On extension of fuzzy measures to aggregation functions

Anna Kolesárová[1] Andrea Stupňanová[2] Juliana Beganová[2]

[1]Institute IAM, FCHFT, Slovak University of Technology,
Radlinského 9, 812 37 Bratislava, Slovakia
e-mail: anna.kolesarova@stuba.sk
[2]Faculty of Civil Engineering, Slovak University of Technology,
Radlinského 11, 813 68 Bratislava, Slovakia
e-mail: {andrea.stupnanova@stuba.sk;juliana.beganova@stuba.sk}

Abstract

In the paper we study a method extending fuzzy measures on the set $N = \{1, \dots, n\}$ to n-ary aggregation functions on the interval $[0, 1]$. The method is based on a fixed suitable n-ary aggregation function and the Möbius transform of the considered fuzzy measure. This approach generalizes the well-known Lovász and Owen extensions of fuzzy measures. We focus our attention on the special class of n-dimensional Archimedean quasi-copulas and prove characterization of all suitable n-dimensional Archimedean quasi-copulas. We also present a special universal extension method based on a suitable associative binary aggregation function. Several examples are included.

Keywords: Aggregation function, Choquet integral, copula, fuzzy measure, n-monotone function, quasi-copula, Archimedean quasi-copula

1. Introduction

In [13] we have introduced a method extending any fuzzy measure on the set $N = \{1, \dots, n\}$ to an n-ary aggregation function by means of a (fixed) suitable aggregation function and the Möbius transform of the considered fuzzy measure. Recall that a *fuzzy measure* m on the set $N = \{1, \dots, n\}$ is a non-decreasing set function $m\colon 2^N \to [0, 1]$ with the properties $m(\emptyset) = 0$ and $m(N) = 1$. An *n-ary aggregation function* ($n \in \mathbb{N}$, $n \geq 2$) on the interval $[0, 1]$ is a function $A\colon [0, 1]^n \to [0, 1]$ which is non-decreasing in each variable and satisfies the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$. We briefly outline the proposed method.
To any fuzzy measure m on N and a given fixed n-ary aggregation function A we assign the function $F_{m,A} : [0, 1]^n \to \mathbb{R}$ defined by

$$F_{m,A}(x_1, \dots, x_n) = \sum_{I \subseteq N} M_m(I)\, A(\mathbf{x}_I), \qquad (1)$$

where $M_m : 2^N \to \mathbb{R}$, $M_m(I) = \sum_{K \subseteq I} (-1)^{|I \setminus K|} m(K)$, is the Möbius transform of m and $\mathbf{x}_I = (u_1, \dots, u_n)$ is the n-tuple assigned to an input n-tuple $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ and a subset I of the set N by

$$u_i = \begin{cases} x_i & \text{if } i \in I, \\ 1 & \text{otherwise.} \end{cases}$$

We say that the function $F_{m,A}$ extends the fuzzy measure m if $F_{m,A}|_{\{0,1\}^n} = m$. However, in general, the functions $F_{m,A}$ defined by (1) are neither aggregation functions (monotonicity can fail) nor extensions of m. Denote the set of all n-ary aggregation functions by $\mathcal{A}_{(n)}$ and the set of all fuzzy measures on N by $\mathcal{M}_{(n)}$. In [13] we have completely characterized all aggregation functions which are suitable for this construction, i.e., all $A \in \mathcal{A}_{(n)}$ which together with any fuzzy measure $m \in \mathcal{M}_{(n)}$ give via (1) an aggregation function $F_{m,A}$ extending m. Such aggregation functions $A \in \mathcal{A}_{(n)}$ are briefly called *suitable aggregation functions*.
For example, all n-dimensional copulas are suitable aggregation functions. On the other hand, not all n-dimensional quasi-copulas possess this property. In this contribution we focus our attention on the special class of n-dimensional Archimedean quasi-copulas and prove characterization of all suitable n-dimensional Archimedean quasi-copulas.
Our approach was motivated by the Lovász and Owen extensions of fuzzy measures, [14, 21]. If $A = Min$, where $Min(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}$, then the expression on the right-hand side of formula (1) coincides with the the expression

