Solution of Fuzzy Differential Equations with generalized differentiability using LU-parametric representation

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Abstract— *The paper uses the LU-parametric representation of fuzzy numbers and fuzzy-valued functions, to obtain valid approximations of fuzzy generalized derivative and to solve fuzzy differential equations. The main result is that a fuzzy differential initial-value problem can be translated into a system of infinitely many ordinary differential equations and, by the LU-parametric representation, the infinitely many equations can be approximated efficiently by a finite set of four ODEs. Some examples are included.*

Keywords— Parametric Fuzzy Partition, Fuzzy Transform, Smoothing Functions, Fuzzy Numbers

1 Introduction

The present paper aims to combine results of [1] and [2] with the parametric representation in [6] to obtain a new approximation results for fuzzy differential equations (FDE) with parametric representation. These results are used for numerically solving FDEs under the parametric representation. The problem is that a fuzzy initial value problem in general can be translated to a system consisting of infinitely many ODEs. However the systems are uncoupled, we cannot solve infinitely many equations. This is why we have to approximate the solutions of FDEs.

Some examples are presented.

We will denote $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers, i.e. normal, fuzzy convex, upper semicontinuous and compactly supported fuzzy sets defined over the real line. The standard Hukuhara difference (H-difference $\ominus_H$) is defined by $u \ominus_H v = w \iff u = v + w$, while the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \iff \begin{cases} & (i) \quad u = v + w \\ or & (ii) \quad v = u - w \end{cases}. \quad (1)$$

If $w = u \ominus_{gH} v$ exists as a fuzzy number, its level cuts $[w_\alpha^-, w_\alpha^+]$ are obtained by $w_\alpha^- = \min\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}$ and $w_\alpha^+ = \max\{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}$ for all $\alpha \in [0,1]$.

2 Fuzzy differentiability

The following fuzzy differentiability concepts used are of generalized type. The first concept was presented in [1] for the fuzzy case and the second in [5].

Definition 1: ([1]) Let $f :]a,b[\to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in]a,b[$. We say that f is strongly generalized Hukuhara differentiable at x_0 (GH-differentiable for short) if there exists an element $f'_{GH}(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that, for all $h > 0$ sufficiently small,

(i) $\exists f(x_0+h) \ominus_H f(x_0)$, $f(x_0) \ominus_H f(x_0-h)$ and

$$\lim_{h \searrow 0} \frac{f(x_0+h) \ominus_H f(x_0)}{h} = \quad (2)$$
$$= \lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0-h)}{h} = f'_{GH}(x_0),$$

or (ii) $\exists f(x_0) \ominus_H f(x_0+h)$, $f(x_0-h) \ominus_H f(x_0)$ and

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0+h)}{(-h)} = \quad (3)$$
$$= \lim_{h \searrow 0} \frac{f(x_0-h) \ominus_H f(x_0)}{(-h)} = f'_{GH}(x_0),$$

or (iii) $\exists f(x_0+h) \ominus_H f(x_0)$, $f(x_0-h) \ominus_H f(x_0)$ and

$$\lim_{h \searrow 0} \frac{f(x_0+h) \ominus_H f(x_0)}{h} = \quad (4)$$
$$= \lim_{h \searrow 0} \frac{f(x_0-h) \ominus_H f(x_0)}{(-h)} = f'_{GH}(x_0),$$

or (iv) $\exists f(x_0) \ominus_H f(x_0+h)$, $f(x_0) \ominus_H f(x_0-h)$ and

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0+h)}{(-h)} = \quad (5)$$
$$= \lim_{h \searrow 0} \frac{f(x_0) \ominus_H f(x_0-h)}{h} = f'_{GH}(x_0).$$

Based on the gH-difference we obtain the following definition (for interval-valued functions, the same definition was suggested in [4] using *inner*-difference):

Definition 2: Let $x_0 \in]a,b[$ and h be such that $x_0 + h \in]a,b[$, then the gH-derivative of a function $f :]a,b[\to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h}[f(x_0+h) \ominus_{gH} f(x_0)]. \quad (6)$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (6) exists, we say that f is *generalized Hukuhara differentiable* (*gH-differentiable* for short) at x_0. Define

$$\Delta_h f(x_0) = \frac{1}{h}[f(x_0+h) \ominus_{gH} f(x_0)].$$

According to the definition of gH-difference (1), we have two possibilities:

$$(i) \qquad h\Delta_h f(x_0) + f(x_0) = f(x_0+h) \quad (7)$$

$$(ii) \qquad f(x_0+h) - h\Delta_h f(x_0) = f(x_0). \qquad (8)$$

If the limit in (6) exists, we say that f is *(i)-gH*-differentiable at x_0 (or *(ii)-gH*-differentiable at x_0) if (i) is true (or (ii) is true, respectively) for small positive and negative values of h. If only the *left-limit* (i.e., for $h \longrightarrow 0^-$, $h < 0$) in (6) exists, we say that f is *(i)-left-gH*-differentiable (or *(ii)-left-gH*-differentiable, respectively) at x_0 if equality (7) is valid (or (8) is valid, respectively) for small negative values of h; analogously, we define *(i)-right-gH*-differentiability and *(ii)-right-gH*-differentiability. The $\langle {(i) \atop (ii)} \rangle$-$\langle {left \atop right} \rangle$-gH derivatives are defined accordingly.

