Two theorems from extreme value theory for interval valued events

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Abstract

In papers [3, 4] we proved the Fisher-Tippett-Gnedenko theorem and the Pickands-Balkema-de Haan theorem on family of intuitionistic fuzzy events. Since between the intuitionistic fuzzy events and the interval valued events exist a connection, so we try to prove these basic theorems from extreme value theory for interval valued events. We define the notion of independence and convergence in distribution for interval valued observables, too.

Keywords: Interval valued event, Interval valued state, Interval valued observable, Joint interval valued observable, Independence, Convergence in distribution, Fisher-Tippett-Gnedenko theorem, Pickands-Balkema-de Haan theorem, Excess interval valued distribution, Maximum domain of attraction, Generalized Pareto distribution, Extreme value theory.

1 Introduction

In papers [3, 4] we proved the Fisher-Tippett-Gnedenko theorem and the Pickands-Balkema-de Haan theorem on family of intuitionistic fuzzy events. These basic theorems are from part of statistic, which is called the extreme value theory. But between the intuitionistic fuzzy events introduced by K.T. Atanassov in [1, 2] and the interval valued events introduced by L.A. Zadeh in [14] exist a connection. In papers [7, 9] the authors studied a connection between the family of intuitionistic fuzzy events

$$\mathcal{F} = \{(\mu_A, \nu_A) \quad ; \quad \mu_A + \nu_A \leq 1_\Omega,\ \mu_A, \nu_A : \Omega \to [0,1] \text{ are } \mathcal{S}-\text{measurable functions}\}$$

with the operations and relation

$$\begin{aligned} \mathbf{A} \leq \mathbf{B} &\Leftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B, \\ \mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} &= \big((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega\big). \end{aligned}$$

and the family of interval valued events

$$\mathcal{K} = \{(\pi_C, \rho_C) \quad ; \quad \pi_C \leq \rho_C,\ \pi_C, \rho_C : \Omega \to [0,1] \text{ are } \mathcal{S}-\text{measurable functions}\}$$

with the operations and relation

$$\mathbf{C} \preceq \mathbf{D} \Leftrightarrow \pi_C \leq \pi_D, \rho_C \leq \rho_D$$

$$\mathbf{C} \widehat{\oplus} \mathbf{D} = \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big)$$

$$\mathbf{C} \widehat{\odot} \mathbf{D} = \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega\big).$$

They showed that these two systems are isomorphic by the mapping $\psi : \mathcal{F} \to \mathcal{K}$ given by

$$\psi\big((\mu_A, \nu_A)\big) = (\mu_A, 1_\Omega - \nu_A)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$. Therefore the following relations hold

$$\begin{aligned} \psi(\mathbf{A} \oplus \mathbf{B}) &= \psi(\mathbf{A}) \widehat{\oplus} \psi(\mathbf{B}), & (1) \\ \psi(\mathbf{A} \odot \mathbf{B}) &= \psi(\mathbf{A}) \widehat{\odot} \psi(\mathbf{B}), & (2) \\ \mathbf{A} \leq \mathbf{B} &\Leftrightarrow \psi(\mathbf{A}) \preceq \psi(\mathbf{B}), & (3) \\ \mathbf{A}_n \nearrow \mathbf{A} &\Leftrightarrow \psi(\mathbf{A}_n) \nearrow \psi(\mathbf{A}), & (4) \end{aligned}$$

for each $\mathbf{A}_n, \mathbf{A}, \mathbf{B} \in \mathcal{F}$. They illustrated the connection between intuitionistic fuzzy state $\mathbf{m} : \mathcal{F} \to [0,1]$ and interval valued state $k : \mathcal{K} \to [0,1]$ and that was $\mathbf{m} = k \circ \psi$.

Further in paper [5] we defined the notion of interval valued observable $z : \mathcal{B}(R) \to \mathcal{K}$ and we displayed the connection to the intuitionistic fuzzy observable $x : \mathcal{B}(R) \to \mathcal{F}$, which was $z = \psi \circ x$.

In paper [5] we defined the product operation and the notion of joint interval valued observable $\widehat{h} : \mathcal{B}(R^2) \to$

$\mathcal{K}$ and we showed the connection to the joint intuitionistic fuzzy observable $h : \mathcal{B}(R^2) \to \mathcal{F}$, which was $\widehat{h} = \psi \circ h$.

In this paper we try to prove the basic theorems from extreme value theory for interval valued events. We define the notion of independence and convergence in distribution for interval valued observables, too.

Remark that in a whole text we use a notation "IF" for short a phrase "intuitionistic fuzzy" and a notation "IV" for short a phrase "interval valued".