$$\sum_{I \subseteq N} M_m(I) \min_{i \in I} x_i,$$

which is equal to the Choquet integral $C - \int_N \mathbf{x}\, \mathrm{d}m$, [5], and also to the Lovász extension of the fuzzy measure m, [15]. Similarly, for $A = \Pi$, where $\Pi(x_1, \dots, x_n) = x_1 \cdots x_n$, the expression on the right-hand side of formula (1) coincides with the the expression

$$\sum_{I \subseteq N} M_m(I) \prod_{i \in I} x_i,$$

which is known as the Owen extension of m. Note that the Lovász and Owen extensions can be applied universally, for any arity n, while the proposed method produces (if A is a suitable n-ary aggregation function) n-ary aggregation functions extending m. In the last section a special universal extension method based on a suitable associative binary aggregation function is proposed. The method is illustrated by examples.

2. Characterization of suitable aggregation functions

As mentioned above, in [13] we have characterized all suitable aggregation functions A. For completeness of information let us recall two main results. The first of them characterizes all aggregation functions $A \in \mathcal{A}_{(n)}$ for which $F_{m,A}$ is an extension of m for each $m \in \mathcal{M}_{(n)}$.

Theorem 1 *Let $A \in \mathcal{A}_{(n)}$. The function $F_{m,A}$ defined by (1) is for each $m \in \mathcal{M}_{(n)}$ an extension of m if and only if A is an aggregation function with zero annihilator.*

Recall that 0 is the annihilator of A if $A(x_1, \ldots, x_n) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$.
In general, extensions $F_{m,A}$ need not be monotone. Before giving conditions ensuring the monotonicity of $F_{m,A}$, we introduce some notations.
Fix $i \in N$. Let $\mathbf{u}$, $\mathbf{v}$ be any elements in $[0,1]^n$, such that

$$\mathbf{u} = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n),$$

$$\mathbf{v} = (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n),$$

with $x_i' > x_i$. For any subset $E \subseteq N \setminus \{i\}$ denote by $\mathbf{u}^E$ and $\mathbf{v}^E$ n-tuples with coordinates

$$\begin{aligned} u_i^E &= x_i \quad & v_i^E &= x_i', \\ u_j^E &= x_j \quad & v_j^E &= 1, \quad \text{if } j \notin E \cup \{i\}, \\ u_j^E &= 0 \quad & v_j^E &= x_j, \quad \text{otherwise.} \end{aligned} \tag{2}$$

Finally, recall that for an n-ary aggregation function A, the A-volume of an n-box $[\mathbf{u}, \mathbf{v}]$ in $[0,1]^n$, $[\mathbf{u}, \mathbf{v}] = [u_1, v_1] \times \ldots \times [u_n, v_n]$, is defined by

$$V_A([\mathbf{u}, \mathbf{v}]) = \sum (-1)^{\alpha(\mathbf{c})} A(\mathbf{c}),$$

where the sum is taken over all vertices $\mathbf{c} = (c_1, \ldots, c_n)$ of the n-box $[\mathbf{u}, \mathbf{v}]$ (i.e., each c_k is equal to either u_k or v_k), and $\alpha(\mathbf{c})$ is the number of indices $k's$ such that $c_k = u_k$.