3 LU-parametric representation

The Lower-Upper (LU) representation of a fuzzy number is a result based on the well known Negoita-Ralescu representation theorem, stating essentially that the membership form and the α-cut form of a fuzzy number u are equivalent and in particular, the α-cuts $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ uniquely represent u, provided that the two functions $\alpha \longrightarrow u_\alpha^-$ and $\alpha \longrightarrow u_\alpha^+$, w.r.t. α, are left continuous for all $\alpha \in]0,1]$, right continuous for $\alpha = 0$, monotonic (u_α^- increasing, u_α^+ decreasing) and $u_1^- \leq u_1^+$ (for $\alpha = 1$).

On the other hand, it is well known that monotonic functions have at most a countable number of points of discontinuity and a countable number of points where the derivative does not exist.

Denote the corresponding points by the increasing sequence $(\alpha_j)_{j \in J}$ with $0 < \alpha_j \leq 1$ and $J = \emptyset$ (empty set) or $J = \{1, 2..., p\}$ (finite set) or $J = \mathbb{N}$ (set of natural numbers).

Then the two functions u_α^-, u_α^+ are differentiable internally to each of the subintervals $]\alpha_{j-1}, \alpha_j[$ i.e., they are formed by a family of differentiable monotonic "pieces" (their restrictions to each subinterval are monotonic and differentiable).

For simplicity, in this presentation we will assume that $J = \emptyset$ (empty set); otherwise, we can repeat the following results on each of the subintervals.

So, u is assumed to be a fuzzy number with α-cuts $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $\alpha \longrightarrow u_\alpha^-$, $\alpha \longrightarrow u_\alpha^+$ monotonic and differentiable w.r.t. α.

For $\alpha \in [0,1]$, let δu_α^- and δu_α^+ denote the first derivatives of u_α^- and u_α^+ w.r.t. α (for $\alpha = 0$ they are right derivatives, for $\alpha = 1$ they are left derivatives).

The following lemma is immediate.

Lemma 1: The two differentiable functions u_α^-, u_α^+ define a fuzzy number if and only if for all $\alpha \in [0,1]$ we have:

$$\begin{cases} u_\alpha^- \leq u_\alpha^+ \\ \delta u_\alpha^- \geq 0 \\ \delta u_\alpha^+ \leq 0 \end{cases} \text{OR} \quad \begin{cases} u_\alpha^- \geq u_\alpha^+ \\ \delta u_\alpha^- \leq 0 \\ \delta u_\alpha^+ \geq 0 \end{cases}.$$

The lemma above is useful to characterize the gH-differentiability of a fuzzy-valued function $f :]a,b[\rightarrow \mathbb{R}_\mathcal{F}$ defined in terms of its α-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$.

Based on the results established in [5], when both $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable w.r.t. x for all α's, then the α-cuts of the gH-derivative of f are

$$\begin{aligned} f'_{gH}(x) \;\; &= \;\; [\min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}, \\ &\qquad \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}] \end{aligned}$$

provided that the two functions $(f'_{gH}(x))_\alpha^- = \min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}$ and $(f'_{gH}(x))_\alpha^+ = \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)$ define (w.r.t. α) a fuzzy number; here, $(f_\alpha^-)', (f_\alpha^+)'$ denote the derivatives w.r.t. x, for given $\alpha \in [0,1]$.

As $f_\alpha^-(x)$ and $f_\alpha^+(x)$ define the α-cuts of the fuzzy number $f(x)$ for each x, clearly they are monotonic and almost everywhere differentiable w.r.t. α and satisfy the conditions of Lemma 1. Assume, for simplicity of presentation, that each function $\alpha \longrightarrow f_\alpha^-(x)$ and $\alpha \longrightarrow f_\alpha^+(x)$ is differentiable w.r.t. α.

Notation: We will use the following notations: $\delta f_\alpha^-(x) = \frac{\partial}{\partial\alpha} f_\alpha^-(x)$, $\delta f_\alpha^+(x) = \frac{\partial}{\partial\alpha} f_\alpha^+(x)$, $(f_\alpha^-)'(x) = \frac{\partial}{\partial x} f_\alpha^-(x)$, $(f_\alpha^+)'(x) = \frac{\partial}{\partial x} f_\alpha^-(x)$, and, for short, given a fuzzy valued function $f(x)$, we will denote by $\delta f(x)$ the pairs of functions $(\delta f_\alpha^-(x), \delta f_\alpha^+(x))_{\alpha \in [0,1]}$; we will assume that the following equalities hold for the mixed derivatives:

$$\begin{aligned} (\delta f_\alpha^-)'(x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial\alpha} f_\alpha^-(x)\right) && (9) \\ &= \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x} f_\alpha^-(x)\right) = \delta\left((f_\alpha^-)'(x)\right) \\ (\delta f_\alpha^+)'(x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial\alpha} f_\alpha^+(x)\right) && (10) \\ &= \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x} f_\alpha^+(x)\right) = \delta\left((f_\alpha^+)'(x)\right). \end{aligned}$$

The following theorem can be proved.