2 Interval valued events, interval valued state and interval valued observables

First we start with definitions of basic notions (see [7, 9]).

Definition 2.1 *Let Ω be a nonempty set. An interval valued set (IV-set) $\mathbf{C}$ on Ω is a pair (π_C, ρ_C) of mappings $\pi_C, \rho_C : \Omega \to [0,1]$ such that $\pi_C \le \rho_C$.*

Definition 2.2 *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An interval valued event (IV-event) is called an IV-set $\mathbf{C} = (\pi_C, \rho_C)$ such that $\pi_C, \rho_C : \Omega \to [0,1]$ are $\mathcal{S}$-measurable. The family of all IV-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{K}$.*

If $\mathbf{C} = (\pi_C, \rho_C) \in \mathcal{K}$, $\mathbf{D} = (\pi_D, \rho_D) \in \mathcal{K}$, then we define the Lukasiewicz binary operations $\widehat{\oplus}, \widehat{\odot}$ on $\mathcal{K}$ by

$$\mathbf{C}\widehat{\oplus}\mathbf{D} = \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big)$$

$$\mathbf{C}\widehat{\odot}\mathbf{D} = \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega\big)$$

and the partial ordering is given by

$$\mathbf{C} \preceq \mathbf{D} \Leftrightarrow \pi_C \le \pi_D, \rho_C \le \rho_D.$$

The continuity is given by

$$\begin{aligned} \mathbf{C} \nearrow \mathbf{D} \quad &\Leftrightarrow \quad \pi_C \nearrow \pi_D, \rho_C \nearrow \rho_D, \\ \mathbf{C} \searrow \mathbf{D} \quad &\Leftrightarrow \quad \pi_C \searrow \pi_D, \rho_C \searrow \rho_D. \end{aligned}$$

In the IV-probability theory instead of the notion of probability we use the notion of state (see [7, 9]).

Definition 2.3 *Let $\mathcal{K}$ be the family of all IV-events in Ω. A mapping $k : \mathcal{K} \to [0,1]$ is called an interval valued state (IV-state), if the following conditions are satisfied:*

(i) $k((1_\Omega, 0_\Omega)) = 1$, $k((0_\Omega, 0_\Omega)) = 0$;

(ii) if $\mathbf{C}\widehat{\odot}\mathbf{D} = (0_\Omega, 0_\Omega)$ and $\mathbf{C}, \mathbf{D} \in \mathcal{K}$, then $k(\mathbf{C}\widehat{\oplus}\mathbf{D}) = k(\mathbf{C}) + k(\mathbf{D})$;

(iii) if $\mathbf{C}_n \nearrow \mathbf{C}$ (i.e. $\pi_{C_n} \nearrow \pi_C, \rho_{C_n} \nearrow \rho_C$), then $k(\mathbf{C}_n) \nearrow k(\mathbf{C})$.

Probably the most useful result in the IV-state theory is the following representation theorem.

Theorem 2.4 *To each IV-state $k : \mathcal{K} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$k(\mathbf{C}) = (1-\alpha) \int_\Omega \pi_C \, dP + \alpha \int_\Omega \rho_C \, dP$$

for each $\mathbf{C} = (\pi_C, \rho_C) \in \mathcal{K}$.

Between IV-states and IF-states is one-one correspondence by the mapping $\psi : \mathcal{F} \to \mathcal{K}$ given by

$$\psi\big((\mu_A, \nu_A)\big) = (\mu_A, 1_\Omega - \nu_A)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

Proposition 2.1 *If $k : \mathcal{K} \to [0,1]$ is an IV-state and $\mathbf{m} = k \circ \psi : \mathcal{F} \to [0,1]$, then $\mathbf{m}$ is an IF-state.*

Recall that by an **intuitionistic fuzzy state** (IF-state) $\mathbf{m}$ we understand each mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ which satisfies the following conditions (see [11]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \le x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets. Now we start with definition of basic notions (see [5]).

Definition 2.5 *By an interval valued observable (IV-observable) on $\mathcal{K}$ we understand each mapping $z : \mathcal{B}(R) \to \mathcal{K}$ satisfying the following conditions:*

(i) $z(R) = (1_\Omega, 1_\Omega)$, $z(\emptyset) = (0_\Omega, 0_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $z(A)\widehat{\odot}z(B) = (0_\Omega, 0_\Omega)$ and $z(A \cup B) = z(A)\widehat{\oplus}z(B)$;

(iii) if $A_n \nearrow A$, then $z(A_n) \nearrow z(A)$.