Theorem 2 *Let $A \in \mathcal{A}_{(n)}$ be an aggregation function with zero annihilator. The function $F_{m,A}$ is for each $m \in \mathcal{M}_{(n)}$ non-decreasing in the ith variable ($i = 1, \ldots, n$) if and only if for all n-tuples $\mathbf{u} = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$, $\mathbf{v} = (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ in $[0,1]^n$ with $x_i' > x_i$, it holds that for each $E \subseteq N \setminus \{i\}$, A-volumes $V_A([\mathbf{u}^E, \mathbf{v}^E])$, where the end-points of n-boxes $[\mathbf{u}^E, \mathbf{v}^E]$ are defined by (2), are non-negative.*

For example, by Theorem 2, for a binary aggregation function A with zero annihilator the function $F_{m,A}$ is non-decreasing for each $m \in \mathcal{M}_{(2)}$ if and only if A-volumes of all possible 2-boxes $[(x_1, x_2), (x_1', 1)]$ and $[(x_1, x_2), (1, x_2')]$ in $[0,1]^2$ with $x_1' > x_1$ and $x_2' > x_2$, are non-negative. Note that A-volumes of 2-boxes of the type $[(0, x_2), (x_1, x_2')]$ and $[(x_1, 0), (x_1', x_2)]$, which in binary case are also obtained from conditions (2), are trivially non-negative because of the monotonicity of A.

From Theorems 1 and 2 we obtain the following characterization.

Corollary 1 *An n-ary aggregation function A is a suitable aggregation function if and only if it has zero annihilator and satisfies the conditions for the monotonicity of $F_{m,A}$ given in Theorem 2 for each variable.*

For example, all n-copulas [24, 19] are suitable aggregation functions for our construction. Recall that *n-copulas* are defined as functions $C\colon [0,1]^n \to [0,1]$ satisfying

(C1) the boundary conditions:
if $0 \in \{x_1, \ldots, x_n\}$ then $C(x_1, \ldots, x_n) = 0$,
$C(1, \ldots, 1, x_j, 1, \ldots, 1) = x_j$ for each $j = 1, \ldots, n$ and each $x_j \in [0,1]$,

(C2) the n-increasing property:
$V_C([\mathbf{u}, \mathbf{v}]) \geq 0$ for each n-box $[\mathbf{u}, \mathbf{v}]$ in $[0,1]^n$.

It is easy to see that aggregation functions described in the following proposition also possess zero annihilator and the A-volumes of all n-boxes in $[0,1]^n$ are non-negative.

Proposition 1 *Let C be an n-copula, $f_i\colon [0,1] \to [0,1]$, non-decreasing functions such that $f_i(0) = 0$, $f_i(1) = 1$, $i = 1, \ldots, n$. Then the function $A\colon [0,1]^n \to [0,1]$ defined by*

$$A(x_1, \ldots, x_n) = C\left(f_1(x_1), \ldots, f_n(x_n)\right),$$

is a suitable n-ary aggregation function.

Not only copulas and distorted copulas from Proposition 1 are suitable aggregation functions.

Example 1 Consider the function $A\colon [0,1]^3 \to [0,1]$, given by

$$A(x, y, z) = xyz \min(1, x + y + z).$$

The function A is a ternary aggregation function with zero annihilator. It is not a copula, because, e.g., for $\mathbf{u} = (0.3, 0.3, 0.3)$ and $\mathbf{v} = (0.35, 0.35, 0.35)$ the A-volume of the corresponding 3-box is $V_A([\mathbf{u}, \mathbf{v}]) = -0.0019 < 0$. After quite tedious computations one obtains that A is a suitable aggregation function. This example can be generalized for any $n > 3$.

The aggregation function A from the previous example is a 3-quasi-copula. In general, *n-quasi-copulas* are functions $Q\colon [0,1]^n \to [0,1]$, which satisfy the same boundary conditions (C1) as n-copulas do and which are non-decreasing (in each variable) and 1-Lipschitz, see [20, 4].

In contrast to n-copulas, not all n-quasi-copulas are suitable aggregation functions.

Example 2 The function $W\colon [0,1]^3 \to [0,1]$ given by $W(x,y,z) = \max\{0, x+y+z-2\}$ is a proper 3-quasi-copula. As the W-volume

$$V_W([(0.5, 0.5, 0.5), (1,1,1)]) = -0.5 < 0,$$

W is not a suitable aggregation function, because Theorem 2 requires the W-volume of the 3-box $[(0.5, 0.5, 0.5), (1,1,1)]$ to be non-negative.