Theorem 1: Let $f :]a,b[\rightarrow \mathbb{R}_\mathcal{F}$ be defined in terms of its α-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$ satisfying conditions (9)-(10). Then

1. f is (i)-gH-differentiable at x if and only if:

$$\begin{cases} (f_1^-)'(x) \leq (f_1^+)'(x) \\ \quad (\delta f_\alpha^-)'(x) \geq 0 \qquad \text{for all } \alpha \in [0,1], ; \\ \quad (\delta f_\alpha^+)'(x) \leq 0 \end{cases} \qquad (11)$$

2. f is (ii)-gH-differentiable at x if and only if:

$$\begin{cases} (f_1^-)'(x) \geq (f_1^+)'(x) \\ \quad (\delta f_\alpha^-)'(x) \leq 0 \qquad \text{for all } \alpha \in [0,1], . \\ \quad (\delta f_\alpha^+)'(x) \geq 0 \end{cases} \qquad (12)$$

Definition 3: A *switching point* $x_0 \in]a,b[$ is such that gH-differentiability changes from type (i) to type (ii) of from type (ii) to type (i). A *switching interval* $]x_0, x_1[$ is such that $x_0 < x_1$ and f is gH-differentiable on its left and on it right but not for all $x \in]x_0, x_1[$.

Clearly, by the use of Theorem 2 below, it is easy to find switching points or the starting point of switching intervals.

Theorem 2: Let $f :]a,b[\rightarrow \mathbb{R}_\mathcal{F}$ be defined in terms of its α-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$ satisfying conditions (9)-(10); let f be gH-differentiable; let $x_0 \in]a,b[$ be given

and let $\xi > 0$ sufficiently small such that $x_0 - \xi \in]a, b[$ and $x_0 + \xi \in]a, b[$.
1) if conditions (11) hold for $x_0 - \xi < x < x_0$ and not for $x = x_0$ then x_0 is a switching point or the left point of a switching interval.
2) if conditions (12) hold for $x_0 - \xi < x < x_0$ and not for $x = x_0$ then x_0 is a switching point or the left point of a switching interval.
3) if conditions (11) hold for $x_0 < x < x_0 + \xi$ and not for $x = x_0$ then x_0 is a switching point or the right point of a switching interval.
4) if conditions (12) hold for $x_0 < x < x_0 + \xi$ and not for $x = x_0$ then x_0 is a switching point or the right point of a switching interval.

Example 1: Consider the fuzzy valued function $f : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ defined by

$$\begin{aligned} f_\alpha^-(x) &= xe^{-x} + \alpha^2(e^{-x^2} + x - xe^{-x}) \\ f_\alpha^+(x) &= e^{-x^2} + x + (1 - \alpha^2)(e^x - x + e^{-x^2}) \end{aligned}$$

We have the following derivatives w.r.t. x of $f_\alpha^-(x)$ and $f_\alpha^+(x)$

$$\begin{aligned} (f_\alpha^-)'(x) &= (1 - x)e^{-x} + \\ &\quad +\alpha^2(1 - 2xe^{-x^2} - e^{-x} + xe^{-x}) \\ (f_\alpha^+)'(x) &= 1 - 2xe^{-x^2} + \\ &\quad +(1 - \alpha^2)(e^x - 1 - 2xe^{-x^2}) \end{aligned}$$

and the following derivatives w.r.t. α

$$\begin{aligned} \delta f_\alpha^-(x) &= 2\alpha(e^{-x^2} + x - xe^{-x}) \\ \delta f_\alpha^+(x) &= -2\alpha(e^x - x + e^{-x^2}). \end{aligned}$$

It follows that

$$\begin{aligned} (\delta f_\alpha^-)'(x) &= 2\alpha(1 - 2xe^{-x^2} - e^{-x} + xe^{-x}) \\ (\delta f_\alpha^+)'(x) &= -2\alpha(e^x - 1 - 2xe^{-x^2}). \end{aligned}$$

At the point $x = 0$, f is (ii)-right-gH-differentiable and it is (ii)-gH-differentiable on the interval $]0, x_1[$ where $x_1 = 0.6103336260$ is such that $(f_1^-)'(x_1) = (f_0^+)'(x_1)$; on the interval $[x_1, x_2]$, where $x_2 = 0.7105062170$ is such that $(f_1^+)'(x_2) = (f_0^-)'(x_2)$, the function is not gH-differentiable; f returns to be (i)-gH-differentiable on the interval $]x_2, 1[$; finally, at $x = 1$ f is (i)-left-gH-differentiable. The interval $[x_1, x_2]$ is a switching interval. Figure 1 shows f and f'_{gH}.

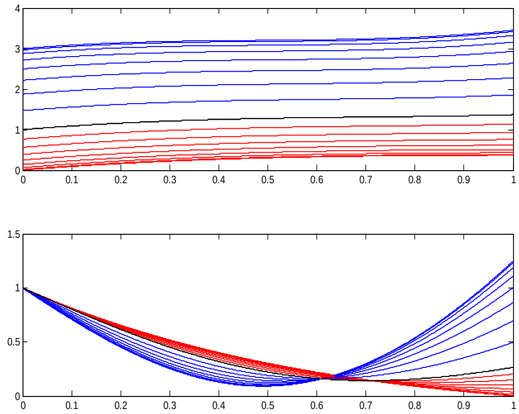

Figure 1. $f(x)$ (top) and $f'_{gH}(x)$ (bottom) for function in example 1.

In the light of using the results above in the setting of fuzzy differential equations, the LU-parametric representation of fuzzy numbers proposed in [6], is shown to have a great application potential. We will recall here its basic elements.