Remark 2.6 *If we denote* $z(A) = \big(z^\flat(A), z^\sharp(A)\big)$ *for each* $A \in \mathcal{B}(R)$*, then* $z^\flat, z^\sharp : \mathcal{B}(R) \to \mathcal{T}$ *are observables, where* $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S}-\text{measurable}\}$*.*

Remark 2.7 *Sometimes we need to work with n-dimensional IV-observable* $z : \mathcal{B}(R^n) \to \mathcal{K}$ *defined as a mapping with the following conditions:*

(i) $z(R^n) = (1_\Omega, 1_\Omega)$*,* $z(\emptyset) = (0_\Omega, 0_\Omega)$*;*

(ii) if $A \cap B = \emptyset$*,* $A, B \in \mathcal{B}(R^n)$*, then* $z(A)\widehat{\odot}z(B) = (0_\Omega, 0_\Omega)$ *and* $z(A \cup B) = z(A)\widehat{\oplus}z(B)$*;*

(iii) if $A_n \nearrow A$*, then* $z(A_n) \nearrow z(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$*.*

If $n = 1$ *we simply say that* z *is an IV-observable.*

Between IV-observable and IF-observable is the connection (see [5]).

Proposition 2.2 *Let* $\psi : \mathcal{F} \to \mathcal{K}$*,* $\psi((u,v)) = (u, 1_\Omega - v)$*. If* $x : \mathcal{B}(R) \to \mathcal{F}$ *is an IF-observable and* $z = \psi \circ x : \mathcal{B}(R) \to \mathcal{K}$*, then* z *is an IV-observable.*

Recall that by **intuitionistic fuzzy observable** (IF-observable) on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions (see [11]):

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1 - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - \text{measurable}\}$.

Theorem 2.8 *Let* $z : \mathcal{B}(R) \to \mathcal{K}$ *be an IV-observable,* $k : \mathcal{K} \to [0,1]$ *be an IV-state. Define the mapping* $k_z : \mathcal{B}(R) \to [0,1]$ *by the formula*

$$k_z(C) = k\big(z(C)\big),$$

for each $C \in \mathcal{B}(R)$*. Then* $k_z : \mathcal{B}(R) \to [0,1]$ *is a probability measure. Moreover*

$$k_z(C) = \mathbf{m}_x(C),$$

where $\mathbf{m}_x = \mathbf{m} \circ x$ *is a probability measure induced by IF-state* $\mathbf{m}$ *and IF-observable* x*.*

Since k_z is a probability measure, we call it the probability distribution of IV-observable. Now we can define the notion of distribution function of IV-observable (see [5]).

Definition 2.9 *If* $z : \mathcal{B}(R) \to \mathcal{K}$ *is an IV-observable, and* $k : \mathcal{K} \to [0,1]$ *is an IV-state, then the interval valued distribution function (IV-distribution function) of* z *is the function* $\widehat{F} : R \to [0,1]$ *defined by the formula*

$$\widehat{F}(t) = k\big(z((-\infty, t))\big)$$

for each $t \in R$*.*

Of course the IV-distribution function fulfils the same properties as a classical distribution function (see [5]).

Theorem 2.10 *Let* $\widehat{F} : R \to [0,1]$ *be the IV-distribution function of an IV-observable* $z : \mathcal{B}(R) \to \mathcal{K}$*. Then* $\widehat{F}$ *is non-decreasing on* R*, left continuous in each point* $t \in R$ *and*

$$\lim_{t\to-\infty} \widehat{F}(t) = 0, \ \lim_{t\to\infty} \widehat{F}(t) = 1.$$

Moreover

$$\widehat{F}(t) = \mathbf{F}(t),$$

for each $t \in R$*, where* $\mathbf{F}$ *is an IF-distribution function of an IF-observable* $x : \mathcal{B}(R) \to \mathcal{F}$*.*

Recall that by **intuitionistic fuzzy distribution function** (IF-distribution function) of an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ we understand each function $\mathbf{F} : R \to [0,1]$ defined by the formula

$$\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$$

for each $t \in R$, where $\mathbf{m} : \mathcal{F} \to [0,1]$ is an IF-state.

Now we can define the IV-mean value and IV-dispersion of an IV-observable.