If we denote by $\mathcal{C}_{(n)}$ the set of all n-copulas, by $\mathcal{Q}_{(n)}$ the set of all n-quasi-copulas and by $\mathcal{F}_{(n)}$ the set of all n-ary aggregation functions suitable for our construction, then, supported by the previous results, we can write $\mathcal{C}_{(n)} \subsetneqq \mathcal{F}_{(n)}$ and $\mathcal{Q}_{(n)} \nsubseteq \mathcal{F}_{(n)}$.

3. Special sets of suitable aggregation functions

In this section we focus our attention on the set of Archimedean n-quasi-copulas. Let us introduce several preparatory notions.

Definition 1 *An n-quasi-copula $Q\colon [0,1]^n \to [0,1]$ given by*

$$Q(x_1, \ldots, x_n) = \varphi^{(-1)}(\varphi(x_1) + \cdots + \varphi(x_n)),$$

where $\varphi\colon [0,1] \to [0,\infty]$ is a continuous strictly decreasing convex function with $\varphi(1) = 0$ and pseudo-inverse $\varphi^{(-1)}$, is called an Archimedean n-quasi-copula.

The function φ is called an additive generator of Q. Its pseudo-inverse $\varphi^{(-1)}\colon [0,\infty] \to [0,1]$ is defined by

$$\varphi^{(-1)}(u) = \varphi^{-1}(\min(\varphi(0), u)).$$

For n-copulas we have the following result, see [17].

Theorem 3 *A function $C\colon [0,1]^n \to [0,1]$ given by*

$$C(x_1, \ldots, x_n) = \varphi^{(-1)}(\varphi(x_1) + \cdots + \varphi(x_n)),$$

where $\varphi\colon [0,1] \to [0,\infty]$ is a continuous strictly decreasing convex function with $\varphi(1) = 0$ and pseudo-inverse $\varphi^{(-1)}$, is an n-copula if and only if there exists an n-monotone function $f\colon [-\infty, 0] \to [0,1]$ such that

$$\varphi^{(-1)}(-x) = f(x), \qquad x \in [-\infty, 0].$$

Note that a real function f is called n-monotone on an interval I if all its differences of order $1, \ldots, n$ are non-negative. This means that f is n-monotone if and only if for each $k \in \{1, \ldots, n\}$, each $x \in I$ and all $\epsilon_1, \ldots \epsilon_k > 0$ such that $x + \epsilon_1 + \cdots + \epsilon_k \in I$

$$\sum_{I \subseteq \{1,\ldots,k\}} f\left(x + \sum_{i\in I} \epsilon_i\right) (-1)^{|I|+k} \geq 0,$$

compare with [2, 16, 11]. Now, we can formulate the result.

Theorem 4 *Let Q be an Archimedean n-quasi-copula. Then the following is equivalent*

(i) *Q is a suitable aggregation function.*
(ii) *Q is an n-copula.*

Proof. (ii) $\Rightarrow$ (i). The claim is evident.

(i) $\Rightarrow$ (ii). Let $k \in \{1, \ldots, n\}$. Let $x_1, \ldots, x_k$ and x_1' be any elements in $[0,1]$ such that $x_1' > x_1$. Consider the n-box $[x_1, x_1'] \times [x_2, 1] \times \ldots \times [x_k, 1] \times [0,1] \times \ldots \times [0,1]$. By Theorem 2 it holds

$$V_Q\left([x_1, x_2, \ldots, x_k, 0, \ldots, 0] \times [x_1', 1, \ldots, 1]\right) \geq 0. \tag{3}$$