First, we choose a family of "standardized" differentiable and increasing shape functions $p : [0, 1] \longrightarrow [0, 1]$, depending on two parameters $\beta_0, \beta_1 \geq 0$ such that

1. $p(0) = 0, p(1) = 1$,
2. $p'(0) = \beta_0, p'(1) = \beta_1$ and
3. $p(t)$ is increasing on $[0, 1]$ if and only if $\beta_0, \beta_1 \geq 0$.

As an example of valid shape functions we can consider rational splines

$$p(t; \beta_0, \beta_1) = \frac{t^2 + \beta_0 t(1 - t)}{1 + (\beta_0 + \beta_1 - 2)t(1 - t)}.$$

Then, the parametric shape functions above are adopted to represent the functions $u_{(.)}^-$ and $u_{(.)}^+$ as "piecewise" differentiable, on a decomposition of the interval $[0, 1]$ into N subintervals $0 = \alpha_0 < \alpha_1 < ... < \alpha_{i-1} < \alpha_i < ... < \alpha_N = 1$. At the extremal points of each subinterval $I_i = [\alpha_{i-1}, \alpha_i]$, the values and the first derivatives (slopes) of the two functions are given

$$\begin{aligned} u^-_{(\alpha_{i-1})} = u^-_{0,i} \ , \ u^+_{(\alpha_{i-1})} = u^+_{0,i} \ , \quad (13) \\ u^-_{(\alpha_i)} = u^-_{1,i} \ , \ u^+_{(\alpha_i)} = u^+_{1,i} \\ u'^-_{(\alpha_{i-1})} = d^-_{0,i} \ , \ u'^+_{(\alpha_{i-1})} = d^+_{0,i} \ , \quad (14) \\ u'^-_{(\alpha_i)} = d^-_{1,i} \ , \ u'^+_{(\alpha_i)} = d^+_{1,i} \end{aligned}$$

and by the transformation $t_\alpha = \frac{\alpha - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}}$, $\alpha \in I_i$, each subinterval I_i is mapped into $[0, 1]$ to determine each piece independently. Globally continuous or more regular $C^{(1)}$ fuzzy numbers can be obtained directly from the values of the parameters.

Let $p_i^{\pm}(t)$ denote the model functions on I_i; we obtain, for example, $p_i^-(t) = p(t; \beta^-_{0,i}, \beta^-_{1,i})$, $p_i^+(t) = p(t; \beta^+_{0,i}, \beta^+_{1,i})$ with $\beta^-_{j,i} = \frac{\alpha_i - \alpha_{i-1}}{u^-_{1,i} - u^-_{0,i}} d^-_{j,i}$ and $\beta^+_{j,i} = -\frac{\alpha_i - \alpha_{i-1}}{u^+_{1,i} - u^+_{0,i}} d^+_{j,i}$ for $j = 0, 1$ so that, for $\alpha \in [\alpha_{i-1}, \alpha_i]$ and $i = 1, 2., , , N$:

$$u^-_\alpha = u^-_{0,i} + (u^-_{1,i} - u^-_{0,i}) p_i^- \left(t_\alpha; \beta^-_{0,i}, \beta^-_{1,i}\right) \quad (15)$$

$$u^+_\alpha = u^+_{0,i} + (u^+_{1,i} - u^+_{0,i}) p_i^+ \left(t_\alpha; \beta^+_{0,i}, \beta^+_{1,i}\right). \quad (16)$$

A fuzzy number with differentiable lower and upper functions is obtained by taking the values and the slopes in appropriate way, i.e. $u^-_{1,i} = u^-_{0,i+1} =: u^-_i$, $u^+_{1,i} = u^+_{0,i+1} =: u^+_i$ and, for the slopes, $d^-_{1,i} = d^-_{0,i+1} =: \delta u^-_i$, $d^+_{1,i} = d^+_{0,i+1} =: \delta u^+_i$. This requires $4(N+1)$ parameters, being $N \geq 1$,

$$u = (\alpha_i; u^-_i, \delta u^-_i, u^+_i, \delta u^+_i)_{i=0,1,...,N} \text{ with} \tag{17}$$

$$u^-_0 \leq u^-_1 \leq ... \leq u^-_N \leq u^+_N \leq u^+_{N-1} \leq ... \leq u^+_0 \tag{18}$$

$$\delta u^-_i \geq 0, \delta u^+_i \leq 0, i = 0,1,...,N \tag{19}$$

and the branches are computed according to (15)-(16).

The gH-difference $w = u \ominus_{gH} v$ has the following LU-parametric form:

$$\begin{aligned}
u \ominus_{gH} v &= (\alpha_i; w^-_i, \delta w^-_i, w^+_i, \delta w^+_i)_{i=0,1,...,N} \text{ with} \\
w^-_i &= \min\{u^-_i - v^-_i, u^+_i - v^+_i\}, \\
0 \leq \delta w^-_i &= \begin{cases} \delta u^-_i - \delta v^-_i & \text{if } w^-_i = u^-_i - v^-_i \\ \delta u^+_i - \delta v^+_i & \text{if } w^-_i = u^+_i - v^+_i \end{cases} \\
w^+_i &= \max\{u^-_i - v^-_i, u^+_i - v^+_i\} \\
0 \geq \delta w^+_i &= \begin{cases} \delta u^-_i - \delta v^-_i & \text{if } w^+_i = u^-_i - v^-_i \\ \delta u^+_i - \delta v^+_i & \text{if } w^+_i = u^+_i - v^+_i \end{cases} .
\end{aligned}$$

