorm.*

(i) *For $(x,y) \in [0,1]^2$ we denote $x \sim_U y$ if $x \preceq_U y$ and $y \preceq_U x$.*

(ii) *For $(x,y) \in [0,1]^2$ we denote $x \parallel_U y$ if neither $x \preceq_U y$ nor $y \preceq_U x$ holds,*
and $dx \nparallel_U y$ if $x \preceq_U y$ or $y \preceq_U x$.

(iii) *For arbitrary $x \in [0,1]$ we denote $x_{\sim_U} = \{z \in [0,1]; z \sim_U x\}$.*

3 Some distinguished families of uninorms and properties of the corresponding pre-orders

We are going to study a relationship between some distinguished families $\mathcal{U}$ of uninorms on the one hand and properties of the corresponding pre-orders $\preceq_U$ for $U \in \mathcal{U}$ on the other hand.

A direct consequence to Lemma 6 is the following.

Corollary 1. *Let U be a uninorm. The following holds for all $x \in [0,1]$:*

(i) *$0 \preceq_U x$ if and only if U is conjunctive,*

(ii) *$1 \preceq_U x$ if and only if U is disjunctive.*

3.1 Locally internal uninorms

In this part we distinguish three types of locally internal uninorms:
– on the boundary,
– on $A(e)$,
– on $[0,e]^2 \cup [e,1]^2$.

Proposition 8. *Let U be a uninorm. It is locally internal on the boundary if and only if for every element $x \in [0,1]$*

$$0 \not\parallel_U x \quad \text{and} \quad 1 \not\parallel_U x.$$

Proposition 9. *Let U be a uninorm with neutral element e. It is locally internal on $A(e)$ if and only if $\preceq_U$ is a partial order and for every element $x \in [0,e]$ and $y \in [e,1]$ we have*

$$x \not\parallel_U y.$$

Remark 9. For an arbitrary uninorm U and for a pair $(x,y) \in [0,1]^2$, we have

$$\begin{aligned} U(x,y) = x \quad &\Rightarrow \quad x \preceq_U y, \\ U(x,y) = y \quad &\Rightarrow \quad y \preceq_U x. \end{aligned}$$

Results in [8] imply that if a uninorm U is locally internal on $A(e)$, there are three possibilities:

(a) $U(x,y) = \min\{x,y\}$ for all $(x,y) \in A(e)$,

(b) $U(x,y) = \max\{x,y\}$ for all $(x,y) \in A(e)$,

(c) there exists a (not necessarily strictly) decreasing function $f : [0,e[\to]e,1]$ such that, for $(x,y) \in [0,e[\times]e,1]$, we have

$$U(x,y) \begin{cases} = x & \text{if } y < f(x), \\ = y & \text{if } y > f(x), \\ \in \{x,y\} & \text{if } y = f(x). \end{cases}$$

Proposition 10. *Let U be a uninorm. If U is locally internal on $[0,e]^2 \cup [e,1]^2$, then $\preceq_U$ is a linear order.*

Example 3. Let $U : [0,1]^2 \to [0,1]$ be defined by

$$U(x,y) = \begin{cases} xy & \text{if } \max\{x,y\} \le 0.6, \\ \min\{1, x+y\} & \text{if } \min\{x,y\} \ge 0.6, \\ \min\{x,y\} & \text{otherwise.} \end{cases} \quad (5)$$

Then U is a uniform with its neutral element $e = 0.6$ and annihilator $a = 0$. U generates the following relation $\preceq_U$

$$x \preceq_U y \;\Leftrightarrow\; \begin{cases} x \le y & \text{and } x < 0.6, \\ x \ge y & \text{and } x,\, y \ge 0.6. \end{cases}$$

The uninorm U is just locally internal on $A(e)$ and not internal, but $\preceq_U$ is a linear order.

3.2 Uninorms with continuous underlying t-norm and t-conorm

Results in [19] imply the following.

Proposition 11. *Let U be a proper uninorm with a neutral element e. Then U has continuous underlying t-norm and t-conorm if and only if the following hold*

$$\begin{aligned} x \le y \;\Rightarrow\; x \preceq_U y \quad &\text{for } (x,y) \in [0,e]^2, \\ y \le x \;\Rightarrow\; x \preceq_U y \quad &\text{for } (x,y) \in [e,1]^2. \end{aligned}$$

Propositions 9 and 11 have the following corollary.

Corollary 2. *Let U be a proper uninorm. Then U is locally internal on $A(e)$ and with continuous underlying t-norm and t-conorm if and only if $\preceq_U$ is a linear order.*

Applying Proposition 2 to the pre-order $\preceq_U$ we get the following characterization of representable uninorms.

Proposition 12. *A uninorm U is representable if and only if for all $(x,y) \in]0,1[^2$ we have $x \sim_U y$.*

Proposition 2 implies the following characterization of uninorms continuous on $]0,1[^2$.

Proposition 13. *Let U be a proper uninorm with neutral element e, which is not representable. Then it is continuous on $]0,1[^2$ if and only if one of the following is valid.*

(i) *There exists $0 < a < e$ such that*

1. *$x \sim_U y$ for all $(x,y) \in]a,1[^2$,*
2. *$x \preceq_U y \;\Leftrightarrow\; x \le y$ for all $(x,y) \in [0,a]^2$.*

(ii) *There exists $e < b < 1$ such that*

1. *$x \sim_U y$ for all $(x,y) \in]0,b[^2$,*
2. *$x \preceq_U y \;\Leftrightarrow\; x \ge y$ for all $(x,y) \in [b,1]^2$.*

3.3 Some other classes of uninorms on $[0,1]$

First, we provide some results on uninorms with an area of constantness in $[0,e]^2$ or $[e,1]^2$.