Theorem 2.11 *Let* $\widehat{F} : R \longrightarrow [0,1]$ *be the IV-distribution function of an IV-observable* $z : \mathcal{B}(R) \to \mathcal{K}$*. Then*

$$\begin{aligned} \widehat{E}(z) &= \int_R t \, d\widehat{F}(t), \\ \widehat{D}^2(z) &= \int_R t^2 \, d\widehat{F}(t) - \big(\widehat{E}(z)\big)^2 = \\ &= \int_R (t - \widehat{E}(z))^2 \, d\widehat{F}(t). \end{aligned}$$

3 Product and joint interval valued observable

In the paper we shall work with independent IV-observables. Of course first we must need the existence of the joint IV-observable. For this reason we shall define the product of IV-events ([6]).

Theorem 3.1 *The operation* $\widehat{\cdot}$ *defined by*

$$(\pi_C, \rho_C)\widehat{\cdot}(\pi_D, \rho_D) = (\pi_C \cdot \pi_D, \rho_C \cdot \rho_D)$$

for each $(\pi_C, \rho_C), (\pi_D, \rho_D) \in \mathcal{K}$ *is product operation on* $\mathcal{K}$*.*

Now we explain the connection between product operations on the family of interval valued events $\mathcal{K}$ and the family of intuitionistic fuzzy events $\mathcal{F}$ and we define the joint interval valued observable (see [6]).

Theorem 3.2 *If the operation $\bullet$ is a product on family of intuitionistic events $\mathcal{F}$ defined by*

$$(\mu_A, \nu_A) \bullet (\mu_B, \nu_B) = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B) = \\ = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big)$$

for each $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ and $\widehat{\cdot}$ is a product operation on a family of interval valued events $\mathcal{K}$ defined by

$$(\pi_C, \rho_C)\widehat{\cdot}(\pi_D, \rho_D) = (\pi_C \cdot \pi_D, \rho_C \cdot \rho_D)$$

for each $\mathbf{C} = (\pi_C, \rho_C), \mathbf{D} = (\pi_D, \rho_D) \in \mathcal{K}$ and $\psi : \mathcal{F} \to \mathcal{K}$ is a function given by $\psi\big((u, v)\big) = (u, 1 - v)$, then

$$\psi(\mathbf{A} \bullet \mathbf{B}) = \psi(\mathbf{A})\widehat{\cdot}\psi(\mathbf{B})$$

for each $\mathbf{A}, \mathbf{B} \in \mathcal{F}$.

Definition 3.3 *Let $z_1, z_2 : \mathcal{B}(R) \to \mathcal{K}$ be two IV-observables. The joint interval valued observable (joint IV-observable) of the IV-observables z_1, z_2 is a mapping $\widehat{h} : \mathcal{B}(R^2) \to \mathcal{K}$ satisfying the following conditions:*

(i) $\widehat{h}(R^2) = (1_\Omega, 1_\Omega)$, $\widehat{h}(\emptyset) = (0_\Omega, 0_\Omega)$;

(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $\widehat{h}(A \cup B) = \widehat{h}(A)\widehat{\oplus}\widehat{h}(B)$ *and* $\widehat{h}(A)\widehat{\odot}\widehat{h}(B) = (0_\Omega, 0_\Omega)$;

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $\widehat{h}(A_n) \nearrow \widehat{h}(A)$;

(iv) $\widehat{h}(C \times D) = z_1(C)\widehat{\cdot}z_2(D)$ *for each* $C, D \in \mathcal{B}(R)$.

In the following proposition we show the connection between the joint interval valued observable and the intuitionistic fuzzy observable (see [6]).

Proposition 3.1 *Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. If $h : \mathcal{B}(R^2) \to \mathcal{F}$ is a joint IF-observable of IF-observables $x_1, x_2 : \mathcal{B}(R) \to \mathcal{F}$ and $\widehat{h} = \psi \circ h : \mathcal{B}(R^2) \to \mathcal{K}$, then $\widehat{h}$ is the joint IV-observable of IV-observables $z_1, z_2 : \mathcal{B}(R) \to \mathcal{K}$, where $z_1 = \psi \circ x_1$, $z_2 = \psi \circ x_2$.*

Recall that by **joint intuitionistic fuzzy observable** (joint IF-observable) we understand each mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions (see [8, 11]):

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A, B \in \mathcal{B}(R^2)$ and $A \cap B = \emptyset$, then $h(A \cup B) = h(A) \oplus h(B)$ and $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$;

(iii) if $A, A_1, \ldots \in \mathcal{B}(R^2)$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in \mathcal{B}(R)$.

Theorem 3.4 *To each two IV-observables $z_1, z_2 : \mathcal{B}(\mathbf{R}) \to \mathcal{K}$ there exists their joint IV-observable.*

If we have several IV-observables and a Borel measurable function, we can define the IV-observable, which is the function of several IV-observables. About this says the following definition (see [6]).