4 FDEs with LU-parametric approximation

Fuzzy differential equations can be considered now with different concepts of differentiability. Using the H-derivative, we can consider the classical Hukuhara differential equation

$$x'_H = f(t,x),\ x(t_0) = x_0; \tag{20}$$

with the GH derivative we can consider the problem

$$x'_{GH} = f(t,x),\ x(t_0) = x_0; \tag{21}$$

with the gH-derivative we have the equation

$$x'_{gH} = f(t,x),\ x(t_0) = x_0. \tag{22}$$

Everywhere $f :]a,b[\times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ is a given function defined in terms of level cuts $[f(t,x)]_\alpha = [f^-_\alpha(t, x^-_\alpha, x^+_\alpha), f^+_\alpha(t, x^-_\alpha, x^+_\alpha)]$, $[x(t)]_\alpha = [x^-_\alpha(t), x^+_\alpha(t)]$.

Assumption A: We fix a decomposition $0 = \alpha_0 < \alpha_1 < ... < \alpha_{i-1} < \alpha_i < ... < \alpha_N = 1$ into N subintervals and we obtain an approximation of $x(t)$ and $f(t,x)$ in the form (17); simply, the functions $x^-_i(t)$, $x^+_i(t)$, $\delta x^-_i(t)$, $\delta x^+_i(t)$ and $f^-_i(t, x^-_i, x^+_i)$, $f^+_i(t, x^-_i, x^+_i)$, $\delta f^-_i(t, x^-_i, x^+_i)$, $\delta f^+_i(t, x^-_i, x^+_i)$ are computed using (9)-(10) from the level cuts $[x(t)]_\alpha = [x^-_\alpha(t), x^+_\alpha(t)]$ and $[f(t,x)]_\alpha = [f^-_\alpha(t, x^-_\alpha, x^+_\alpha), f^+_\alpha(t, x^-_\alpha, x^+_\alpha)]$, and for the required values of $\alpha = \alpha_i$, $i = 0, ..., N$.

The choice among (20), (21) or (22) of the fuzzy derivative $x'(t)$ influences the equation. As we have seen, the derivative $x'(t)$ has level cuts $[x'(t)]_\alpha = [(x^-_\alpha)'(t), (x^+_\alpha)'(t)]$ or $[x'(t)]_\alpha = [(x^+_\alpha)'(t), (x^-_\alpha)'(t)]$ when GH or gH differentiability is considered. We analyze the problem (22) at the moment. Following Bede and Gal ([1], Lemma 20), when $f(t,x)$ is continuous, the differential equation (22) is equivalent to the following integral equation

$$x(t) \ominus_{gH} x_0 = \int_{t_0}^{t} f(s, x(s)) ds. \tag{23}$$

If we write (23) in terms of level cuts, we obtain

$$\text{case (i): } \begin{cases} x^-_\alpha(t) = (x_0)^-_\alpha + \phi^-_\alpha(t, x^-_\alpha, x^+_\alpha) \\ x^+_\alpha(t) = (x_0)^+_\alpha + \phi^+_\alpha(t, x^-_\alpha, x^+_\alpha) \end{cases}$$

$$\text{case (ii): } \begin{cases} x^-_\alpha(t) = (x_0)^-_\alpha + \phi^+_\alpha(t, x^-_\alpha, x^+_\alpha) \\ x^+_\alpha(t) = (x_0)^+_\alpha + \phi^-_\alpha(t, x^-_\alpha, x^+_\alpha) \end{cases}$$

where $\phi^-_\alpha(t, x^-_\alpha, x^+_\alpha) = \int_{t_0}^{t} f^-_\alpha(s, x^-_\alpha, x^+_\alpha) ds$ and $\phi^+_\alpha(t, x^-_\alpha, x^+_\alpha) = \int_{t_0}^{t} f^+_\alpha(s, x^-_\alpha, x^+_\alpha) ds$. In both cases, the two integral equations above are independent for different values of $\alpha \in [0,1]$, and are equivalent to the ordinary differential equations

$$\text{(i): } \begin{cases} (x^-_\alpha)' = f^-_\alpha(t, x^-_\alpha, x^+_\alpha), & x^-_\alpha(t_0) = (x_0)^-_\alpha \\ (x^+_\alpha)' = f^+_\alpha(t, x^-_\alpha, x^+_\alpha), & x^+_\alpha(t_0) = (x_0)^+_\alpha \end{cases}$$

$$\text{(ii): } \begin{cases} (x^-_\alpha)' = f^+_\alpha(t, x^-_\alpha, x^+_\alpha), & x^-_\alpha(t_0) = (x_0)^-_\alpha \\ (x^+_\alpha)' = f^-_\alpha(t, x^-_\alpha, x^+_\alpha), & x^+_\alpha(t_0) = (x_0)^+_\alpha. \end{cases} .$$

