Fuzzy and Fourier Transforms

Irina Perfilieva,[1] Petra Hoďáková[2]

[1]University of Ostrava, Inst. for Research and Applications of Fuzzy Modeling, Irina.Perfilieva@osu.cz
[2]University of Ostrava, Inst. for Research and Applications of Fuzzy Modeling, Petra.Hodakova@osu.cz

Abstract

The fuzzy transform (F-transform for short) is a universal tool for a fuzzy modeling with convincing applications to image processing. The aim of this contribution is to explain the effect of the F-transform in image processing. With this purpose, we investigate properties of the Fourier transform over the F-transform components. We prove that the direct F-transform is a low-pass filter. This explains specific tools and methodologies that are developed in the F-transform applications to the image processing.

Keywords: Convolutions of functions, discrete F-transform, discrete Fourier transform

1. Introduction

Various kinds of transforms are used as powerful methods for solving many problems, including image processing. The main idea of them consists in transforming the original model into a special space where a computation is simpler. In this contribution, we will discuss the Fourier transform and the F-transform.

The Fourier transform is a well known method that is widely used in image processing. In general, we can say that the Fourier transform converts a function (image), considered in a time or spatial domain, into another function, considered in a frequency domain. In the case of images, the number of frequencies in a frequency domain is equal to the number of pixels in the image or spatial domain.

Transformation to a frequency domain is a very important tool in many applications. For example, applying filters to images in a frequency domain is computationally faster than doing the same in an image domain. Spectrum analysis is also widely used in speech analysis, image compression, search of periodicity in a wide variety of data in economics, biology, physics, etc.

In particular, the Fourier image analysis has several useful properties. For example, the operation of convolution in a spatial domain corresponds to the operation of multiplication in a frequency domain. This is important because multiplication is a simpler mathematical operation than convolution.

The F-transform is another technique discussed in this contribution. It performs a transformation of an original universe of functions into a universe of vectors. In more details, the F-transform establishes a correspondence between a set of continuous functions on an interval of real numbers and the set of n-dimensional (real) vectors.

The F-transform proves to be a successful methodology with various applications in image compression and reconstruction ([4], [5]), image fusion ([2], [3]), numeric solution of differential equations ([7]), time-series procession ([6]). It turned out that the F-transform is very general and as powerful in many applications as conventional transforms. Moreover, sometimes the F-transform can be more efficient than its counterparts.

The structure of this paper is as following: Section 2 introduces notions of a fuzzy partition and a generating function of an h-uniform fuzzy partition. In this section, the direct form of a discrete F-transform is reminded and its representation in the form of a convolution is introduced. In Section 3, the properties of a convolution are recalled. Section 4 reminds a definition of the discrete Fourier transform. In Section 5, an application of the discrete Fourier transform to the F-transform components is discussed. Section 6 presents examples, and Section 7 concludes the contribution.

2. F-transform as Convolution

In this section, we aim at expressing the F-transform in a form of a convolution of two functions. We will start with reminding basic definitions regarding the F-transform. We will focus on the discrete F-transform only.

2.1. Discrete F-transform

Let us consider the discrete F-transform [1]. We choose an interval $[a, b]$ as a universe, and assume that a function f is given at points $p_0, \dots, p_{l-1} \in [a, b]$.

Below, we recall the definition of a fuzzy partition. Let $a = x_0 < \dots < x_n = b$, $n \geq 3$ be fixed nodes within $[a, b]$. Fuzzy sets $A_1, \dots, A_{n-1}$ identified with their membership functions $A_1, \dots, A_{n-1}$, defined on $[a, b]$, establish a *fuzzy partition* of $[a, b]$ if they fulfill the following conditions for $k = 1, \dots, n-1$:

1) $A_k : [a, b] \to [0, 1]$, $A_k(x_k) = 1$;
2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$, $k = 1, \dots, n-1$;
3) $A_k(x)$ is continuous;

4) $A_k(x)$ strictly increase on $[x_{k-1}, x_k]$, $k = 1, \ldots, n-1$; and strictly decrease on $[x_k, x_{k+1}]$, $k = 1, \ldots, n-1$;
5) $\sum_{k=1}^{n} A_k(x) = 1$, $x \in [x_1, x_{n-1}]$.