Definition 3.5 *Let $z_1, \ldots, z_n : \mathcal{B}(R) \to \mathcal{K}$ be IV-observables, $\widehat{h}_n$ their joint IV-observable and $g_n : R^n \to R$ a Borel measurable function. Then we define the IV-observable $\widehat{y}_n = g_n(z_1, \ldots, z_n) : \mathcal{B}(R) \to \mathcal{K}$ by the formula*

$$\widehat{y}_n = g_n(z_1, \ldots, z_n)(A) = \widehat{h}_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

Example 3.6 *Let $z_1, \ldots, z_n : \mathcal{B}(R) \to \mathcal{K}$ be IV-observables and $\widehat{h}_n : \mathcal{B}(R^n) \to \mathcal{K}$ be their joint IV-observable. Then the IV-observable*

$$\widehat{y}_n = \frac{1}{a_n}\big(\max(z_1, \ldots, z_n) - b_n\big)$$

is defined by the equality

$$\widehat{y}_n = \widehat{h}_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}(\max(u_1, \ldots, u_n) - b_n)$.

Between a function of several IV-observables $\widehat{y}_n = g_n(z_1, \ldots, z_n)$ and a function of several IF-observables $y_n = g_n(x_1, \ldots, x_n)$ exists a connection (see [6]).

Proposition 3.2 *Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. If $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ is a function of several IF-observables $x_1, \ldots, x_n$ and $\widehat{y}_n = \psi \circ y_n : \mathcal{B}(R) \to \mathcal{K}$, then $\widehat{y}_n = g_n(z_1, \ldots, z_n)$ is a function of several IV-observables $z_1, \ldots, z_n$, where $z_i = \psi \circ x_i$, $i = 1, \ldots, n$.*

Recall that by **a function of several intuitionistic fuzzy observables** we understand the IF-observable defined by

$$y_n = g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$, where h_n is a joint IF-oservable of IF-observables $x_1, \ldots, x_n$.

4 Independence and convergence in distribution

In this section we define the notion of independence of interval valued observables.

Definition 4.1 *Let k be an IV-state. The IV-observables $z_1, \ldots, z_n : \mathcal{B}(R) \to \mathcal{K}$ are independent if for n-dimensional IV-observable $\widehat{h}_n : \mathcal{B}(R^n) \to \mathcal{K}$ there holds*

$$k\big(\widehat{h}_n(A_1 \times \ldots \times A_n)\big) = k\big(z_1(A_1)\big) \cdot \ldots \cdot k\big(z_n(A_n)\big)$$

for each $A_1, \ldots, A_n \in \mathcal{B}(R)$.

Now we explain the connection between independence of IV-observables and independence of IF-observables. Recall that **IF-observables** $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ **are independent** if for n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ there holds

$$\mathbf{m}\big(h_n(A_1 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, \ldots, A_n \in \mathcal{B}(R)$, where $\mathbf{m}$ is an IF-state.

Proposition 4.1 *The IV-observables $z_1, \ldots, z_n : \mathcal{B}(R) \to \mathcal{K}$ are independent if and only if the IF-observables $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ are independent. There $z_i = \psi \circ x_i$, $i = 1, \ldots, n$.*

Proof. "$\Rightarrow$" Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. If the IV-observables $z_1, \ldots, z_n$ are independent, then by *Definition 4.1* for n-dimensional IV-observable $\widehat{h}_n$ there holds

$$k\big(\widehat{h}_n(A_1 \times \ldots \times A_n)\big) = k\big(z_1(A_1)\big) \cdot \ldots \cdot k\big(z_n(A_n)\big)$$

for each $A_1, \ldots, A_n \in \mathcal{B}(R)$. But using *Proposition 3.1*, *Proposition 2.2* and *Proposition 2.1* we have

$$\widehat{h}_n = \psi \circ h_n, \ z_i = \psi \circ x_i, \ \mathbf{m} = k \circ \psi$$

where h_n is n-dimensional IF-observable, x_i, $i = 1, \ldots, n$ are IF-observables and $\mathbf{m}$ is IF-state. Therefore

$$\mathbf{m}\big(h_n(A_1 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, \ldots, A_n \in \mathcal{B}(R)$. Hence IF-observables $x_1, \ldots, x_n$ are independent.

The proof of "$\Leftarrow$" is a analogue to the proof of "$\Rightarrow$". □

We need the notion of convergence in distribution of IV-observables yet.