As a final step, we represent all the fuzzy quantities by the LU-fuzzy parametrization, with the meaning of the symbols as in (17); for each $\alpha = \alpha_i$ and $i = 0, 1, ..., N$ we obtain the differential equations

$$\text{(i): } \begin{cases} (x^-_i)' = f^-_i(t, x^-_i, x^+_i), & x^-_i(t_0) = (x_0)^-_i \\ (x^+_i)' = f^+_i(t, u^-_i, u^+_i), & x^+_i(t_0) = (x_0)^+_i \end{cases}$$

$$\begin{cases} (\delta x^-_i)' = \delta f^-_i(t, x^-_i, x^+_i), & \delta x^-_i(t_0) = (\delta x_0)^-_i \\ (\delta x^+_i)' = \delta f^+_i(t, x^-_i, x^+_i), & \delta x^+_i(t_0) = (\delta x_0)^+_i \end{cases}$$

$$\text{(ii): } \begin{cases} (x^-_i)' = f^+_i(t, x^-_i, ux^+_i), & x^-_i(t_0) = (x_0)^-_i \\ (x^+_i)' = f^-_i(t, u^-_i, u^+_i), & x^+_i(t_0) = (x_0)^+_i \end{cases}$$

$$\begin{cases} (\delta x^-_i)' = \delta f^+_i(t, x^-_i, x^+_i), & \delta x^-_i(t_0) = (\delta x_0)^-_i \\ (\delta x^+_i)' = \delta f^-_i(t, x^-_i, x^+_i), & \delta x^+_i(t_0) = (\delta x_0)^+_i. \end{cases}$$

They are $N+1$ independent (systems) of ordinary differential equations (four equations) with given initial conditions; they can be solved by any of the existing efficient methods for ODE.

In the following theorem we use the results in [2] to obtain a parametrization of an FDE with GH-derivative.

We denote by $D(x,y)$ the usual Hausdorff distance between two fuzzy numbers $x, y \in \mathbb{R}_\mathcal{F}$.

For simplicity of notation, the LU-parametric representations of a fuzzy number $x \in \mathbb{R}_\mathcal{F}$ and of a function $f(t,x) :]a,b[\times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ as in Assumption A, will be denoted by $x_N = (\alpha_i; x^-_i, \delta x^-_i, x^+_i, \delta x^+_i)$ and $f_N(t,x) = (\alpha_i; f^-_i, \delta f^-_i, f^+_i, \delta f^+_i)$, respectively; clearly, each of the elements f^-_i, δf^-_i, f^+_i and δf^+_i can be considered as functions of the five variables t, x^-_i, δx^-_i, x^+_i and δx^+_i for $i = 0, ..., N$.

Theorem 3: Let $\varepsilon > 0$ be arbitrary and let $x_0 \in \mathbb{R}_\mathcal{F}$ and $f :]a,b[\times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$. Assume that the conditions of

Theorem 3.1 in [2] for the solvability of the fuzzy initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{24}$$

are satisfied. Then there exist LU-parametrizations x_{N_ε} and $f_{N_\varepsilon}(t,x)$ with N_ε subintervals, such that the two solutions of (24) can be approximated with absolute error $D(x(t), x_{N_\varepsilon}(t)) < \varepsilon$ on some interval $[t_0, t_0 + k]$, by solving the following two sets of ODEs ($i = 0, ..., N$):

$$(i) \left\{ \begin{array}{c} \left(x_i^-\right)' = f_i^-(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(\delta x_i^-\right)' = \delta f_i^-(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(x_i^+\right)' = f_i^+(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(\delta x_i^+\right)' = \delta f_i^+(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ x_i^-(t_0) = x_{0,i}^-, x_i^+(t_0) = x_{0,i}^+ \\ \delta x_i^-(t_0) = \delta x_{0,i}^-, \delta x_i^+(t_0) = \delta x_{0,i}^+ \end{array} \right. , \tag{25}$$

$$(ii) \left\{ \begin{array}{c} \left(x_i^-\right)' = f_i^+(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(\delta x_i^-\right)' = \delta f_i^+(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(x_i^+\right)' = f_i^-(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ \left(\delta x_i^+\right)' = \delta f_i^-(t, x_i^-, x_i^+, \delta x_i^-, \delta x_i^+) \\ x_i^-(t_0) = x_{0,i}^-, x_i^+(t_0) = x_{0,i}^+ \\ \delta x_i^-(t_0) = \delta x_{0,i}^-, \delta x_i^+(t_0) = \delta x_{0,i}^+ \end{array} \right. . \tag{26}$$

After the preceding results, the LU parametric representation can be adopted to solve any FDE with fuzzy valued functions having at most a finite number of discontinuities (or points where the derivative w.r.t. α does not exist); in fact, it is sufficient that the points $\widehat{\alpha}$ where there is a discontinuity or a non differentiability is included as a point α_i in the decomposition $0 \leq \alpha_1 \leq ... \leq \alpha_N$ of the interval $[0,1]$ used for the parametric representation.

Furthermore, it is possible to intercept the switching points (and the switching intervals) by the following simple rule, based on one of the theorems above.

Rule to detect a switching: Consider a time iteration of the FDE solver, i.e. we have evaluated the solution $x_i^-(\widehat{t} - h), x_i^+(\widehat{t} - h)$ at $t = \widehat{t} - h$ and we perform the next iteration to obtain the solution at $t = \widehat{t}$ (i.e. with a step size h and assume that the discretization error of the solver is sufficiently small); suppose that $x_i^-(\widehat{t} - h) < x_i^+(\widehat{t} - h)$ for all $i \leq N$;
Solve the systems (25) or (26) starting with $i = N$ and decreasing i from N to 0;
If, for a given $i \leq N$, we find that $x_i^+(\widehat{t}) < x_i^-(\widehat{t})$ then a switching point exists between $\widehat{t} - h$ and $\widehat{t}$.