$A_1, \ldots, A_{n-1}$ are called *basic functions.*

We say that the fuzzy partition given by $A_1, \ldots, A_{n-1}$, is an *h-uniform fuzzy partition* if the nodes $x_k = a + hk$, $k = 0, \ldots, n$, are equidistant, $h = (b-a)/n$ and two additional properties are met:

6) $A_k(x_k - x) = A_k(x_k + x)$, $x \in [0, h]$, $k = 1, \ldots, n-1$;
7) $A_k(x) = A_{k-1}(x - h)$, $k = 2, \ldots, n-1$, $x \in [x_{k-1}, x_{k+1}]$.

Assume that fuzzy sets $A_1, \ldots, A_{n-1}$ establish a fuzzy partition of $[a, b]$ and $f : P \longrightarrow \mathbb{R}$ is a discrete real valued function defined on the set $P = \{p_0, \ldots, p_{l-1}\}$ where $P \subseteq [a, b]$ and $l > n$. The following vector of real numbers $\mathbf{F}_n[f] = [F_1, \ldots, F_{n-1}]$ is the *(direct) discrete F-transform* of f w.r.t. $A_1, \ldots, A_{n-1}$ where the k−th component F_k is defined by

$$F_k = \frac{\sum_{j=0}^{l-1} A_k(p_j) f(p_j)}{\sum_{j=0}^{l-1} A_k(p_j)}, \ k = 1, \ldots, n-1. \quad (1)$$

By using an inversion formula we can approximately reconstruct function f from the vector of components of its direct discrete F-transform. We define [1] the *inverse discrete F-transform* as

$$f_{F,n}(p_j) = \sum_{k=1}^{n-1} F_k A_k(p_j), \ j = 0, \ldots, l-1.$$

Moreover, the following Theorem 1 says that the inverse discrete F-transform $f_{F,n}$ can approximate the original function f at common nodes with an arbitrary precision (proved in [1]).

Theorem 1
Let a function f be given at nodes $p_0, \ldots, p_{l-1}$ constituting the set $P \subseteq [a, b]$. Then, for any $\varepsilon > 0$, there exist n_ε and a fuzzy partition $A_1, \ldots, A_{n_\varepsilon}$ of $[a, b]$ such that P is sufficiently dense with respect to $A_1, \ldots, A_{n_\varepsilon}$ and for all $p \in \{p_0, \ldots, p_{l-1}\}$

$$|f(p) - f_{F,n_\varepsilon}(p)| < \varepsilon$$

holds true.

2.2. F-Transform as Convolution

Let us assume that the interval $[a, b]$ is h-uniformly partitioned by fuzzy sets $A_1, \ldots, A_{n-1}$, f is a discrete function, and the F-transform of a discrete function f is given by $\mathbf{F}_n[f]$ with components obtained by(1).

It is easy to see that if the fuzzy partition $A_1, \ldots, A_{n-1}$ of $[a, b]$ is h-uniform, then there exists an even function

$$A : [-h, h] \longrightarrow [0, 1]$$

such that for all $k = 1, \ldots, n-1$,

$$A_k(x) = A(x - x_k) = A(x_k - x), \ x \in [x_{k-1}, x_{k+1}].$$

We call A a *generating function* of an h-uniform fuzzy partition.

The example of a triangular generating function A and the respective h-uniform fuzzy partition $A_1, \ldots, A_{n-1}$ is given in Figure 1.

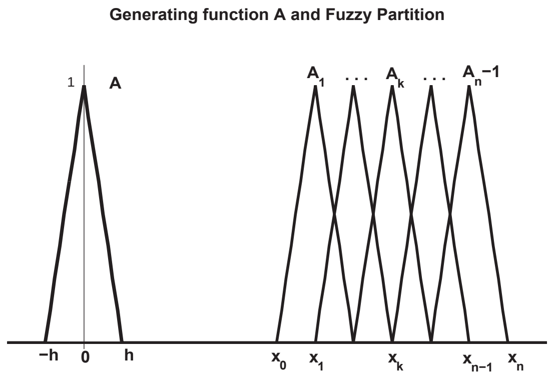

Figure 1: Generating function A of an h-uniform fuzzy partition.