Definition 4.2 *Let $(\widehat{y}_n)_n$ be a sequence of IV-observables and k be a IV-state. We say that $(\widehat{y}_n)_n$ converges in distribution to a function $\Psi : R \to [0, 1]$, if for each $t \in R$*

$$\lim_{n\to\infty} k\big(\widehat{y}_n((-\infty, t))\big) = \Psi(t).$$

5 Extreme value theory for interval valued observables

The extreme value theory is a part of statistics, which deals with examination of probability of extreme and rare events with a large impact. The extreme value theory search endpoints of the distributions. In this Section we show the modification of Fisher-Tippet-Gnedenko theorem and the modification of Pickands-Balkema-de Hann theorem for interval valued observables.

Let $z_1, z_2, \ldots$ be an independent, equally distributed IV-observables on $\mathcal{K}$. Denote $\widehat{M}_n$ maximum of n IV-observables

$$\widehat{M}_1 = z_1, \ \widehat{M}_n = \max(z_1, \ldots, z_n),$$

for $n \geq 2$.

Theorem 5.1 (Fisher-Tippett-Gnedenko) *Let $z_1, z_2, \ldots$ be a sequence of independent, equally distributed IV-observables such that $\widehat{D}^2(z_n) = \sigma^2$, $\widehat{E}(z_n) = a$, $(n = 1, 2, \ldots)$. If there exists the sequences of real constant $a_n > 0$, b_n and a non-degenerate distribution function H, such that for $\widehat{y}_n = \frac{1}{a_n}\big(\widehat{M}_n - b_n\big)$ holds*

$$\lim_{n\to\infty} k\big(\widehat{y}_n((-\infty, t))\big) = H(t),$$

then H is the distribution function one of the following three types of distributions:

1. *Gumbel*

$$H_{\mu,\sigma}(t) = \exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right), \ \ t \in R,$$

2. *Fréchet*

$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} 0, & \text{for } t \leq \mu \\ \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^{-\alpha}\right), & \text{for } t > \mu, \alpha > 0 \end{cases}$$

3. *Weibull*

$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} \exp\left(-\left(-\frac{t-\mu}{\sigma}\right)^{\alpha}\right), & \text{for } t \leq \mu, \alpha > 0 \\ 1, & \text{for } t > \mu. \end{cases}$$

There a parameter $\mu \in R$ is the location parameter and a parameter $\sigma > 0$ is the scale parameter.

Proof. Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. Let $z_1, z_2, \ldots$ be a sequence of independent, equally distributed IV-observables such that $\widehat{D}^2(z_n) = \sigma^2$, $\widehat{E}(z_n) = a$, $(n = 1, 2, \ldots)$. Then by *Proposition 4.1* $x_1, x_2, \ldots$ is a sequence of independent IF-observables, where $x_n = \psi^{-1} \circ z_n$ (i.e. $z_n = \psi \circ x_n$), $n = 1, \ldots$. Moreover using *Theorem 2.10* we have

$$\begin{aligned} a &= \widehat{E}(z_n) = \int_R t \; d\widehat{F}(t) = \int_R t \; d\mathbf{F}(t) = \mathbf{E}(x_n), \\ \sigma^2 &= \widehat{D}^2(z_n) = \int_R (t - \widehat{E}(z_n))^2 \; d\widehat{F}(t) = \\ &= \int_R (t - \mathbf{E}(x_n))^2 \; d\mathbf{F}(t) = \mathbf{D}^2(x_n). \end{aligned}$$

Hence by the Fisher-Tippett-Gnedenko theorem for IF-case (see *Theorem 7* in [3]) there exists the sequences of real constant $a_n > 0$, b_n and a non-degenerate distribution function H, such that

$$\lim_{n\to\infty} \mathbf{m}\Big(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\Big) = H(t), \qquad (5)$$

where $\mathbf{M}_n$ is a maximum of n IF-observables $x_1, \ldots, x_n$ given by

$$\mathbf{M}_1 = x_1, \; \mathbf{M}_n = \max(x_1, ..., x_n),$$

for $n \geq 2$. Put

$$\begin{aligned} \widehat{y}_n &= \frac{1}{a_n}\big(\widehat{M}_n - b_n\big) = \widehat{h}_n \circ g_n^{-1}, \\ y_n &= \frac{1}{a_n}\big(\mathbf{M}_n - b_n\big) = h_n \circ g_n^{-1}, \end{aligned}$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}(\max(u_1, \ldots, u_n) - b_n)$, $\widehat{h}_n$ is joint IV-observable of IV-observables $z_1, \ldots, z_n$ and h_n is joint IF-observable of IF-observables $x_1, \ldots, x_n$.