On the other hand, as 0 is the annihilator of Q and Q is generated by φ ($\varphi(1) = 0$), we obtain

$$\begin{aligned} & V_Q\left([x_1, x_2, \ldots, x_k, 0, \ldots, 0] \times [x_1', 1, \ldots, 1]\right) \\ &= \sum_{I \subseteq \{1,\ldots,k\}} \varphi^{(-1)}\left(\sum_{i\in I} \varphi(x_i) + \varphi(x_1')\mathbf{1}_{I^c}(1)\right) (-1)^{|I|} \end{aligned} \tag{4}$$

If we denote

$$a = \varphi(x_1'), \quad b_1 = \varphi(x_1) - \varphi(x_1'),$$
$$b_2 = \varphi(x_2), \ldots, b_k = \varphi(x_k),$$

then from (4) and (3) we obtain that for each $k \leq n$ it holds

$$\sum_{I \subseteq \{1,\ldots,k\}} \varphi^{(-1)}\left(a + \sum_{i\in I} b_i\right) (-1)^{|I|} \geq 0. \tag{5}$$

For $x \in [-\infty, 0]$ define $f(x) = \varphi^{(-1)}(-x)$ and denote $u = -\left(a + \sum_{i=1}^{k} b_i\right)$. Then

$$\begin{aligned} \varphi^{(-1)}\left(a + \sum_{i\in I} b_i\right) &= f\left(-a - \sum_{i\in I} b_i\right) \\ &= f\left(u + \sum_{i\in I} b_i\right) (-1)^k. \end{aligned} \tag{6}$$

Finally, from (5) and (6) it follows that

$$\sum_{I \subseteq \{1,\ldots,k\}} f\left(u + \sum_{i\in I} b_i\right) (-1)^{|I|+k} \geq 0, \tag{7}$$

which means that f is n-monotone and by Theorem 3, Q is an n-copula.

$\square$

Remark 1 Observe that based on the results presented in [25], Theorem 4 can be generalized to the case of associative n-quasi-copulas. The associativity of n-ary functions in the Post sense [22] is considered, i.e., the associativity of an n-ary quasi-copula Q means that for all $x_1, \ldots, x_{2n-1} \in [0,1]$ it holds

$$\begin{aligned} & Q\left(Q(x_1,\ldots,x_n), x_{n+1},\ldots,x_{2n-1}\right) \\ = \; & Q\left(x_1, Q(x_2,\ldots,x_{n+1}),\ldots,x_{2n-1}\right) = \ldots \\ = \; & Q\left(x_1,\ldots,x_{n-1}, Q(x_n,\ldots,x_{2n-1})\right). \end{aligned}$$

An associative n-quasi-copula is a suitable aggregation function if and only if it is an ordinal sum of n-ary Archimedean copulas. For the later concept see [18].

4. A universal extending method

As mentioned in Introduction, the Lovász and Owen extensions can be applied universally, independently of the arity n. To obtain another universal extension method, consider a suitable binary aggregation function $A\colon [0,1]^2 \to [0,1]$. In [13] we proved the following characterization of suitable binary aggregation functions. Note that Example 2 shows that this characterization is valid for binary case only.

Theorem 5 *Let $A \in \mathcal{A}_{(2)}$. The function $F_{m,A}$ given by (1) is for each $m \in \mathcal{M}_{(2)}$ an aggregation function extending m if and only if for each $(x,y) \in [0,1]^2$ it holds*

$$A(x,y) = Q(f(x), g(y)), \tag{8}$$

where Q is a 2-quasi-copula and f, g are non-decreasing $[0,1] \to [0,1]$ functions with $f(0) = g(0) = 0$, $f(1) = g(1) = 1$.