Let us assume that points $p_0, \ldots, p_{l-1}$ are equidistant in the interval $[a, b]$ and moreover $p_j = a + jh/m$; $j = 0, \ldots, l-1$, where m and l are connected by the following equality: $l = nm + 1$. Thus chosen points $p_0, \ldots, p_{l-1}$ assure that the nodes $x_0, \ldots, x_n$ are among them, i.e. for each $k = 0, \ldots, n$, there exists j such that $x_k = p_j$. Moreover, the following *Lemma 1* holds true.

Lemma 1
Let $A_1, \ldots, A_{n-1}$ establish an h-uniform fuzzy partition of $[a, b]$ and points $p_0, \ldots, p_{l-1}$ from $[a, b]$ are chosen as above. Then there exists a constant $c > 0$ such that for all $k = 1, \ldots, n-1$,

$$\sum_{j=0}^{l-1} A_k(p_j) = c. \quad (2)$$

PROOF: In order to prove (2), it is sufficient to show that for all $k = 1, \ldots, n-2$,

$$\sum_{j=0}^{l-1} A_{k+1}(p_j) = \sum_{j=0}^{l-1} A_k(p_j). \quad (3)$$

Indeed, the uniformity of our partition and the fact that

$$A_{k+1}(p_{j+m}) = A_k(p_j), j = 0, \ldots, l-1-m,$$

leads to

$$\sum_{j=0}^{l-1} A_{k+1}(p_j) = A_{k+1}(p_{km}) + \cdots + A_{k+1}(p_{(k+2)m}) =$$

$$A_k(p_{(k-1)m}) + \cdots + A_k(p_{(k+1)m}) = \sum_{j=0}^{l-1} A_k(p_j),$$

$$k = 1, \ldots, n-2.$$

□

Remark 1
Let us remark that (2) is not the generalized Ruspini condition, because the sum is taken over points $p_0, \ldots, p_{l-1}$. Actually, the sum in (2) is taken over those points that are covered by a single basic function $A_k, k = 1, \ldots, n-1$.

By (2), the expression (1) can be rewritten as follows:

$$F_k = \frac{\sum_{j=0}^{l-1} A(x_k - p_j) f(p_j)}{c}; \; k = 1, \ldots, n-1. \quad (4)$$

Let us consider F_k as a value of a discrete function F, defined on the set $\mathbf{Z}_{n-1} = \{1, \ldots, n-1\}$ with values from $\mathbb{R}$ such that $F : \mathbf{Z}_{n-1} \longrightarrow \mathbb{R}$ and $F(k) = F_k$. We will use (4) for an analytic extension of F from $\mathbf{Z}_{n-1}$ to $\mathbf{Z}_l = \{0, 1, \ldots, l-1\}$, so that

$$F(t) = \frac{\sum_{j=0}^{l-1} A(p_t - p_j) f(p_j)}{c}; \; t = 0, \ldots, l-1. \quad (5)$$

Similarly, we can assume that functions A and f are defined on the set $\mathbf{Z}_l$ and rewrite (5) into

$$F(t) = \frac{\sum_{j=0}^{l-1} A(t - j) f(j)}{c}; \; t = 0, \ldots, l-1. \quad (6)$$

Finally, we will normalize values of A dividing them by c and keep the same denotation A for the normalized function. Then without loss of generality, we will continue working with the below given expression for F:

$$F(t) = \sum_{j=0}^{l-1} A(t - j) f(j); \; t = 0, \ldots, l-1. \quad (7)$$

Analyzing (7), we see that the function $F : \mathbf{Z}_l \longrightarrow \mathbb{R}$ is a convolution (see e.g., [8], [9]) of two discrete functions A and f. Let us remark that F contains the F-transform components F_k, $k = 1, \ldots, n-1$ among its values.