5 Linear fuzzy differential equations

Consider the following linear fuzzy differential equations where, for all $t \in [t_0, T]$, $a(t)$ and $b(t)$ are fuzzy numbers having $\alpha - cuts$ $[a(t)]_\alpha = [a_\alpha^-(t), a_\alpha^+(t)]$, $[b(t)]_\alpha = [b_\alpha^-(t), b_\alpha^+(t)]$:

1. $x'(t) = a(t)x(t) + b(t)$,
2. $x'(t) - a(t)x(t) = b(t)$,
3. $x'(t) - a(t)x(t) - b(t) = 0$,
4. $x'(t) - b(t) = a(t)x(t)$.

The initial condition is $x(t_0) = u$ where u is a given fuzzy number.

We can write the four equations, in terms of gH-derivative, as

$$1. \left\{ \begin{array}{l} \min\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^- + b_\alpha^-(t) \\ \max\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^+ + b_\alpha^+(t) \end{array} \right.$$

$$2. \left\{ \begin{array}{l} \min\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \\ \max\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \end{array} \right.$$

$$3. \left\{ \begin{array}{l} \min\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^+ - b_\alpha^+(t) \\ \max\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^- - b_\alpha^-(t) \end{array} \right.$$

$$4. \left\{ \begin{array}{l} \min\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \\ \max\{(x_\alpha^-(t))', (x_\alpha^+(t))'\} = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \end{array} \right.$$

Splitting the four FDE in terms of (i) or (ii) gH-differentiability, we obtain:

$$1\text{(i)} \left\{ \begin{array}{l} (x_\alpha^-(t))' = (a(t)x(t))_\alpha^- + b_\alpha^-(t) \\ (x_\alpha^+(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^+(t) \end{array} \right. ,$$

$$1\text{(ii)} \left\{ \begin{array}{l} (x_\alpha^+(t))' = (a(t)x(t))_\alpha^- + b_\alpha^-(t) \\ (x_\alpha^-(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^+(t) \end{array} \right. ,$$

$$2\text{(i)} \left\{ \begin{array}{l} (x_\alpha^-(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \\ (x_\alpha^+(t))' = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \end{array} \right. ,$$

$$2\text{(ii)} \left\{ \begin{array}{l} (x_\alpha^+(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \\ (x_\alpha^-(t))' = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \end{array} \right. ,$$

$$3\text{(i)} \left\{ \begin{array}{l} (x_\alpha^-(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^+(t) \\ (x_\alpha^+(t))' = (a(t)x(t))_\alpha^- + b_\alpha^-(t) \end{array} \right. ,$$

$$3\text{(ii)} \left\{ \begin{array}{l} (x_\alpha^+(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^+(t) \\ (x_\alpha^-(t))' = (a(t)x(t))_\alpha^- + b_\alpha^-(t) \end{array} \right. ,$$

$$4\text{(i)} \left\{ \begin{array}{l} (x_\alpha^-(t))' = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \\ (x_\alpha^+(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \end{array} \right. ,$$

$$4\text{(ii)} \left\{ \begin{array}{l} (x_\alpha^+(t))' = (a(t)x(t))_\alpha^- + b_\alpha^+(t) \\ (x_\alpha^-(t))' = (a(t)x(t))_\alpha^+ + b_\alpha^-(t) \end{array} \right. .$$

Considering that (i)-gH-differentiability requires $(x_\alpha^-(t))' \leq (x_\alpha^+(t))'$ $\forall\alpha$ and (ii)-gH-differentiability requires $(x_\alpha^+(t))' \leq (x_\alpha^-(t))'$ $\forall\alpha$, we see immediately the following cases, in combination with the necessary validity conditions:

(VC): for all t, $x_1^-(t) \leq x_1^+(t)$ and, w.r.t. α, $x_\alpha^-(t)$ is increasing and $x_\alpha^+(t)$ is decreasing.

Observe that

- equations 1(i) always have an acceptable fuzzy (i)-gH-differentiable solution, with increasing support length;
- equations 1(ii) have an acceptable fuzzy (ii)-gH-differentiable solution, with decreasing support length, if the validity conditions (VC) are also satisfied;
- equations 2(i) have an acceptable (i)-gH-differentiable solution only if (VC) are satisfied and

$$(a(t)x(t))_\alpha^+ + b_\alpha^-(t) \leq (a(t)x(t))_\alpha^- + b_\alpha^+(t);$$

- equations 2(ii) have an acceptable (ii)-gH-differentiable solution only if (VC) are satisfied and

$$(a(t)x(t))_\alpha^- + b_\alpha^+(t) \leq (a(t)x(t))_\alpha^+ + b_\alpha^-(t);$$