Suppose that the considered suitable binary aggregation function A is associative. The associativity of A means that for all $x, y, z \in [0,1]$ it holds $A(A(x,y),z) = A\left(x, A(y,z)\right)$, i.e.,

$$\begin{aligned} & Q\left(f\left(Q\left(f(x), g(y)\right)\right), g(z)\right) \\ & = Q\left(f(x), g\left(Q\left(f(y), g(z)\right)\right)\right). \end{aligned}$$

Putting $y = z = 1$ one obtains $f(x) = f\left(f(x)\right)$. Similarly, the equality $g(z) = g\left(g(z)\right)$ can be proved. To obtain a continuous extension, the continuity of f and g is required, and so the only possibility for f and g is the identity function. Thus $A = Q$, where Q is an associative 2-quasi-copula. Its n-ary extension is a suitable aggregation function if and only if it is an n-copula. Following the results in [18] we can conclude:

Theorem 6 *An associative binary aggregation function $A\colon [0,1]^2 \to [0,1]$ generates a continuous extended aggregation function $\overline{A}\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ such that for each $n \geq 2$, $\overline{A}|_{[0,1]^n}$ is a suitable n-ary aggregation function, if and only if A is an ordinal sum of Archimedean copulas, $A = (\langle a_k, b_k, C_k\rangle)$, where each C_k is generated by an additive generator $\varphi_k\colon [0,1] \to [0,\infty]$, such that the function $f_k\colon [-\infty,0] \to [0,1]$, given by $f_k(x) = \varphi_k^{(-1)}(-x)$, is totally monotone, i. e., f_k has all derivatives on $]-\infty,0[$ which are non-negative.*

Example 3 (i) Let $A = T_0^H$ be the Hamacher product given by $A(x,y) = \frac{xy}{x+y-xy}$. A is an Archimedean copula generated by the additive (strict) generator $\varphi\colon \varphi(x) = \frac{1}{x} - 1$. The function $f(x) = \varphi^{-1}(-x) = \frac{1}{1-x}$, $x \in [-\infty, 0]$, has derivatives $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$, $n \in \mathbb{N}$, which are non-negative. The function

$$A(x_1,\ldots,x_n) = \left(\sum_{i=1}^{n} \frac{1}{x_i} - n + 1\right)^{-1}$$

is a suitable n-ary aggregation function and moreover, for any n.

(ii) Let A be the copula ordinal sum, $A = (\langle 0, 1/2, \Pi\rangle)$, i.e.,

$$A(x,y) = \begin{cases} 2xy & (x,y) \in [0,1/2],^2 \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

The product copula Π is generated by the additive generator $\varphi(x) = -\log x$. For the function $f(x) = \varphi^{-1}(-x) = e^x$, $x \in [-\infty, 0]$, all derivatives are non-negative. The function

$$\begin{aligned} & A(x_1,\ldots,x_n) \\ & = \begin{cases} \frac{1}{2}\prod\limits_{i=1}^{n} \min\{2x_i, 1\} & \text{if } \min\{x_1,\ldots,x_n\} \leq \frac{1}{2}, \\ \min\{x_1,\ldots,x_n\} & \text{otherwise,} \end{cases} \end{aligned}$$

is a suitable aggregation function for any n.
Observe that the extension of fuzzy measures based on A can be seen as a mixture of the Lovász and Owen extensions in the following sense: if $\mathbf{x} \in [1/2,1]^n$ then $F_{m,A}(\mathbf{x}) = F_{m,Min}(\mathbf{x})$, i.e., $F_{m,A}$ is just the Lovász extension, and if $\mathbf{x} \in [0,1/2]^n$ then $F_{m,A}(\mathbf{x}) = \frac{1}{2}F_{m,\Pi}(2\mathbf{x})$, i.e., $F_{m,A}$ is a linear transform of the Owen extension.

Acknowledgment The authors acknowledge the support of the grant VEGA 1/0198/09 and the project of Science and Technology Assistance Agency under the contract No. APVV–0012–07.

References

[1] C. Alsina, R.B. Nelsen, B. Schweizer, On the characterization of a class of binary operations on distributions functions, *Stat. Probab. Lett.*, 17:85–89, 1993.

[2] A. G. Bronevich: On the closure of families of fuzzy measures under eventwise aggregation, *Fuzzy Sets and Systems*, 153:45–70, 2005.