Therefore using *Proposition 3.2*, *Proposition 2.1* and (5) we obtain

$$\begin{aligned} &\lim_{n\to\infty} k\big(\widehat{y}_n((-\infty, t))\big) = \lim_{n\to\infty} k \circ \psi \circ y_n((-\infty, t)) = \\ &= \lim_{n\to\infty} \mathbf{m} \circ y_n((-\infty, t)) = \\ &= \lim_{n\to\infty} \mathbf{m}\Big(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\Big) = H(t). \end{aligned}$$

□

Gumbel, Frechet and Weibull distribution from *Theorem 5.1* can be described with using a **generalized distribution of extreme values - GEV**:

$$H_{\mu,\sigma,\varepsilon}(t) = \begin{cases} \exp\left[-\left(1 + \varepsilon\left(\frac{t-\mu}{\sigma}\right)\right)^{-\frac{1}{\varepsilon}}\right], \varepsilon \neq 0, \\ \qquad\qquad 1 + \varepsilon\left(\frac{t-\mu}{\sigma}\right) > 0, \\ \exp\left(-\exp\left(-\frac{t-\mu}{\sigma}\right)\right), \; t \in R, \varepsilon = 0. \end{cases}$$

A parameter ε is called the *shape parameter*.

The Fisher-Tippet-Gnedenko theorem says about convergence in distribution of maximums of independent, equally distributed IV-observables. An alternative to the maximal observation method is the method that models all observations that exceed any predefined boundary (ie. threshold).

Such the extremes occur "near" the upper end of distribution support, hence intuitively asymptotic behavior of $\widehat{M}_n$ must be related to the distribution function $\widehat{F}$ in its right tail near the right endpoint.

Let z be an IV-observable on $\mathcal{K}$ and $\widehat{F}$ be an IV-distribution function of z. We denote by

$$t_{\widehat{F}} = \sup\{t \in R : \widehat{F}(t) < 1\}$$

the right endpoint of IV-distribution function $\widehat{F}$.

Definition 5.2 (Maximum domain of attraction for IV-case) *We say that the IV-distribution function $\widehat{F}$ of IV-observable z belongs to the maximum domain of attraction of the extreme value distributions H if there exists constants $a_n > 0$, $b_n \in R$ such that*

$$\lim_{n\to\infty} k\Big(\frac{1}{a_n}\big(\widehat{M}_n - b_n\big)\big((-\infty, t)\big)\Big) = H(t),$$

holds. We write $\widehat{F} \in \widehat{MDA}(H)$.

Definition 5.3 (Excess interval valued distribution function) *Let $\widehat{F}$ be an interval valued distribution function with right endpoint $t_{\widehat{F}}$. For fixed $u < t_{\widehat{F}}$, $u > 0$,*

$$\widehat{F}_u(t) = \frac{\widehat{F}(t+u) - \widehat{F}(u)}{1 - \widehat{F}(u)}, \quad 0 \leq t \leq t_{\widehat{F}} - u$$

is the excess interval valued distribution function of the interval valued observable z (of the IV-distribution function $\widehat{F}$) over the threshold u.

Definition 5.4 (Generalized Pareto distribution - GPD) *Define the distribution function $G_{\varepsilon,\beta}$ by*

$$G_{\varepsilon,\beta}(t) = \begin{cases} 1 - \left(1 + \varepsilon \cdot \frac{t}{\beta}\right)^{-\frac{1}{\varepsilon}}, & \text{if } \varepsilon \neq 0, \\ 1 - e^{-\frac{t}{\beta}}, & \text{if } \varepsilon = 0, \end{cases}$$

where

$$\begin{aligned} t &\geq 0 && \text{if } \varepsilon \geq 0, \\ 0 \leq \; t &\leq -\frac{\beta}{\varepsilon} && \text{if } \varepsilon < 0 \end{aligned}$$

and $\beta > 0$ is the scale parameter. $G_{\varepsilon,\beta}$ is called the generalised Pareto distribution. We can extend the family by adding a location parameter $\nu \in R$. Then we get the function $G_{\varepsilon,\nu,\beta}$ by replacing the argument t above by $t - \nu$ in $G_{\varepsilon,\beta}$. The support has to be adjusted accordingly.

Remark 5.5 *The GPD transforms into a number of other distributions depending on the value of ε. When $\varepsilon > 0$, it takes the form of the ordinary Pareto distribution. This case would be most relevant for financial time series data as it has a heavy tail. If $\varepsilon = 0$, the GPD corresponds to exponential distribution, and it is called a short-tailed, Pareto II type distribution for $\varepsilon < 0$.*

Theorem 5.6 (Pickands-Balkema-de Haan) *For every $\varepsilon \in R$,*

$$\widehat{F} \in \widehat{MDA}(H_\varepsilon) \Leftrightarrow$$

$$\Leftrightarrow \lim_{u \to t_{\widehat{F}}} \sup_{0<t<t_{\widehat{F}}-u} |\widehat{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0$$

for some positive function β.