3. Convolution of Functions

Let us briefly remind the general definition of a convolution of functions (see e.g., [8]) and its properties. Let two functions $h, g : \mathbf{Z}_l \longrightarrow \mathbf{Z}_l$ be defined on the set of natural numbers $\mathbf{Z}_l = \{0, 1, \ldots, l-1\}$. Then a discrete convolution $h * g$ is a function $h * g : \mathbf{Z}_l \longrightarrow \mathbf{Z}_l$ defined by

$$(h * g)(t) = \sum_{j=0}^{l-1} h(t - j) g(j).$$

The important property is that the (discrete) Fourier transform (see below) of a convolution of functions is the product of their Fourier transforms, i.e.

$$\widehat{h * g} = \hat{h} \cdot \hat{g}, \quad (8)$$

where symbols $\widehat{h * g}$, $\hat{h}$, $\hat{g}$ denote the Fourier transforms of $h * g$, h, g, respectively.

4. Discrete Fourier Transform

In this section, we recall the definition of the discrete Fourier transform (see e.g., [8]) as well as some properties which will be used further on. Let $h : \mathbf{Z}_l \longrightarrow \mathbf{C}$ be a function from the set $\mathbf{Z}_l = \{0, 1, \ldots, l-1\}$ to the set of complex numbers $\mathbf{C}$. Then the discrete Fourier transform $\hat{h} : \mathbf{Z}_l \longrightarrow \mathbf{C}$ of h has the following representation:

$$\hat{h}(u) = \sum_{t=0}^{l-1} h(t) \cdot \exp(-2\pi i t u / l); \; u \in \mathbf{Z}_l. \quad (9)$$

The inversion formula recovers the function h from its discrete Fourier transform $\hat{h}$. It is defined by

$$h(t) = \frac{1}{l} \sum_{u=0}^{l-1} \hat{h}(u) \cdot \exp(2\pi i t u / l); \; t \in \mathbf{Z}_l. \quad (10)$$

5. Discrete Fourier Transform of F-transform Components

Let the function $F : \mathbf{Z}_l \longrightarrow \mathbb{R}$ be given by (7) and coincide with the F-transform components at certain nodes. The discrete Fourier transform of F is equal to:

$$\hat{F}(u) = \sum_{t=0}^{l-1} F(t) \cdot \exp(-2\pi i t u / l); \; u = 0, \ldots, l-1.$$

Using the inversion formula of the Fourier transform we will obtain the following representation of the function F:

$$F(t) = \frac{1}{l} \sum_{u=0}^{l-1} \hat{F}(u) \cdot \exp(2\pi i t u / l); \; t = 0, \ldots, l-1, \quad (11)$$

where expressions

$$\exp(2\pi i t u / l), \; u = 0, \ldots, l-1 \quad (12)$$

represent basis functions of the Fourier decomposition (11).

Our purpose is to estimate values of $\hat{F}(u)$ for each frequency u, $u = 0, \ldots, l-1$.

Main Result

Theorem 2
Let $\hat{f} : \mathbf{Z}_l \longrightarrow \mathbb{R}$ be the Fourier transform of a function $f : \mathbf{Z}_l \longrightarrow \mathbb{R}$. Let $n \geq 3$ and $A_1, \ldots, A_{n-1}$ be an h-uniform fuzzy partition of $[a, b]$ where $h = \frac{b-a}{n}$. Assume that the fuzzy partition $A_1, \ldots, A_{n-1}$ has $A : [-h, h] \longrightarrow [0, 1]$ as a generating function and moreover, A is of a triangular shape, i.e. $A(x)$ is defined on $[-h, h]$ by

$$A(x) = \begin{cases} 1 - \frac{|x|}{h}, & |x| \leq h, \\ 0, & |x| > h. \end{cases} \quad (13)$$

Let $F : \mathbf{Z}_l \longrightarrow \mathbb{R}$ be the discrete function given by (7), which contains the F-transform components of f among its values. Then the Fourier transform of F is given by

$$\hat{F}(0) = \hat{f}(0);$$
$$\hat{F}(u) \approx \frac{mn^2}{2\pi^2u^2}\exp(-2\pi iu/n)(1-\cos\frac{2\pi u}{n})\cdot\hat{f}(u);\quad u = 1,\dots,l-1,$$

where m is a fixed parameter.