[3] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation Operators: Properties, Classes and Construction Methods. In *Aggregation Operators. New Trends and Applications*, T. Calvo, G. Mayor, R. Mesiar, eds. Physica-Verlag, Heidelberg, pp. 3-107, 2002.

[4] I. Cuculescu, R. Theodorescu, Copulas: diagonals and tracks, *Revue Rumaine de Mathématique Pures et Appliquées*, 46:731-742, 2001.

[5] A. Chateauneuf, J.Y. Jaffray, Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion, *Math. Soc. Sciences*, 17:263-283, 1989.

[6] G. Choquet, Theory of capacities, *Ann. Inst. Fourier*, 5:131-295, 1953-54.

[7] C. Genest, J.J. Quesada Molina, J.A. Rodríguez-Lallena, and C. Sempi, A characterization of quasi-copulas, *J. Multivariate Anal.*, 69:193-205, 1999.

[8] M. Grabisch, T. Murofushi, and M. Sugeno, *Fuzzy Measures and Integrals. Theory and Applications*, Physica Verlag, Heidelberg, 2000.

[9] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, Cambridge University Press, Cambridge, 2009.

[10] E.P. Klement, R. Mesiar, and E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[11] E.P. Klement, M. Manzi, R. Mesiar, Aggregation functions with stronger types of monotonicity. In E. Hüllermeier, R. Kruse, F. Hoffmann, eds., *Proc. IPMU 2010*, LNAI 6178, Springer-Verlag, Berlin Heidelberg, pp. 418–424, 2010.

[12] A. Kolesárová, 1–Lipschitz aggregation operators and quasi-copulas, *Kybernetika* , 9:615-629, 2003.

[13] A. Kolesárová, A. Stupňanová, and J. Beganová, Aggregation-based extensions of utility functions. *Fuzzy Sets and Systems*, submitted.

[14] L. Lovász, Submodular function and convexity. In *Mathematical Programming: The state of the art.* Springer, Berlin, pp. 235-257, 1983.

[15] J.-L. Marichal, Aggregation of interacting criteria by means of the discrete Choquet integral. In *Aggregation Operators. New Trends and Applications*, T. Calvo, G. Mayor, R. Mesiar, eds. Physica-Verlag, Heidelberg, pp. 224-244, 2002.

[16] M. Marinacci, L. Montrucchio: Ultramodular functions, *Math. Oper. Res.*, 30:311–332, 2005.

[17] A.J. McNeil, J. Nešlehová, Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions, *The Annals of Statistics*, 37:3059-3097, 2009.

[18] R. Mesiar, C. Sempi, Ordinal sums and idempotents of copulas, *Aequationes Mathematicae*, 79:39-52 2010.

[19] R.B. Nelsen, *An Introduction to Copulas*, Lecture Notes in Statistics 139, Springer Verlag, New York, 1999.

[20] R.B. Nelsen, J.J. Quesada-Molina, B. Schweizer, and C. Sempi, Derivability of some operations on distributions functions. In L. Rüschendorf, B. Schweizer, M.D. Taylor, eds., *Distributions with Fixed Marginals and Related Topics*, CA: IMS Lecture Notes - Monograph Series Number 28, Hayward, pp. 233-243, 1996.

[21] G. Owen, Multilinear extensions of games. In A.E. Roth, ed., *The Shapley value. Essays in Honour of Lloyd S. Shapley*, Cambridge University Press, pp. 139-151, 1988.

[22] E.-L. Post, Polyadic groups, *Trans. Amer. Math. Soc.*, 48:208-350, 1940.

[23] D. Schmeidler, Integral representation without additivity, *Proc. Amer. Math. Soc.*, 97:255-261, 1986.

[24] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris*, 8:229-231, 1959.

[25] A. Stupňanová, A. Kolesárová, Associative n-dimensional copulas. *Kybernetika*, in press, 2011.

[26] Z. Wang, G.J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.