- equations 3(i) cannot have an acceptable fuzzy (i)-gH-differentiable solution (unless all is crisp);
- equations 3(ii) cannot have an acceptable fuzzy (ii)-gH-differentiable solution (unless all is crisp);
- equations 4(i) produce an acceptable (i)-gH-differentiable solution only if (VC) are satisfied and

$$(a(t)x(t))^-_\alpha + b^+_\alpha(t) \le (a(t)x(t))^+_\alpha + b^-_\alpha(t);$$

- equations 4(ii) produce an acceptable (ii)-gH-differentiable solution only if (VC) are satisfied and

$$(a(t)x(t))^-_\alpha + b^+_\alpha(t) \le (a(t)x(t))^+_\alpha + b^-_\alpha(t).$$

We conclude this section by considering the case of the linear FDE on $[a, b] = [0, 2]$

$$\begin{aligned} x'(t) &= (1-t)x(t) + e^{-t}\langle 0, 0.5, 1\rangle \\ x(0) &= \langle -0.5, 0.5, 1\rangle \end{aligned}$$

in the case where $a(t) = (1-t)$ is a crisp (non fuzzy) coefficient function and $\langle 0, 0.5, 1\rangle$ is a triangular symmetric fuzzy number with support $[0, 1]$; we have $b^-_\alpha(t) = e^{-t}\frac{\alpha}{2}$ and $b^+_\alpha(t) = e^{-t}(1-\frac{\alpha}{2})$. The initial value ($t = 0$) is the triangular fuzzy number with support $[-0.5, 1]$ and core $\{0.5\}$. For this equation, the two solutions are obtained by solving two sets of ordinary differential equations, depending on the sign of $a(t)$.

The (i)-gH-differentiable solution is obtained by solving the ODEs

$$\begin{cases} (x^-_\alpha(t))' = \left\langle \begin{array}{ll} (1-t)x^-_\alpha(t) + e^{-t}\frac{\alpha}{2} & t \le 1 \\ (1-t)x^+_\alpha(t) + e^{-t}\frac{\alpha}{2} & t > 1 \end{array} \right. \\ (x^+_\alpha(t))' = \left\langle \begin{array}{ll} (1-t)x^+_\alpha(t) + e^{-t}(1-\frac{\alpha}{2}) & t \le 1 \\ (1-t)x^-_\alpha(t) + e^{-t}(1-\frac{\alpha}{2}) & t > 1 \end{array} \right. \\ x(0) = \langle -0.5, 0.5, 1\rangle \end{cases}$$

and the solution, obtained numerically using the LU-parametric representation with $N = 8$, is the Hukuhara-type standard solution with increasing support (see Fig. 2).

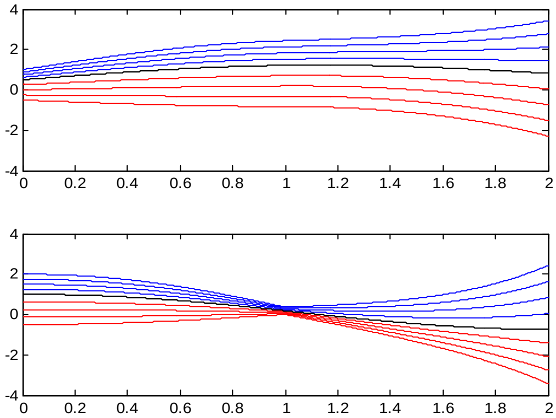

Figure 2. H-differentiable solution $x(t)$ (top) and its gH-derivative $x'_H(t)$ (bottom).

The (ii)-gH-differentiable solution is obtained by solving the ODEs

$$\begin{cases} (x^-_\alpha(t))' = \left\langle \begin{array}{ll} (1-t)x^+_\alpha(t) + e^{-t}(1-\frac{\alpha}{2}) & t \le 1 \\ (1-t)x^-_\alpha(t) + e^{-t}(1-\frac{\alpha}{2}) & t > 1 \end{array} \right. \\ (x^+_\alpha(t))' = \left\langle \begin{array}{ll} (1-t)x^-_\alpha(t) + e^{-t}\frac{\alpha}{2} & t \le 1 \\ (1-t)x^+_\alpha(t) + e^{-t}\frac{\alpha}{2} & t > 1 \end{array} \right. \\ x(0) = \langle -0.5, 0.5, 1\rangle \end{cases}.$$

The (ii)-gH-differentiable solution, obtained numerically using the LU-parametric representation with $N = 8$, is a valid fuzzy number for $t \in [0, t_1]$, where $t_1 \approx 0.478$; if, at $t = t_1$, the FDE is switched to (i)-gH-differentiability the solution is continued up to $t = 2$. The point $t_1 \approx 0.478$ is a switching point and the found solution is (ii)-gH-differentiable on $[0, t_1]$ and (i)-gH-differentiable on $[t_1, 2]$. The gH-differentiable solution on the interval $[0, 2]$ is illustrated in Fig. 3.

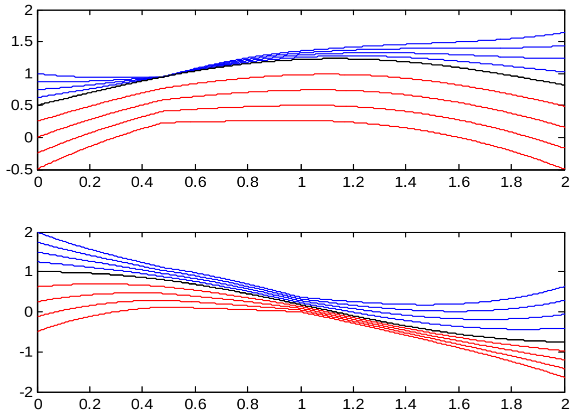

Figure 3. gH-differentiable solution $x(t)$ (top) and its gH-derivative $x'_{gH}(t)$ (bottom).

6 Conclusion

Following the ideas recently developed in [3], we propose here to investigate fuzzy differential equations with gH-differentiability and we suggest a numerical procedure using the LU-parametric representation.

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