PROOF: Let us consider F in the form of the convolution (7). Using the property (8), we can write

$$\hat{F}(u) = \hat{A}(u)\cdot\hat{f}(u). \tag{14}$$

Now we will estimate $\hat{A}(u)$ and leave $\hat{f}(u)$ as it is. Recall that in (14), the function A is normalized. We use the general expression (9) to compute $\hat{A}(u)$:

$$\hat{A}(u) = \sum_{t=0}^{l-1} A(t)\cdot\exp(-2\pi itu/l);\ u = 0,\dots,l-1.$$

In particular, $A(0) = 1$, which easily follows from normalization of A. For other values $u = 1,\dots,l-1$, the expression above will be approximated by respective integrals, so that

$$\hat{A}(u) \approx \frac{m}{h}\exp(-2\pi iu/n)\int_{-h}^{h} A(x)\cos\frac{2\pi xu}{nh}dx - i\,\frac{m}{h}\exp(-2\pi iu/n)\int_{-h}^{h} A(x)\sin\frac{2\pi xu}{nh}dx;\quad u = 1,\dots,l-1.$$

Because A is an even function on $[-h,h]$ (cf. (13)), the second integral in the expression above is 0. By direct integration of $\int_{-h}^{h} A(x)\cos\frac{2\pi xu}{nh}dx$, we obtain the following approximate values of $\hat{A}(u),\ u = 1,\dots,l-1$:

$$\hat{A}(u) \approx \frac{mn^2}{2\pi^2u^2}\exp(-2\pi iu/n)(1-\cos\frac{2\pi u}{n}). \tag{15}$$

Substitution of (15) into (14) gives us the desired expression:

$$\hat{F}(u) \approx \frac{mn^2}{2\pi^2u^2}\exp(-2\pi iu/n)(1-\cos\frac{2\pi u}{n})\cdot\hat{f}(u);\quad u = 1,\dots,l-1.$$

□

Corollary 1
Let the assumptions of the Theorem 2 be fulfilled. Then the influence of the Fourier coefficient $\hat{f}(u)$ in the representation (11) is weakened by the factor $\frac{1}{u^2}$.

In other words, Corollary 1 states that every F-transform component works as a low-pass filter of an original function.

6. Graphical Example

Below, we illustrate the idea described above on a particular example. We take an interval $[0,2\pi]$ as a universe and two discrete functions $\sin x$, $\sin 5x$, both defined at points $0 = p_0,\dots,p_{80} = 2\pi$, where $p_j = \frac{j\pi}{40}$, $j = 0,\dots,80$. We form a h-uniform fuzzy partition of the interval $[0,2\pi]$ represented by triangular basic functions $A_1,\dots,A_7$ over the nodes $x_0,\dots,x_8$, where the distance between each two neighboring nodes $h = \frac{\pi}{4}$.

For both functions we compute the direct discrete F-transform and the inverse discrete F-transform with respect to the given fuzzy partition of the interval $[0,2\pi]$. The function $\sin x$ with its inverse F-transform and the F-transform components is depicted on Figure 2 and the function $\sin 5x$ with the corresponding inverse F-transform and the F-transform components is shown on Figure 3. Both functions and their F-transforms are represented at points p_j, $j = 0,\dots,80$, although graphs seem to be continuous.

It is easy to see that the oscillation of $\sin 5x$ is higher than that of $\sin x$. Therefore by Lemma 2 from [1], for the same partition, the approximation of $\sin x$ by its inverse F-transform is closer to the original function than the approximation of $\sin 5x$ by its inverse F-transform.

In the frequency domain of the Fourier spectra, peaks of a high oscillating function give evidence of a presence of high frequencies. As can be seen from Figure 3, the F-transform components of $\sin 5x$ reduce the influence of high frequencies in the respective approximation given by the inverse F-transform.

Therefore, in order to increase the quality of approximation of a high oscillating function by its inverse F-transform it is necessary to increase the value of n leaving all other parameters unchanged, as can be seen on Figure 4. However, this requires a thorough analysis of the expression (15).

7. Conclusion

Our investigation was focused on the discrete F-transform and its effect in image processing. After a brief introduction, the discrete F-transform was presented in the form of a convolution. We investigated properties of the discrete Fourier transform of the direct discrete F-transform. We proved that every F-transform component works as a low-pass filter of an original function.