Proof. Let $\psi : \mathcal{F} \to \mathcal{K}$, $\psi((u, v)) = (u, 1_\Omega - v)$. Let k be an IV-state and $(z_n)_n$ be a sequence of independent IV-observables in $\mathcal{K}$ with the same IV-distribution $\widehat{F}$. Then by *Proposition 4.1* $x_1, x_2, \ldots$ is a sequence of independent IF-observables, where $x_n = \psi^{-1} \circ z_n$ (i.e. $z_n = \psi \circ x_n$), $n = 1, \ldots$. Moreover by *Theorem 2.10* we have $\widehat{F} = \mathbf{F}$. Hence $t_{\widehat{F}} = t_{\mathbf{F}}$, $\widehat{F}_u = \mathbf{F}_u$ and

$$\lim_{u \to t_{\widehat{F}}} \sup_{0<t<t_{\widehat{F}}-u} |\widehat{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| =$$
$$= \lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} |\mathbf{F}_u(t) - G_{\varepsilon,\beta(u)}(t)|$$

for some positive function β. There $\mathbf{F}$ is an IF-distribution function of IF-observables $x_1, x_2, \ldots$. Put

$$\widehat{y}_n = \frac{1}{a_n}\big(\max(z_1, \ldots, z_n) - b_n\big) = \widehat{h}_n \circ g_n^{-1},$$
$$y_n = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big) = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}(\max(u_1, \ldots, u_n) - b_n)$, $\widehat{h}_n$ is joint IV-observable of IV-observables $z_1, \ldots, z_n$ and h_n is joint IF-observable of IF-observables $x_1, \ldots, x_n$. Therefore using *Proposition 3.2* and *Proposition 2.1* we obtain

$$k\bigg(\frac{1}{a_n}\big(\widehat{M}_n - b_n\big)\big((-\infty, t)\big)\bigg) = k\big(\widehat{y}_n((-\infty, t))\big) =$$
$$= k \circ \psi \circ y_n((-\infty, t)) = \mathbf{m} \circ y_n((-\infty, t)) =$$
$$= \mathbf{m}\bigg(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\bigg).$$

Thus we have for every $\varepsilon \in R$,

$$\widehat{F} \in \widehat{MDA}(H_\varepsilon) \Leftrightarrow \mathbf{F} \in \mathbf{MDA}(H_\varepsilon).$$

Recall that $\mathbf{F} \in \mathbf{MDA}(H)$ if there exists constants $a_n > 0$, $b_n \in R$ such that

$$\lim_{n \to \infty} \mathbf{m}\bigg(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\bigg) = H(t),$$

holds (see *Definition 5.2* in [4]). Finally from the Pickands-Balkema-de Haan theorem for IF-case (see *Theorem 5.7* in [4]) we obtain

$$\mathbf{F} \in \mathbf{MDA}(H_\varepsilon) \Leftrightarrow$$

$$\Leftrightarrow \lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} |\mathbf{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0.$$

Therefore for every $\varepsilon \in R$,

$$\widehat{F} \in \widehat{MDA}(H_\varepsilon) \Leftrightarrow$$

$$\Leftrightarrow \lim_{u \to t_{\widehat{F}}} \sup_{0<t<t_{\widehat{F}}-u} |\widehat{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0$$

for some positive function β. □

Remark 5.7 *Theorem 5.6 say that for some function β to be estimated from the data, the excess interval valued distribution $\widehat{F}_u$ converges to the generalized Pareto distribution $G_{\varepsilon,\beta}$ for large u.*

Remark 5.8 *The Generalized distribution of extreme values describes the limit distribution of normalized maxima.*
The Generalized Pareto distribution appears as the limit distribution of scaled excesses over high thresholds.

6 Conclusion

The classical theory of extreme values is based on the practical needs of astronomers, hydrologists and technicians. It is used in rainfall modelling, air pollution modelling, flood forecasting and so on. In recent years it is used in the economic field to model risk in the financial sector.

In some cases we need to consider the imprecise values. Therefore we can used the interval valued sets. In this paper we have proved a very important assertion of mathematical statistics for interval valued events.

The results can be applied in areas like catastrophe bonds and insurance or a malfunctions of the pipeline (see [10, 12, 13]).

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