Proposition 14 ([14])**.** *Let U be a proper uninorm having e as neutral element. Let $y > e$ be an idempotent element of U. If there exists $x < e$ such that $U(x,y) = \tilde{x} \in]x,e[$ then*

$$U(z,y) = \tilde{x} \text{ and } U(z,x) = U(\tilde{x},x) \quad \text{for all } z \in [x,\tilde{x}].$$

Dually to Proposition 14 we get

Proposition 15. *Let U be a proper uninorm having e as neutral element. Let $y < e$ be an idempotent element of U. If there exists $x > e$ such that $U(x,y) = \tilde{x} \in]e,x[$ then*

$$U(z,y) = \tilde{x} \text{ and } U(z,x) = U(\tilde{x},x) \quad \text{for all } z \in [\tilde{x},x].$$

Propositions 14 and 15 have the following corollary.

Corollary 3. *Let U be a proper uninorm having e as neutral element.*

(i) *Assume $x < e$ is an idempotent element of U. Then either $x \not\parallel_U y$ for all $y \in [e, 1]$ or there exists an interval $]a, b] \subset [e, 1]$ such that $x \parallel_U z$ for all $z \in]a, b]$.*

(ii) *Assume $x > e$ is an idempotent element of U. Then either $x \not\parallel_U y$ for all $y \in [0, e]$ or there exists an interval $[a, b[\subset [0, e]$ such that $x \parallel_U z$ for all $z \in [a, b[$.*

Kalina and Král' [18] introduced uninorms which are strictly increasing on $]0, 1[^2$, but not continuous. The construction method was further studied in [1, 27]. Since we are not able to distinguish among continuous t-norms T (t-conorms S) by means of the relation $\preceq_T$ ($\preceq_S$), we are not able to characterize unambiguously uninorms which are strictly increasing on $]0, 1[^2$. We present the main idea of the construction method, paving, in case the basic 'brick' is the product t-norm T_π:

(a) we split the interval $]0, 1[$ into infinitely countably many disjoint right-closed subintervals $\{I_j; j \in \mathcal{J}\}$, where $\mathcal{J}$ is an index set and $(\mathcal{J}, \circledast, j_0)$ is a commutative increasing monoid and j_0 is its neutral element,

(b) $\vartheta_j : I_j \to]0, 1]$ is an increasing bijection.

The resulting uninorm is defined by:

$$
\begin{aligned}
U_p(x, y) =& \vartheta^{-1}_{i \circledast j}\big(T_\pi(\vartheta_i(x), \vartheta_j(y))\big) \text{ for } x \in J_i,\ y \in J_j,\\
& 0 \quad \text{if } \min\{x, y\} = 0,\\
& 1 \quad \text{otherwise.}
\end{aligned}
\tag{6}
$$

Concerning the properties of $\preceq_{U_p}$ there are two possibilities depending whether $(\mathcal{J}, \circledast, j_0)$ is a group or not.

Proposition 16. *Let U_p be a uninorm defined by (6), $(\mathcal{J}, \circledast, j_0)$ be a commutative group and $\{I_j; j \in \mathcal{J}\}$ be a system of disjoint right-closed intervals whose union is $]0, 1[$. Then:*

(i) *for every $j \in \mathcal{J}$ and all $(x, y) \in I_j^2$ we have*

$$x \preceq_{U_p} y \;\Leftrightarrow\; x \leq y,$$

(ii) *for all $i, j \in \mathcal{J}$, $i \neq j$, all $x \in J_i$ and $y \in J_j$ we have*

$$
\begin{aligned}
x \sim_{U_p} y \quad &\Leftrightarrow \quad \vartheta_j(y) = \vartheta_i(x),\\
x \preceq l_{U_p} y \quad &\Leftrightarrow \quad \vartheta_i(x) \leq \vartheta_j(y).
\end{aligned}
$$

Proposition 17. *Let U_p be a uninorm defined by (6), $(\mathcal{J}, \circledast, j_0)$ be a commutative monoid without inverse elements, with the neutral element j_0 and $\{I_j; j \in \mathcal{J}\}$ be a system of disjoint right-closed intervals whose union is $]0, 1[$. Then:*

(i) *for every $j \in \mathcal{J}$ and all $(x, y) \in I_j^2$ we have*

$$x \preceq_{U_p} y \;\Leftrightarrow\; x \leq y,$$

(ii) *for all $i, j \in \mathcal{J}$, $i \neq j$, all $x \in J_i$ and $y \in J_j$ we have*
$x \preceq_{U_p} y \;\Leftrightarrow\; \vartheta_i(x) \leq \vartheta_j(y)$ and $(\exists k \in \mathcal{J})(j \circledast k = i)$,
$x \parallel_{U_p} y$ if and only if one of the following holds
$(\nexists k \in \mathcal{J})(j \circledast k = i \;\vee\; i \circledast k = j)$,
$(\exists k \in \mathcal{J})(j \circledast k = i)$ and $\vartheta_i(x) > \vartheta_j(y)$,
$(\exists k \in \mathcal{J})(i \circledast k = j)$ and $\vartheta_i(x) < \vartheta_j(y)$.

We could formulate dual theorems to Propositions 16 and 17 for the case when the basic 'brick' is the probabilistic sum t-conorm.

4 Conclusion

The results presented in this paper are aimed to characterize classes of uninorms by means of a pre-order the induce. We have succeeded in getting full characterization for uninorms which are continuous on $]0, 1[^2$, as well as for uninorms with continuous underlying t-norm and t-conorm, and for those which are locally internal on the boundary and on $A(e)$. Further, it is possible to distinguish whether a uninorm is conjunctive or disjunctive. In some other cases we have partial results.

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