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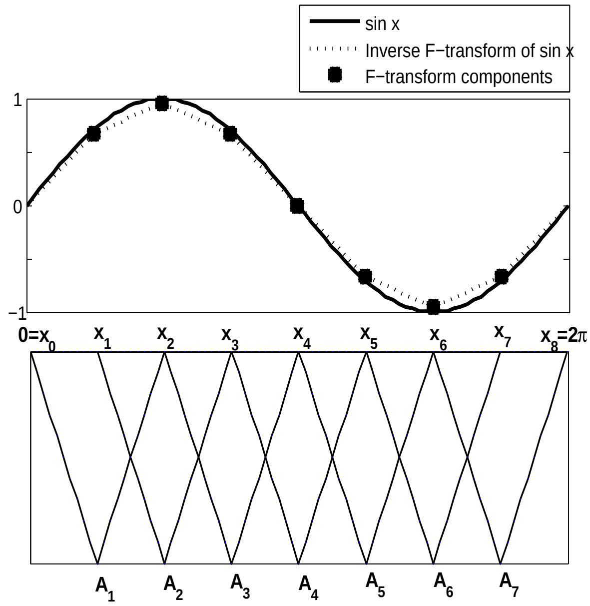

Figure 2: *Above:* Function $\sin x$, its inverse *F*-transform and corresponding 7 components of direct F-transform; *Below:* Fuzzy partition of $[0, 2\pi]$.

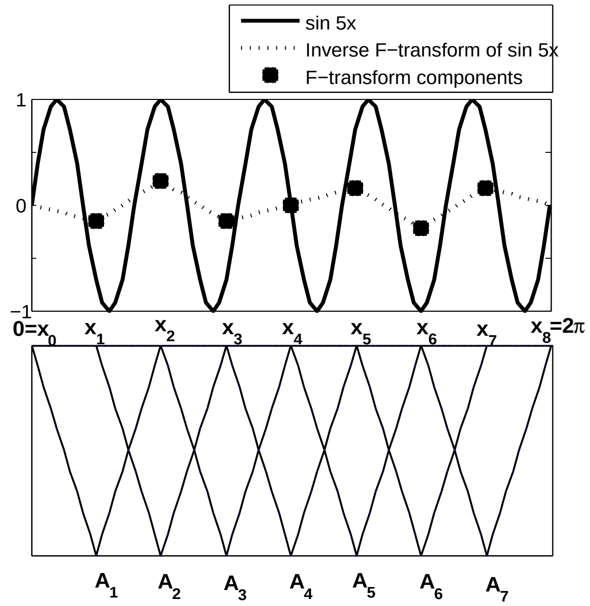

Figure 3: *Above:* Function $\sin(5x)$, its inverse *F*-transform and corresponding 7 components of direct F-transform; *Below:* Fuzzy partition of $[0, 2\pi]$.

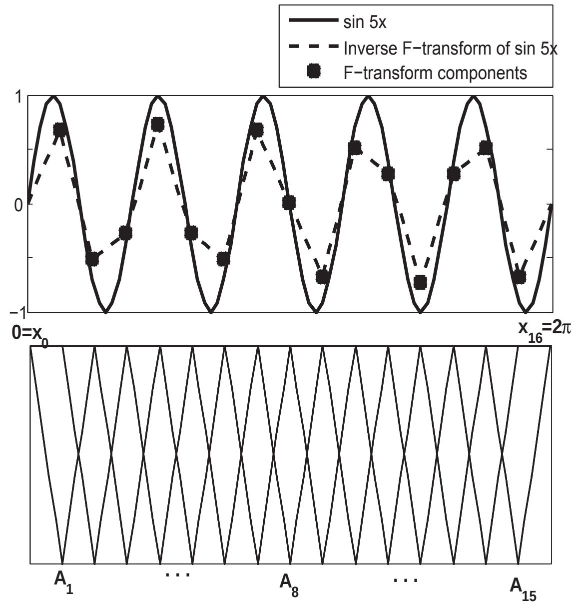

Figure 4: *Above:* Function $\sin(5x)$, its inverse *F*-transform and corresponding 15 components of direct F-transform; *Below:* Fuzzy partition of $[0, 2\pi]$.

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